Spectral analysis of sub-Riemannian Laplacians, Weyl measures

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Projet ANR SRGI (Sub-Riemannian Geometry and Interactions)
(\(M, D, g\)) sub-Riemannian (sR) structure:

- \(M\) (smooth) connected manifold of dimension \(n\)
- \(D \subset TM\) subsheaf (horizontal distribution)
- \(g\) Riemannian metric on \(D\)

Locally: \(D = \text{Span}(X_1, \ldots, X_m)\),

\[
\forall q \in M \quad \forall v \in D_q \\
g_q(v, v) = \inf \left\{ \sum_{i=1}^{m} u_i^2 \mid v = \sum_{i=1}^{m} u_i X_i(q) \right\}
\]

\((g_q: \text{positive definite quadratic form on } D_q)\)
Sub-Riemannian Laplacian

\( \mu \): arbitrary smooth volume on \( M \)

**Definition**

\[ -\triangle_{sR} = \text{selfadjoint nonnegative operator on } L^2(M, \mu) \text{ defined as the Friedrichs extension of the Dirichlet integral} \]

\[ Q(\phi) = \int_M \|d\phi\|_{g^*}^2 d\mu, \quad \phi \in C^\infty(M) \]

\( g^* \): cometric associated with \( g \)

\[ g^*(\xi, \xi) = \max_{v \in D_q \setminus \{0\}} \frac{\langle \xi, v \rangle^2}{g_q(v, v)} \]

**Equivalent definitions:**

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**Equivalent definitions:**

\[ \triangle_{sR} \phi = \text{div}_\mu (\nabla_{sR} \phi) \quad \forall \phi \in C^\infty(M) \]

where

- \( \text{div}_\mu \) defined by \( L_X d\mu = \text{div}_\mu (X) d\mu \quad \forall X \text{ vector field on } M \)

- horizontal gradient \( \nabla_{sR} \) defined by \( g_q(\nabla_{sR} \phi(q), v) = d\phi(q).v \quad \forall v \in D_q \)

note that \( \| d\phi \|^* = \| \nabla_{sR} \phi \|_g \)
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**Equivalent definitions:**

If \((X_1, \ldots, X_m)\) is a local \( g \)-orthonormal frame of \( D \) then \( \nabla_{sR} \phi = \sum_{i=1}^m (X_i \phi) X_i \) and

\[ \triangle_{sR} = -\sum_{i=1}^m X_i^* X_i = \sum_{i=1}^m \left( X_i^2 + \text{div}_\mu (X_i) X_i \right) \]

Setting \( h_X(q, p) = \langle p, X(q) \rangle \), note that \( \sigma_P(-\triangle_{sR}) = \sum_{i=1}^m (h_{X_i})^2 \) and that \( \sigma_{\text{sub}}(-\triangle_{sR}) = 0 \)
\[
\triangle = \triangle_{sR} = -\sum_{i=1}^{m} X_i^* X_i = \sum_{i=1}^{m} \left( X_i^2 + \text{div}_\mu(X_i)X_i \right)
\]

For \( M \) compact, under Hörmander’s assumption \( \text{Lie}(D) = TM \), the operator \(-\triangle\) is hypoelliptic (and even subelliptic), has a compact resolvent and thus a discrete spectrum

\[
0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty
\]

Let \( (\phi_k)_{k \in \mathbb{N}^*} \) be an orthonormal eigenbasis of \( L^2(M, \mu) \).

**Objectives**
- Establish heat kernel asymptotics.
- Derive a (micro-)local Weyl law for \( \triangle \).
- Identify (micro-)local Weyl measure.
- Establish Quantum Ergodicity (QE) properties.
More generally: $X_0$ smooth vector field on $M$, $c$ smooth bounded function on $M$.

$X = (X_0, X_1, \ldots, X_m)$

$$\triangle_X = \sum_{i=1}^{m} X_i^2 + X_0 + c \text{id}$$

$\rightarrow$ operator on $L^2(M, \mu)$.

Remark: $\triangle_X$ is selfadjoint $\iff X_0 = \sum_{i=1}^{m} \text{div}_\mu (X_i) X_i$. 
Sequence of subsheafs $D^k \subset TM$:
- $D^0 = \{0\}$
- $D^1 = D = \text{Span}(X_1, \ldots, X_m)$
- $D^{k+1} = D^k + [D, D^k]$ for $k \geq 1$

sR flag at $q$:

$$\{0\} = D^0_q \subset D_q = D^1_q \subset D^2_q \subset \ldots \subset D^{r(q)-1}_q \subsetneq D^r_q = T_q M$$

$q$ is said **regular** if the flag at $q$ is regular.
The sR structure is said **equiregular** if all points are regular.
Sequence of subsheaves $D^k \subset TM$:
- $D^0 = \{0\}$
- $D^1 = D = \text{Span}(X_1, \ldots, X_m)$
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sR flag at $q$:

$$\{0\} = D^0_Q \subset D^1_Q \subset D^2_Q \subset \ldots \subset D^{r(q)-1}_Q \subsetneq D^r_Q = T_q M$$

Example: 3D contact case

$$X_1 = \partial_x, \quad X_2 = \partial_y + x \partial_z$$
$$[X_1, X_2] = \partial_z$$
$$\rightarrow \text{equiregular}$$
$$n_1 = 2, \quad n_2 = 3$$
$$Q = 4$$

Example: 3D Martinet case

$$X_1 = \partial_x, \quad X_2 = \partial_y + x^2 \partial_z$$
$$[X_1, X_2] = x \partial_z, \quad [X_1, [X_1, X_2]] = \partial_z$$
$$\rightarrow \text{singular at } x = 0 \text{ (and contact outside)}$$
$$n_1(0) = n_2(0) = 2, \quad n_3(0) = 3$$
$$Q(0) = 5$$
Nilpotentization of the sR structure \((M, D, g)\) at \(q_0 \in M\): (intrinsic) sR structure \((\hat{M}^{q_0}, \hat{D}^{q_0}, \hat{g}^{q_0})\), where

- \(\hat{M}^{q_0}\) smooth connected manifold of dimension \(n\)
- \(\hat{D}^{q_0} = \text{Span}(\hat{X}_{q_0}^1, \ldots, \hat{X}_{q_0}^m)\)
- sR metric \(\hat{g}^{q_0}\) is defined, accordingly, by

\[
\forall x \in \hat{M}^{q_0}, \forall v \in \hat{D}^{q_0}_x, \quad \hat{g}^{q_0}_x(v, v) = \inf \left\{ \sum_{i=1}^{m} u_i^2 : v = \sum_{i=1}^{m} u_i \hat{X}_{q_0}^i(x) \right\},
\]

(equivalence class under the action of sR isometries on sR structures)

If \(q_0\) is regular then \(\hat{M}^{q_0} \sim \mathcal{G}^{q_0}\) (nilpotent Lie group of diffeomorphisms of \(\mathbb{R}^n\) generated by the \(\exp(t\hat{X}_{q_0}^i)\)) is a Carnot group, endowed with a left-invariant sR structure.

In a local chart \(\psi_{q_0} : U_{q_0} \subset M \to V_0 \subset \mathbb{R}^n\) of privileged coordinates:

- Dilation \(\delta_{\varepsilon}(x) = (\varepsilon^{w_1(q_0)} x_1, \ldots, \varepsilon^{w_n(q_0)} x_n)\)
- Nilpotentization at \(q_0\) of the vector fields: \(\hat{X}_{q_0}^i = \lim_{\varepsilon \to 0} \varepsilon \delta_{\varepsilon}^*(\psi_{q_0}) \ast X_i\)
- Nilpotentization at \(q_0\) of the measure: \(\hat{\mu}^{q_0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon \mathcal{Q}(q_0)} \delta_{\varepsilon}^*(\psi_{q_0}) \ast \mu = \text{Cst}(q_0) \, dx\)
Nilpotentization

Example:

\[ X_1 = \partial_x, \quad X_2 = \partial_y + x \partial_z + x^2 \partial_z \]
\[ \hat{X}_1^0 = \partial_x, \quad \hat{X}_2^0 = \partial_y + x \partial_z \]

Example:

\[ X_1 = \partial_x, \quad X_2 = \partial_y + \frac{x^2}{2} \partial_z + x^3 \partial_z \]
\[ \hat{X}_1^0 = \partial_x, \quad \hat{X}_2^0 = \partial_y + \frac{x^2}{2} \partial_z \]

Nilpotentization \( \hat{X}_i^{q_0} \) of \( X_i \) = “first-order” terms of the “nonholonomic Taylor expansion”

\( (\hat{M}^{q_0}, \hat{D}^{q_0}, \hat{g}^{q_0}) \) = tangent space (in the metric sense of Gromov)

(Carnot group if \( q_0 \) is regular)

Carnot groups are to sR geometry as Euclidean spaces are to Riemannian geometry.
In contrast to Riemannian geometry where all tangent spaces are isometric, two nilpotentizations at \( q_0 \) and \( q_1 \) may not be sR-isometric.
Heat kernels of $\triangle$ and of $\hat{\triangle}^{q_0}$:

- $D(\triangle) = \{ f \in L^2(M, \mu) \mid \triangle f \in L^2(M, \mu) \}$. 
  
  $(\triangle, D(\triangle))$ generates $(e^{t\triangle})_{t \geq 0}$ on $L^2(M, \mu)$. 

Let $e = e_{\triangle, \mu} : (0, +\infty) \times M \times M \to \mathbb{R}$ be the heat (Schwartz) kernel.

- Nilpotentized sR Laplacian: $\hat{\triangle}^{q_0} = \sum_{i=1}^{m} \left( \hat{X}^{q_0}_i \right)^2$ 
  
  $D(\hat{\triangle}^{q_0}) = \{ f \in L^2(\hat{M}^{q_0}, \hat{\mu}^{q_0}) \mid \hat{\triangle}^{q_0} f \in L^2(\hat{M}^{q_0}, \hat{\mu}^{q_0}) \}$. 

  $\to$ semigroup $(e^{t\hat{\triangle}^{q_0}})_{t \geq 0}$ on $L^2(\hat{M}^{q_0}, \hat{\mu}^{q_0})$. 

Let $\hat{e} = e_{\hat{\triangle}^{q_0}, \hat{\mu}^{q_0}} : (0, +\infty) \times \hat{M}^{q_0} \times \hat{M}^{q_0} \to \mathbb{R}$ be the heat (Schwartz) kernel.

Note that (homogeneity)

$$e(t, x, x') = \varepsilon^{Q(q_0)} \hat{e}(\varepsilon^2 t, \delta_{\varepsilon}(x), \delta_{\varepsilon}(x'))$$
Fundamental lemma

$q_0 \in M$ arbitrary, and $\mu$ arbitrary smooth measure on $M$. In a local chart of privileged coordinates at $q_0$:

$$\forall k \in \mathbb{N} \quad t^{Q(q_0)/2} \, e(t, \delta_{\sqrt{t}}(x), \delta_{\sqrt{t}}(x'))$$

$$= \tilde{e}(1, x, x') + t \, a_1(x, x') + \cdots + t^k \, a_k(x, x') + o(t^k),$$

as $t \to 0^+$, in $C^\infty(M \times M)$, with $a_j$ smooth.

- $q_0$ need not be regular.
- If $q_0$ is regular, then the above convergence and asymptotic expansion are also locally uniform with respect to $q_0$.
- Still valid for $\Delta_X = \sum_{i=1}^{m} X_i^2 + X_0 + c \, \text{id}$, with an expansion in $t^{k/2}$, provided that:
  - either $X_0$ smooth section of $D$;
  - or $X_0$ smooth section of $D^2$, and then replace $\hat{\Delta}_{q_0}$ with $\hat{\Delta}_{q_0} + \hat{X}_0^{q_0}$.
Heat kernel asymptotics

Fundamental lemma

$q_0 \in M$ arbitrary, and $\mu$ arbitrary smooth measure on $M$. In a local chart of privileged coordinates at $q_0$:

$$\forall k \in \mathbb{N} \quad t^{Q(q_0)/2} e(t, \delta \sqrt{t}(x), \delta \sqrt{t}(x'))$$

$$= \tilde{e}(1, x, x') + t a_1(x, x') + \cdots + t^k a_k(x, x') + o(t^k),$$

as $t \to 0^+$, in $C^\infty(M \times M)$, with $a_j$ smooth.

- Taking $x = x' = 0$, we get the expansion of the kernel along the diagonal, and

$$e(t, q_0, q_0) d\mu(q_0) \sim \frac{\tilde{e}(1, 0, 0)}{t^{Q(q_0)/2}} d\mu(q_0) = \tilde{e}(t, 0, 0) d\mu(q_0) = \tilde{e}(t, 0, 0) d\nu(q_0) \quad \forall \nu$$

→ useful to derive the local Weyl law.
Generalization of results by Métivier (1976), Ben Arous (1989).

- estimations near the diagonal → microlocal Weyl law and singular sR structures.
General question in sR geometry: define an intrinsic volume in a sR manifold.

[Agrachev Barilari Boscain 2012]

- **Hausdorff** (standard or spherical)  
  Mitchell 1985, Gromov 1996
- **Popp**  
  Montgomery 2002
- a new one: **Weyl**
Let $q \in M$, regular or not.

**Popp volume** (Montgomery 2002): canonical smooth volume form associated with the sR metric $g$ and the flag structure, defined by

$$|dP(q)| = \Phi^* |\nu_1 \wedge \cdots \wedge \nu_r|$$

with the canonical isomorphism

$$\Lambda^n(T_q^* M) \xrightarrow{\Phi} \Lambda^n \left( \bigoplus_{k=1}^{r(q)} D_q^k / D_q^{k-1} \right)^*$$

and with $\nu_k$ = canonical volume form on $D_q^k / D_q^{k-1}$ induced by the Euclidean structure coming from the surjection $D^\otimes q^k \rightarrow D_q^k / D_q^{k-1}$ (take Lie brackets modulo $D_q$).

- $P$ is invariant under (local) sR isometries
- $P$ commutes with nilpotentization: $\widehat{P^q}_M = P^q_{\widehat{M}} \Rightarrow P$ is "doubly intrinsic"
- $P$ is smooth near regular points

Explicit expression in [Barilari Rizzi 2013] (in the equiregular case)
\( -\Delta \phi_k = \lambda_k \phi_k, \quad (\phi_k)_{k \in \mathbb{N}^*} \text{ orthonormal eigenbasis of } L^2(M, \mu), \quad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty \)

Spectral counting function: \( N(\lambda) = \# \{ n \mid \lambda_k \leq \lambda \} \)

\( M \) compact

**Local Weyl measure** = probability measure \( w_\Delta \) on \( M \) defined (if the limit exists) by

\[
\int_M f \, dw_\Delta = \lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \int_M f|\phi_k|^2 \, d\mu \quad \forall f \in C^0(M)
\]

i.e.,

\[
w_\Delta = \text{weak } \lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} |\phi_k|^2 \, \mu
\]

(Cesàro mean)

- If it exists:
  - \( w_\Delta \) does not depend on the choice of \( (\phi_k)_{k \in \mathbb{N}^*} \)
  - \( w_\Delta \) is invariant under sR isometries of \( M \)

- In the equiregular case: \( w_\Delta \) exists, is smooth (cf further), does not depend on \( \mu \), and commutes with nilpotentization: \( \overline{w_\Delta}^q = "w_\Delta^q" \)
(Micro-)local Weyl measure

\[-\triangle \phi_k = \lambda_k \phi_k, \quad (\phi_k)_{k \in \mathbb{N}^*} \text{ orthonormal eigenbasis of } L^2(M, \mu), \quad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty\]

Spectral counting function: \( N(\lambda) = \#\{n \mid \lambda_k \leq \lambda\} \quad M \text{ compact} \)

Local Weyl measure = probability measure \( w_\triangle \) on \( M \) defined (if the limit exists) by

\[
\int_M f \, dw_\triangle = \lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \int_M |\phi_k|^2 \, d\mu \quad \forall f \in C^0(M)
\]

Microlocal Weyl measure = probability measure on \( S^*M = S(T^*M) \) defined (if the limit exists) by

\[
\int_{S^*M} a \, dW_\triangle = \lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_k \leq \lambda} \langle \text{Op}(a)\phi_k, \phi_k \rangle_{L^2(M, \mu)} \quad \forall a \in S^0(S^*M)
\]

Note that \( \pi_* W_\triangle = w_\triangle \) with \( \pi : T^*M \to M \) canonical projection.

Lemma: We always have \( \text{supp}(W_\triangle) \subset S\Sigma \), where \( \Sigma = D^\perp \) (annihilator of \( D \)).

General problem: identify these measures, compare them with Hausdorff, Popp.
Defining the *averaged correlation* of eigenfunctions by

\[ C(x, y) = \lim_{\lambda \to +\infty} \frac{1}{N(\lambda)} \sum_{\lambda k \leq \lambda} \phi_k(x + y/2) \bar{\phi}_k(x - y/2), \]

for all \((x, y) \in M \times M\) (when the limit exists), the correlation is the Fourier transform with respect to \(\xi\) of the microlocal Weyl density, i.e.,

\[ C(x, y) = \int e^{-iy \cdot \xi} \frac{dW_\triangle(x, \xi)}{dx} \]
Local Weyl law in the equiregular case

Assume that \((M, D, g)\) is equiregular \(\Rightarrow Q(q) = Q\) constant.

\(\forall q \in M\), consider the sR structure \((\hat{M}^q, \hat{D}^q, \hat{g}^q)\), that is the nilpotentization of \((M, D, g)\) at \(q\). It induces the sR Laplacian \(\hat{\triangle}^q = \sum_{i=1}^{m} (\hat{X}_i^q)^2\).

\(\hat{e}^q = e_{\hat{\triangle}^q, \hat{P}^q} = \) heat kernel on \((0, +\infty) \times \hat{M}^q \times \hat{M}^q\) associated with \(\hat{\triangle}^q\) and with the Popp measure \(\hat{P}^q\) on \(\hat{M}^q\).

Since (along the diagonal)

\[ t^{Q/2} e(t, q, q) \, d\mu(q) \longrightarrow \hat{e}^q(1, 0, 0) \, dP(q), \]

as \(t \rightarrow 0^+\), we get

**Theorem**

\[
\int_M e_{\triangle_X, \mu}(t, q, q) f(q) \, d\mu(q) = \frac{\int_M \hat{e}^q(1, 0, 0) f(q) \, dP(q)}{t^{Q/2}} + o \left( \frac{1}{t^{Q/2}} \right) \quad \forall f \in C_c^0(M)
\]
Local Weyl law in the equiregular case

By inverse Laplace transform (Karamata), we get:

**Corollary**

Assumptions: $M$ compact, equiregular. Then:

$$\sum_{\lambda_k \leq \lambda} \int_M f|\phi_k|^2 \, d\mu = \frac{1}{\Gamma(Q/2 + 1)} \lambda^{Q/2} \int_M \hat{e}^q(1, 0, 0) f(q) \, dP(q) + o(\lambda^{Q/2}) \quad \forall f \in C_0^0(M)$$

as $\lambda \to +\infty$. In particular:

$$N(\lambda) \sim \int_M \hat{e}^q(1, 0, 0) \, dP(q) \frac{1}{\Gamma(Q/2 + 1)} \lambda^{Q/2}$$

and the local Weyl measure $w_\triangle$ exists, is smooth, and

$$dw_\triangle(q) = \hat{e}^q(1, 0, 0) \, dP(q)$$

Remark: Since $w_\triangle$ is smooth, it differs in general from $\mathcal{H}_S$ (which is not smooth in general for $n \geq 5$, see [AgrachevBarilariBoscain 2012])
Local Weyl law in the equiregular case

Equiregular case: \[ dw_\triangle (q) = \hat{e}^q(1, 0, 0) \, dP(q) \]

Hence: \( w_\triangle = P \) (up to some multiplying scalar; i.e., Weyl = Popp)

\[ \Leftrightarrow \hat{e}^q(1, 0, 0) = \text{Cst} \Leftrightarrow \text{all nilpotent sR structures } (\hat{M}^q, \hat{D}^q, \hat{g}^q) \text{ are sR-isometric} \]

\[ \Leftrightarrow \text{Iso}_{sR}(M) \text{ of sR isometries of } M \text{ acts transitively on } M \]

This is so in the following cases:

- The sR structure on \( M \) is free
- The sR structure on \( M \) is nilpotent and equiregular, and \( \dim M \leq 5 \), with:
  - dimension 3: growth vector \((2, 3)\) (Heisenberg)
  - dimension 4: growth vector \((2, 3, 4)\) (Engel) and \((3, 4)\) (quasi-Heisenberg)
  - dimension 5: growth vector \((2, 3, 5)\) (Cartan), \((2, 3, 4, 5)\) (Goursat rank 2), \((3, 5)\) (corank 2), \((3, 4, 5)\) (Goursat rank 3)

[\text{Agrachev Barilari Boscain 2012}]

Remark: In the 3D contact case, we have \( \hat{e}^q(1, 0, 0) = 1/16. \)

Remark: \( w_\triangle \neq P \) in general (even in the equiregular case; in which both are smooth)
Microlocal Weyl law in the equiregular case

Define $\Sigma^i = (D^i)\perp \subset T^*M$ (annihilator of $D^i$) for $i = 1, \ldots, r$.

**Theorem**

**Assumptions:** $M$ compact, equiregular.

For every $A \in \Psi^0(M)$, we have

$$\sum_{\lambda_k \leq \lambda} \langle A\phi_k, \phi_k \rangle = \frac{C}{\Gamma(\frac{Q}{2} + 1)} \lambda^{\frac{Q}{2}} \int_{S\Sigma^{r-1}} a dW_\triangle + o(\lambda^{\frac{Q}{2}}),$$

where $S\Sigma = (\Sigma \setminus \{0\})/\mathbb{R}^+$. In particular:

$$\text{supp}(W_\triangle) \subset S\Sigma^{r-1}$$

and

$$dW_\triangle (q, p) = \frac{1}{(2\pi)^n \int_M \hat{e}^q(1, 0, 0) dP(q)} \left( \int_{T^*_q M/\Sigma_q^{r-1}} K_q \right) \delta_{\Sigma^{r-1}}(q, p),$$

with $K_q(p) = \int_M e^{iq' \cdot p} \hat{e}^q(1, q', q) dP(q') = \mathcal{F} \left( q' \mapsto \hat{e}^q(1, q', q) \right)(p)$. 

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Singular sR structures

Martinet case (without singularity):
- $M$ compact 3D manifold, endowed with a smooth measure $\mu$
- $D = \ker \alpha$, with $\alpha \wedge d\alpha$ vanishing on $S$ (Martinet surface), $D$ transverse to $S$.

$$\Leftrightarrow D^2 \neq D^3, \quad D^2 = TM \text{ outside of } S, \quad D^3 = TM \text{ along } S, \quad D \cap TS \text{ line bundle over } S.$$ 

Local model: $\alpha = dz - \frac{x^2}{2} dy$, $S = \{x = 0\}$.

In $M \setminus S$, we have $dP = \frac{1}{|x|} dx \, dy \, dz = \frac{1}{|x|} dx \otimes d\nu$ with $\nu$ smooth.

**Theorem**

We have $(\pi|_{\Sigma} : S\Sigma \to M$ double covering$)$

$$N(\lambda) \sim \frac{\nu(S)}{32} \lambda^2 \ln \lambda, \quad w_\Delta = \frac{\nu}{\nu(S)}, \quad W_\Delta = \frac{1}{2} \pi^*_\Sigma w_\Delta$$

“Spectral concentration occurs on the Martinet (singular) surface.”

**Two-terms expansion:**

$$\int_M e_{\Delta, \mu}(t, q, q)f(q) \, d\mu = \frac{1}{t^2} \left( \left( \frac{|\ln t|}{16} + A \right) \int_S f \, d\nu + \text{P.V.} \int_M fh \, dP + o(1) \right)$$
Singular sR structures

**Grushin case:** almost-Riemannian structure on a surface

- $M$ compact 2D manifold, endowed with a smooth measure $\mu$
- Locally, $X$ and $Y$ generate $TM$ except along a 1D submanifold $S$

Local model: $X = \partial_x$, $Y = x \partial_y$. \hspace{1cm} $-\triangle = X^*X + Y^*Y$

In $M \setminus S$, we have $dP = \frac{1}{|x|} \, dx \, dy = \frac{1}{|x|} \, dx \otimes d\nu$ with $\nu$ smooth.

**Theorem**

We have \hspace{1cm} $(\pi|_{\Sigma} : \Sigma \rightarrow M$ double covering$)$

$$N(\lambda) \sim \frac{\nu(S)}{4\pi} \lambda \ln \lambda, \quad W_\triangle = \frac{\nu}{\nu(S)}, \quad W_\triangle = \frac{1}{2} \pi^{\star}|_{\Sigma} W_\triangle$$

Two-terms expansion:

$$\int_M e_{\triangle,\mu}(t, q, q) f(q) \, d\mu = \frac{1}{4\pi t} \left( (|\ln t| + \gamma + 4 \ln 2) \int_S f \, d\nu + \text{P.V.} g \int_M f \, dx_g + o(1) \right)$$
Generalization to any singular structure: ongoing.

\[
\lim_{\lambda \to +\infty} \frac{\ln N(\lambda)}{\ln \lambda} = \text{(sR spherical) Hausdorff dimension}
\]

This may a priori allow \( N(\lambda) \sim \lambda^\alpha \ln \lambda \ln \ln \lambda \) (open question).
Quantum Limits (QL)

- $M$ locally compact space, endowed with a Radon measure $\mu$
- $T$ selfadjoint nonnegative operator on $L^2(M, \mu)$, with compact resolvent
  - discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to +\infty$
  - $(\phi_j)_{j \in \mathbb{N}^*}$ orthonormal eigenbasis of $L^2(M, \mu)$

**A quantum limit** (on the base) is a (weak) limit of the sequence of probability measures $|\phi_j^2| \, dx$.

More generally (pseudo-diff. version), a QL is a probability measure on $S^* M$, closure point of the sequence of measures $\mu_j(a) = \langle \text{Op}(a)\phi_j, \phi_j \rangle$ ($a$: symbol of order 0)

- Does the energy **equidistribute** at high frequencies (i.e., $\mu_j \rightharpoonup$ uniform measure)
- Or, at the contrary, may the energy **concentrate**? (for instance, $\mu_j \rightharpoonup$ Dirac)

→ insight on the way eigenfunctions concentrate

General question in quantum physics, quantum chaos: what are the possible QLs?

Y. Colin de Verdière, L. Hillairet, E. Trélat

Spectral analysis of sub-Riemannian Laplacians
Examples of quantum limits for the Dirichlet-Laplacian

- **1D case** $\Omega = (0, \pi)$: $\phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$, $\sin^2(jx) \rightarrow 1/2$
  $\Rightarrow$ Quantum Unique Ergodicity (QUE)

- **2D square** $\Omega = (0, \pi)$: $\phi_{j,k}(x) = \frac{2}{\pi} \sin(jx) \sin(ky)$ $\rightarrow$ many QLs

- **2D disk**: Bessel functions

Whispering galleries phenomenon

All quantum limits in the disk: [Hillairet Privat Trélat]

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All quantum limits in the disk: [Hillairet Privat Trélat]
Integrable systems:

QLs must be invariant under the geodesic flow.

On the sphere: every invariant measure is a QL (Zelditch).

Chaotic systems:
Quantum Ergodicity (QE)

Water wave in a stadium-shaped basin:

Laser wave in an optic fiber:

Spectral analysis of sub-Riemannian Laplacians

Y. Colin de Verdière, L. Hillairet, E. Trélat
We say we have QE for \( (T, (\phi_n)_{n \in \mathbb{N}^*}) \) if there exist a probability measure \( \nu \) on \( M \) and a subsequence \( (n_j)_{j \in \mathbb{N}^*} \) of density one such that

\[
|\phi_{n_j}|^2 \, d\mu \rightharpoonup d\nu \quad \text{as } j \to +\infty
\]

More generally (pseudo-diff. version),

\[
\left\langle \text{Op}(a) \phi_{n_j}, \phi_{n_j} \right\rangle_{L^2(M, \mu)} \to \int_{\Sigma} a \, d\tilde{\nu} \quad \forall a \in S^0
\]
Quantum Ergodicity

Historical example of QE, by A. Shnirelman in 1974 (incomplete proof):

On \((M, g)\) compact Riemannian manifold, if the geodesic flow is ergodic, then we have QE for any orthonormal basis of eigenfunctions of the Laplace-Beltrami operator \(\Delta\), with \(d\nu = \text{normalized Riemannian volume}\).

(ergodicity for instance if the curvature of \((M, g)\) is \(< 0\))

More generally, in pseudo-diff. version: \(\tilde{\nu} = \text{Liouville measure on } S^* M\)


Other existing results (Schrödinger operators, graphs), but always for elliptic operators and never in hypoelliptic situations.
QUE conjecture (Rudnick-Sarnak 1994): every compact manifold having negative sectional curvature satisfies QUE.

If QUE fails, we may have **scars**: energy concentration phenomena (exceptional subsequences may converge to other invariant measures, like, for instance, measures carried by closed geodesics)
The 3D contact case

\( M \) compact connected 3D manifold (without boundary), arbitrary smooth measure \( \mu \), Riemannian contact structure \( \Rightarrow \) sub-Riemannian Laplacian \( \triangle_{SR} \)

\( (\phi_n, \lambda_n)_{n \in \mathbb{N}^\ast} \) orthonormal eigenbasis of \( L^2(M, \mu) \)

**Theorem**: If the Reeb flow is ergodic on \( M \) for the Popp measure, then we have QE.

We identify \( S^*M = U^*M \cup S\Sigma \), with \( U^*M = \{g^* = 1\} \) (cylinder bundle).

**Theorem**: Without any ergodicity assumption,

1. \( \forall \beta \) quantum limit associated with \( (\phi_n)_{n \in \mathbb{N}^\ast} \),
   
   \[ \beta = \beta_0 + \beta_\infty \text{ with } \beta_0 \perp \beta_\infty \text{ and } \]
   
   - \( \text{supp}(\beta_0) \subset U^*M \), and \( \beta_0 \) invariant under the sR geodesic flow
   - \( \text{supp}(\beta_\infty) \subset S\Sigma \), and \( \beta_\infty \) invariant under the (lift to \( S\Sigma \) of the) Reeb flow

2. \( \exists (n_j)_{j \in \mathbb{N}^\ast} \) of density one s.t. \( \forall \beta \) quantum limit associated with \( (\phi_{n_j})_{j \in \mathbb{N}^\ast} \), we have \( \text{supp}(\beta) \subset S\Sigma \) (i.e., \( \beta_0 = 0 \))
Proposition: In the 3D Heisenberg flat case:

- If $\gamma$ is a QL for the flat torus $\mathbb{R}^2/\sqrt{2}\pi\mathbb{Z}^2$, then $\beta = \gamma \otimes |dz| \otimes \delta_{p_z=0}$ is a QL for $\triangle$ (with $\beta\infty = 0$).
- Any probability measure $\beta$ on $\Sigma$ that is invariant under the Reeb flow is a QL for $\triangle$ (with $\beta\infty = 0$).
- There exist QLs $\beta_0$ and $\beta\infty$ for $\triangle$, with $\beta_0$ invariant under the sR flow and $\beta\infty$ supported on $\Sigma$ and invariant under the Reeb flow, such that, for every $a \in [0, 1]$, $\beta_a = a\beta_0 + (1 - a)\beta\infty$ is also a QL for $\triangle$.

Model: compact Heisenberg flat case

- Heisenberg group $G = \mathbb{R}^3$ with product rule $(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z' - xy')$
- Contact form $\alpha = dz + x dy$
- Vector fields $X = \partial_x$ and $Y = \partial_y - x\partial_z$ left-invariant on $G$
- $\Gamma = \{(x, y, z) \in G \mid x, y \in \sqrt{2}\pi\mathbb{Z}, z \in 2\pi\mathbb{Z}\}$ discrete co-compact subgroup of $G$
- $M = \Gamma \backslash G$, $D = \ker\alpha$ endowed with $g$ s.t. $(X, Y)$ $g$-orthonormal frame of $D$
- Reeb vector field $Z = -[X, Y] = \partial_z$
- Lebesgue volume $d\mu = dx \, dy \, dz = dP$ Popp volume
- $\triangle = X^2 + Y^2$

$$\text{Spec}(-\triangle_{sR}) = \{\lambda_{\ell,m} = (2\ell + 1)|m| \mid m \in \mathbb{Z} \setminus \{0\}, \ell \in \mathbb{N}\} \cup \{\mu_{j,k} = 2\pi(j^2 + k^2) \mid j, k \in \mathbb{Z}\},$$

where $\lambda_{\ell,m}$ is of multiplicity $|m|$
A general path towards QE

(see Zelditch)

\[ N(\lambda) = \#\{n \mid \lambda_n \leq \lambda \} \]

First step: establish a microlocal Weyl law

(and identify the invariant measure \( \nu \))

\[ E(A) \overset{\text{def}}{=} \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leq \lambda} \langle A\phi_n, \phi_n \rangle = \bar{a} = \int_{ST^* M} a \, dW_\Delta \]

\( \forall A \in \Psi^0 \text{ with } a = \sigma_P(A). \)

(\( E(A) = \text{Cesáro mean} \))

\( \rightarrow \) Cesáro convergence property, under weak assumptions (without ergodicity):

\[ \langle (A - \bar{a} \text{id})\phi_n, \phi_n \rangle \to 0 \quad \text{in Cesáro mean} \]
A general path towards QE

(see Zelditch)

\[ N(\lambda) = \# \{ n \mid \lambda_n \leq \lambda \} \]

**Second step: prove a variance estimate**

\[ V(A - \bar{a} \text{id}) \overset{def}{=} \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \leq \lambda} |\langle (A - \bar{a} \text{id})\phi_n, \phi_n \rangle|^2 = 0. \]

i.e.

\[ |\langle (A - \bar{a} \text{id})\phi_n, \phi_n \rangle|^2 \to 0 \quad \text{in Cesáro mean} \]

→ Combine the microlocal Weyl law with ergodicity properties of some associated classical dynamics and with an Egorov theorem.
A general path towards QE

(see Zelditch)

End of the proof of QE:

Lemma (Koopman and Von Neumann)

Given a bounded sequence \((u_n)_{n \in \mathbb{N}}\) of nonnegative real numbers:

\[
\frac{1}{n} \sum_{k=0}^{n-1} u_k \xrightarrow{n \to +\infty} 0 \iff \exists S \subset \mathbb{N} \text{ of density one s.t. } u_k \xrightarrow{k \to +\infty} 0 \quad \forall k \in S
\]

(density one meaning that \(\frac{1}{n} \# \{k \in S \mid k \leq n - 1\} \xrightarrow{n \to +\infty} 1\))

Hence, there exists a density-one sequence \((n_j)_{j \in \mathbb{N}^*}\) s.t.

\[
\lim_{j \to +\infty} \langle A\phi_{n_j}, \phi_{n_j} \rangle = \bar{a}.
\]

Conclusion with a diagonal argument, using the fact that \(S^0\) admits a countable dense subset.
The 3D contact case

Normal form in the Heisenberg flat case:

\[-\Delta_{sR} = R\Omega = \Omega R\]

with

- \( R = \sqrt{Z^*Z} \) acting by multiplication by \(|m|\) on the functions \( e^{imz} f(x, y) \)
- \( \Omega = U^2 + V^2, \quad U = \frac{1}{i} R^{-\frac{1}{2}} X = \text{Op}^W(\frac{h_x}{\sqrt{|h_z|}}), \quad V = \frac{1}{i} R^{-\frac{1}{2}} Y = \text{Op}^W(\frac{h_y}{\sqrt{|h_z|}}) \)

harmonic oscillator

(pseudo of order 1/2 in the cone \( C_c = \{ p_z^2 > c(p_x^2 + p_y^2) \} \))

- \([U, V] = \pm \text{id}, \quad [R, \Omega] = 0, \quad e^{2i\pi \Omega} = \text{id}\)

In terms of symbols,

\[\sigma(-\Delta_{sR}) = h^2_X + h^2_Y = |h_z| \left( \left( \frac{h_x}{\sqrt{|h_z|}} \right)^2 + \left( \frac{h_y}{\sqrt{|h_z|}} \right)^2 \right) \]

\[ R \quad \quad U^2 + V^2 = \Omega \]
The 3D contact case

General case: Birkhoff normal form

\((M, \triangle_{sR})\) arbitrary 3D contact case \(\sim\) \((M_H, \triangle_H)\) 3D Heisenberg flat case

**Theorem (classical normal form, see also Melrose 1984)**

\[ \forall q \in M \quad \exists C_q \text{ conical neighbourhood of } \Sigma_q^\pm \]

\[ \exists \chi : C_q \to M_H \text{ homogeneous symplectic diffeo., with } \chi(\Sigma^\pm \cap C_q) \subset \Sigma_H, \text{ s.t.} \]

\[ \sigma_P(-\triangle_H) \circ \chi = \sigma_P(-\triangle_{sR}) + O_\Sigma(\infty) \]

**Proof** by symplectic reductions, Darboux-Weinstein lemma, cohomological equations. By quantization:

**Corollary (quantum normal form)**

In a microlocal neighborhood of \(\Sigma_q\):

\[ -\triangle_{sR} = R\Omega + V_0 + O_\Sigma(\infty) \]

- \(V_0 \in \Psi^0\), \(R, \Omega \in \Psi^1\) selfadjoint
- \(\sigma_P(R) = |h_Z| + O_\Sigma(2)\), \(\sigma_P(\Omega) = \frac{h_x^2}{|h_Z|} + \frac{h_y^2}{|h_Z|} + O_\Sigma(3)\)
- \([R, \Omega] = 0 \mod \Psi^{-\infty}\), \(\exp(2i\pi\Omega) = \text{id} \mod \Psi^{-\infty}\)
3D contact case: are the Reeb periods spectral invariants?

5D contact case: two harmonic oscillators, hence resonances

⇒ failure of Birkhoff normal form at infinite order along \( \Sigma \)

QE (and QLs) in more general cases:
- Grushin: we have QE if the singular curve is connected.
- Martinet: ergodicity of the singular flow (in Martinet surface) ⇒ QE?
- Quasi-contact in dim 4 (equiregular):
  magnetic lines = projections of singular geodesics.
  ergodicity of the magnetic vector field ⇒ QE?

Existence of \( w_\Delta \) and \( W_\Delta \) in general (singular cases)

Controllability, observability of subelliptic heat or wave equations.