Strichartz estimates and local dispersion on the Heisenberg group

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Paris, September 2020

Joint with H. Bahouri and D. Barilari

Strichartz estimates for the Schrödinger equation on $\mathbb{R}^n$

Consider the Schrödinger equation on $\mathbb{R}^n$

$$(S) \qquad \begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \,. \end{cases}$$

Notice that

$$\|u(t,\cdot)\|_{L^2} = \|u_0\|_{L^2} \,.$$

The **dispersive estimate** writes (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|u_0\|_{L^1(\mathbb{R}^n)} \,,$$

and can be easily derived from the explicit expression of the solution

$$u(t,\cdot) = \frac{e^{i\frac{|\cdot|^2}{4t}}}{(4\pi it)^{\frac{n}{2}}} \star u_0 \,.$$

Strichartz estimates for the Schrödinger equation on $\mathbb{R}^n$

The dispersive estimate

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}}\|u_0\|_{L^1(\mathbb{R}^n)}\,,$$

gives rise by a duality argument (TT^*) to the **Strichartz estimate**

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\|u_0\|_{L^2(\mathbb{R}^n)}$$

as soon as

$$\frac{2}{q}+\frac{n}{p}=\frac{n}{2} \quad \text{with} \quad q\geq 2 \quad \text{and} \quad (n,q,p)\neq(2,2,\infty)\,.$$

Strichartz estimates for the Schrödinger equation on $\mathbb{R}^n$

More generally

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\Big(\|u_0\|_{L^2(\mathbb{R}^n)} + \|i\partial_t u - \Delta u\|_{L^{q_1'}(\mathbb{R},L^{p_1'}(\mathbb{R}^n))}\Big)\,,$$

which is a key ingredient to solving

$$(NLS) \qquad \begin{cases} i\partial_t u - \Delta u = f(u) \\ u_{|t=0} = u_0 \end{cases}$$

by a fixed-point procedure.

Geometries with no dispersion

- compact Riemannian manifolds (Bourgain '93, Burq, Gérard and Tzvetkov '04)
- bounded domains (Ivanovici, Lebeau and Planchon '14)
- **the Heisenberg group** (Bahouri, Gérard, Xu '99) : there exists a function u_0 in the Schwartz class $\mathcal{S}(\mathbb{H}^d)$ such that the solution to the free Schrödinger equation (S) satisfies

$$\forall (Y,s) \in \mathbb{H}^d\,,\ \forall t \in \mathbb{R}\,, \quad u(t,Y,s) = u_0(Y, s + 4td)\,.$$

Plan of the talk

- Strichartz estimates on the Heisenberg group : the Fourier restriction method

- Local dispersion on the Heisenberg group

The Heisenberg group

The Heisenberg group $\mathbb{H}^d$ can be defined as $\mathbb{R}^{2d} \times \mathbb{R}$ endowed with the noncommutative product law

$$(Y,s)\cdot(Y',s') \stackrel{\text{def}}{=} \big(Y+Y', s+s'+2\langle \eta, y'\rangle - 2\langle \eta', y\rangle\big)\,,$$

where $w=(Y,s)=(y,\eta,s)$ and $w'=(Y',s')=(y',\eta',s')$ are elements of $\mathbb{H}^d$. Convolution product

$$f \star g(w) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w\cdot v^{-1}) g(v)\, dv$$

Distance to the origin

$$\rho_{\mathbb{H}}(w) \stackrel{\text{def}}{=} \big(|Y|^4 + 4s^2\big)^{\frac{1}{4}}\,.$$

Define $Q \stackrel{\text{def}}{=} 2d+2$ the homogeneous dimension of $\mathbb{H}^d$.

The Heisenberg group

Define the sublaplacian

$$\Delta_{\mathbb{H}} u \stackrel{\text{def}}{=} \sum_{j=1}^{d} (\mathcal{X}_j^2 u + \Xi_j^2 u)\,,$$

where

$$\mathcal{X}_j \stackrel{\text{def}}{=} \partial_{y_j} + 2\eta_j \partial_s\,, \quad \Xi_j \stackrel{\text{def}}{=} \partial_{\eta_j} - 2y_j \partial_s\,.$$

The Schrödinger equation in $\mathbb{H}^d$ writes

$$(S) \qquad \begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0\,. \end{cases}$$

Strichartz estimates for Schrödinger in $\mathbb{H}^d$

Consider

$$(S) \qquad \begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0 \,. \end{cases}$$

Theorem

Given (p, q) and (p_1, q_1) belonging to the admissible set

$$\Big\{(p,q) \in [2,\infty]^2 \,/\, q \leq p \quad \text{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2}\Big\},$$

there is a constant C such that the solution to the Schrödinger equation associated with radial data satisfies the following Strichartz estimate

$$\|u\|_{L^\infty_s L^q_t L^p_Y} \leq C\Big(\|u_0\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_s L^{q_1'}_t L^{p_1'}_Y}\Big)\,.$$

Strichartz estimate via Fourier restriction in $\mathbb{R}^n$

If u solves $i\partial_t u - \Delta u = 0$ then $\widehat{u}(t,\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t,x)dx$ solves

$$i\partial_t \widehat{u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi)$$

so

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi \,.$$

This can be seen as the **restriction of the Fourier transform on** $\mathbb{R}^{n+1}$ to

$$\widehat{S} \stackrel{\text{def}}{=} \left\{ (\alpha,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid \alpha = |\xi|^2 \right\}.$$

Strichartz estimate via Fourier restriction in $\mathbb{R}^n$

Recall that

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi$$

and

$$\widehat{S} = \left\{ (\alpha, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid \alpha = |\xi|^2 \right\}.$$

We endow $\widehat{S}$ with the measure $d\sigma = d\xi$ induced by the projection $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ onto the second factor.

Define $g \stackrel{\text{def}}{=} \widehat{u}_0 \circ \pi|_{\widehat{S}}$ so that $g(|\xi|^2, \xi) = \widehat{u}_0(\xi)$. Then

$$u(t,x) = \int_{\widehat{S}} e^{iz\cdot\zeta} g(\zeta) d\sigma(\zeta) = \mathcal{F}^{-1}(g d\sigma)(t,x)$$

where $z = (t,x)$ and $\zeta = (\alpha, \xi)$. Note that

$$\|g\|_{L^2(\widehat{S}, d\sigma)} = \|u_0\|_{L^2(\mathbb{R}^n)}.$$

Strichartz estimate via Fourier restriction in $\mathbb{R}^n$

Recall that

$$u(t,x) = \mathcal{F}^{-1}(g d\sigma)(t,x)$$

with $g = \widehat{u}_0 \circ \pi|_{\widehat{S}}$, and

$$\|g\|_{L^2(\widehat{S},d\sigma)} = \|u_0\|_{L^2(\mathbb{R}^n)} \,.$$

Question : for which r does one have

$$\|\mathcal{F}^{-1}(g d\sigma)\|_{L^r(\mathbb{R}^{n+1})} \leq C_p \|g\|_{L^2(\widehat{S},d\sigma)} \,?$$

Strichartz estimate via Fourier restriction in $\mathbb{R}^n$

Restriction problem : Given a hypersurface $\widehat{S} \subset \mathbb{R}^m$ endowed with a smooth measure $d\sigma$, for which pairs (p, q) does an inequality of the form

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^q(\widehat{S},d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^m)}$$

hold ? In the following we focus on the case $q = 2$.

Tomas-Stein theorem '75 : if $\widehat{S}$ is a smooth (compact) hypersurface in $\mathbb{R}^m$ with non vanishing Gaussian curvature at every point, then the inequality holds for every $p \leq (2m+2)/(m+3)$ (optimal).

The dual form of the Tomas-Stein inequality is

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^{p'}(\mathbb{R}^m)} \leq C_p \, \|g\|_{L^2(\widehat{S},d\sigma)} \, ,$$

for all $g \in L^2(\widehat{S}, d\sigma)$ and all $p' \geq 2(m+1)/(m-1)$.

Strichartz estimate via Fourier restriction in $\mathbb{R}^n$

If $\widehat{u}_0$ is supported on the unit ball (corresponding to compact support for $d\sigma$) then the Tomas-Stein inequality gives

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C \,\|g\|_{L^2(\widehat{S},d\sigma)} \,,$$

so

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C\|u_0\|_{L^2(\mathbb{R}^n)} \,,$$

for all $p' \geq 2(n+2)/n$.

Set $u_{0,R} \stackrel{\mathrm{def}}{=} u_0(R^{-1}\cdot)$, then $u(t,x) = u_R(R^2t, Rx)$ and

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} R^{-\frac{2}{p'}-\frac{n}{p'}} \leq C\|u_0\|_{L^2(\mathbb{R}^n)} R^{-\frac{n}{2}}$$

and the density of spectrally localized functions in $L^2(\mathbb{R}^n)$, we get for all $u_0 \in L^2(\mathbb{R}^n)$

$$\|u\|_{L^{\frac{2n+4}{n}}(\mathbb{R},L^{\frac{2n+4}{n}}(\mathbb{R}^n))} \leq C\|u_0\|_{L^2(\mathbb{R}^n)} \,.$$

The Fourier transform on $\mathbb{H}^d$

Define (see Bahouri, Chemin, Danchin '19)

$$\mathcal{F}f(\widehat{w}) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} \overline{e^{is\lambda}\mathcal{W}(\widehat{w}, Y)}\, f(Y,s)\, dY\, ds$$

with $\widehat{w} \stackrel{\text{def}}{=} (n, m, \lambda)$ in $\widehat{\mathbb{H}}^d \stackrel{\text{def}}{=} \mathbb{N}^{2d} \times \mathbb{R} \setminus \{0\}$, and $\mathcal{W}$ is the Wigner transform of the (renormalized) Hermite functions

$$\mathcal{W}(\widehat{w}, Y) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z\rangle} H_{n,\lambda}(y+z) H_{m,\lambda}(-y+z)\, dz\,.$$

Here $H_{m,\lambda}(x) \stackrel{\text{def}}{=} |\lambda|^{\frac{d}{4}} H_m(|\lambda|^{\frac{1}{2}} x)$, with

$$-(\Delta - |x|^2)H_m = (2|m| + d)H_m\,, \quad H_0(x) \stackrel{\text{def}}{=} \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}\,.$$

The Fourier transform on $\mathbb{H}^d$

Inversion :

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widehat{\mathbb{H}}^d} e^{is\lambda} \mathcal{W}(\widehat{w}, Y) \mathcal{F}f(\widehat{w})\, d\widehat{w}\,,$$

Fourier-Plancherel :

$$(f|g)_{L^2(\mathbb{H}^d)} = \frac{2^{d-1}}{\pi^{d+1}} (\mathcal{F}f|\mathcal{F}g)_{L^2(\widehat{\mathbb{H}}^d)}\,,$$

with the notation

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{w})\, d\widehat{w} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \sum_{(n,m)\in\mathbb{N}^{2d}} \theta(n,m,\lambda)\, |\lambda|^d\, d\lambda\,.$$

The Fourier transform on $\mathbb{H}^d$

If f is radial ($f(Y,s) = f(|Y|,s)$) then

$$\mathcal{F}(f)(n,m,\lambda) = \mathcal{F}(f)(|n|,|n|,\lambda)\delta_{n,m}$$

and

$$\mathcal{F}(f)(\ell,\ell,\lambda) = \binom{\ell+d-1}{\ell}^{-1} \int_{\mathbb{H}^d} \overline{e^{is\lambda}\widetilde{\mathcal{W}}(\ell,\lambda,Y)} f(Y,s)\,dY\,ds\,,$$

with

$$\widetilde{\mathcal{W}}(\ell,\lambda,Y) \stackrel{\text{def}}{=} \sum_{\substack{n\in\mathbb{N}^d \\ |n|=\ell}} \mathcal{W}(n,n,\lambda,Y)\,.$$

Absence of dispersion on $\mathbb{H}^d$

Notice that

$$\mathcal{F}(\Delta_{\mathbb{H}} f)(\widehat{w}) = -4|\lambda|(2|m| + d)\mathcal{F}(f)(\widehat{w}) \, .$$

The function

$$\Theta_\lambda(Y,s) \stackrel{\text{def}}{=} e^{is\lambda} e^{-\lambda|Y|^2} = e^{is\lambda} \widetilde{W}(0,\lambda,Y)$$

satisfies $-\Delta_{\mathbb{H}}\Theta_\lambda = 4\lambda d\, \Theta_\lambda$ so

$$(i\partial_t - \Delta_{\mathbb{H}})\big(\Theta_\lambda(Y, s + 4td)\big) = 0$$

and

$$u(t,Y,s) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \Theta_\lambda(Y, s+4td) g(\lambda) \, |\lambda|^d d\lambda \, .$$

with $g \in \mathcal{D}(]0,\infty[)$, solves the free Schrödinger equation with no dispersion.

The Schrödinger equation on $\widehat{\mathbb{H}}^d$

In Fourier variables,

$$\begin{cases} i\dfrac{d}{dt}\mathcal{F}(u)(t,n,m,\lambda) &= -4|\lambda|(2|m|+d)\mathcal{F}(u)(t,n,m,\lambda) \\ \mathcal{F}(u)_{|t=0} &= \mathcal{F}u_0\,. \end{cases}$$

By integration, if u_0 is radial this leads to

$$\mathcal{F}(u)(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\,.$$

Finally

$$u(t,Y,s) = \frac{2^{d-1}}{\pi^{d+1}}\int_{\widehat{\mathbb{H}}^d} e^{is\lambda}\,\mathcal{W}(\widehat{w},Y)\,e^{4it|\lambda|(2|n|+d)}\,\mathcal{F}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\,d\widehat{w}\,.$$

So we define

$$\Sigma \stackrel{\text{def}}{=} \Big\{(\alpha,\widehat{w}) = \big(\alpha,(n,n,\lambda)\big) \in \mathbb{R}\times\widehat{\mathbb{H}}^d \,/\, \alpha = 4|\lambda|(2|n|+d)\Big\}\,.$$

Fourier restriction on the sphere of $\widehat{\mathbb{H}}^d$

Consider $\mathbb{S}_{\widehat{\mathbb{H}}^d} \stackrel{\text{def}}{=} \Big\{(n,n,\lambda) \in \widehat{\mathbb{H}}^d \,/\, (2|n|+d)|\lambda| = 1\Big\}$. Then

$$\langle d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}}, \theta\rangle_{\mathcal{S}'(\widehat{\mathbb{H}}^d)\times\mathcal{S}(\widehat{\mathbb{H}}^d)} = \sum_{n\in\mathbb{N}^d} \frac{1}{(2|n|+d)^{d+1}}\Big(\theta\big(n,n,\frac{1}{2|n|+d}\big) + \theta\big(n,n,\frac{-1}{2|n|+d}\big)\Big)\,.$$

Theorem [Müller '90]

If $1 \leq p \leq 2$, then

$$\|\mathcal{F}(f)_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widehat{\mathbb{H}}^d})} \leq C\|f\|_{L^p_Y L^1_s}\,,$$

for all radial functions f in $\mathcal{S}_{\rm rad}(\mathbb{H}^d)$, and thus

$$\|\mathcal{F}^{-1}(\theta_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}})\|_{L^{p'}_Y L^\infty_s} \leq C\,\|\theta_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widehat{\mathbb{H}}^d})}\,.$$

The Schrödinger equation on $\mathbb{H}^d$

Set $\mathbb{D} \stackrel{\text{def}}{=} \mathbb{R} \times \mathbb{H}^d$, $\widehat{\mathbb{D}} \stackrel{\text{def}}{=} \mathbb{R} \times \widehat{\mathbb{H}^d}$,

$$\Sigma = \Big\{(\alpha, \widehat{w}) = \big(\alpha, (n, n, \lambda)\big) \in \widehat{\mathbb{D}} \,/\, \alpha = 4|\lambda|(2|n| + d)\Big\} ,$$

and

$$\int_\Sigma \Theta(\alpha, \widehat{w})\, d\Sigma_{\rm loc}(\alpha, \widehat{w}) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{N}^d} c_n^{d+1}$$

$$\times \int_0^\infty \Big(\Theta\big(\alpha, n, n, \alpha c_n\big) + \Theta\big(\alpha, n, n, -\alpha c_n\big)\Big)\, \alpha^d\, \psi(\alpha)\, d\alpha ,$$

where $c_n \stackrel{\text{def}}{=} \big(4(2|n| + d)\big)^{-1}$, and with ψ any smooth, nonnegative, even function, compactly supported in $\mathbb{R}$ with an L^∞ norm at most 1.

Fourier restriction on Σ_{loc}

Theorem

If $1 \leq q \leq p \leq 2$, then

$$\|\mathcal{F}_{\mathbb{D}}(f)_{|\Sigma_{\mathrm{loc}}}\|_{L^2(d\Sigma_{\mathrm{loc}})} \leq C\|f\|_{L^1_s L^q_t L^p_Y},$$

for all radial functions f in $\mathcal{S}_{\mathrm{rad}}(\mathbb{R} \times \mathbb{H}^d)$.

Proof : TT^* argument, estimates on twisted convolutions with Laguerre polynomials.

Strichartz estimates for Schrödinger in $\mathbb{H}^d$

Theorem

Given (p, q) and (p_1, q_1) belonging to the admissible set

$$\left\{(p,q) \in [2,\infty]^2 \,/\, q \leq p \quad \text{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2}\right\},$$

there is a constant C such that the solution to the Schrödinger equation associated with radial data satisfies the following Strichartz estimate

$$\|u\|_{L^\infty_s L^q_t L^p_Y} \leq C\Big(\|u_0\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_s L^{q_1'}_t L^{p_1'}_Y}\Big).$$

Local dispersion for Schrödinger in $\mathbb{H}^d$

Recall that if

$$(S) \qquad i\partial_t u - \Delta u = 0\,, \quad u_{|t=0} = u_0\,,$$

on $\mathbb{R}^n$, then

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|u_0\|_{L^1(\mathbb{R}^n)}\,.$$

Theorem

Let $u_0 \in \mathcal{D}(B_{\mathbb{H}}(0,R_0))$. For any $0 < R < \sqrt{4d}$, the solution to (S) in $\mathbb{H}^d$ satisfies

$$\|u(t,\cdot)\|_{L^\infty(B_{\mathbb{H}}(0,R\sqrt{|t|}))} \leq \frac{C}{|t|^{\frac{Q}{2}}} \|u_0\|_{L^1(\mathbb{H}^d)}$$

for all $|t| \geq \left(\frac{R_0}{\sqrt{4d}-R}\right)^2$.

Dispersion for Schrödinger in $\mathbb{R}^n$

The dispersion inequality in $\mathbb{R}^n$ can easily be derived from the explicit expression of the solution

$$u(t,\cdot) = \frac{\mathrm{e}^{i\frac{|\cdot|^2}{4t}}}{(4\pi it)^{\frac{n}{2}}} \star u_0$$

which comes from the fact that in $\mathcal{S}'(\mathbb{R}^n)$

$$\mathcal{F}^{-1}\big(\mathrm{e}^{it|\cdot|^2}\big) = \frac{\mathrm{e}^{-i\frac{|\cdot|^2}{4t}}}{(-4\pi it)^{\frac{n}{2}}} \cdot$$

Dispersion for Schrödinger in $\mathbb{R}^n$

The fact that

$$\mathcal{F}^{-1}\big(\mathrm{e}^{it|\cdot|^2}\big) = \frac{\mathrm{e}^{-i\frac{|\cdot|^2}{4t}}}{(-4\pi it)^{\frac{n}{2}}}$$

can be proved by writing the corresponding formula for the heat equation, along with some complex analysis arguments : for any x in $\mathbb{R}^n$, the two maps

$$z \longmapsto H_1(z) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} e^{-z|\xi|^2}\, d\xi \quad \text{and} \quad z \longmapsto H_2(z) \stackrel{\mathrm{def}}{=} \Big(\frac{\pi}{z}\Big)^{\frac{n}{2}} e^{-\frac{|x|^2}{4z}}$$

are holomorphic on $D := \{\mathrm{Re}\, z > 0\}$ and coincide on $\mathbb{R} \cap D$, and thus on D. The conclusion comes by choosing a sequence in D converging to $-it$ for $t \neq 0$.

Local dispersion for Schrödinger in $\mathbb{H}^d$

Theorem

If u solves the free Schrödinger equation in $\mathbb{H}^d$ then $u(t) = u_0 \star S_t$, where

$$S_t(Y,s) = \frac{1}{(-4i\pi t)^{\frac{Q}{2}}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(-\frac{\tau s}{2t} - i\frac{|Y|^2\tau}{2t\tanh 2\tau}\right) d\tau\,,$$

provided that $\rho(w) < \sqrt{4d|t|}$.

Local dispersion for Schrödinger in $\mathbb{H}^d$

The proof consists in using the kernel of the heat equation : if

$$(H) \qquad \partial_t u - \Delta u = 0\,, \quad u_{|t=0} = u_0\,,$$

by Fourier analysis $u(t) = u_0 \star h_t$, where

$$h_t(y,\eta,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_{n\in\mathbb{N}^d} \int_{\mathbb{R}} e^{is\lambda} \mathcal{W}(\widehat{w}, Y) e^{-4t|\lambda|(2|n|+d)} \delta_{n,m} |\lambda|^d d\lambda$$

which we show coincides with the formula obtained by Gaveau '77

$$h_t(y,\eta,s) = \frac{1}{(4\pi t)^{\frac{Q}{2}}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(i\frac{\tau s}{t} - \frac{|Y|^2\tau}{2t\tanh 2\tau}\right) d\tau$$

Local dispersion for Schrödinger in $\mathbb{H}^d$

Then observe that the maps

$$H_z^1(Y,s) \stackrel{\text{def}}{=} \frac{2^{d-1}}{\pi^{d+1}} \sum_{m\in\mathbb{N}^d} \int_{\mathbb{R}} e^{is\lambda} e^{-4z|\lambda|(2|m|+d)} \mathcal{W}((m,m,\lambda),Y)|\lambda|^d d\lambda$$

$$H_z^2(Y,s) \stackrel{\text{def}}{=} \frac{1}{(4\pi z)^{\frac{Q}{2}}} \int_{\mathbb{R}} \left(\frac{2\tau}{\sinh 2\tau}\right)^d \exp\left(i\frac{\tau s}{z} - \frac{|Y|^2\tau}{2z\tanh 2\tau}\right) d\tau$$

are, for all (Y,s) in $\mathbb{H}^d$, holomorphic on $\widetilde{D}_{\rho(w)} \stackrel{\text{def}}{=} \left\{|z| > \frac{\rho^2(w)}{4d}\right\}$.

Since they coincide on the intersection of the real line with $\Omega_{\rho(w)} \stackrel{\text{def}}{=} D \cap \widetilde{D}_{\rho(w)}$, they coincide on the whole domain $\Omega_{\rho(w)}$.

The conclusion follows by considering a sequence $(z_p)_{p\in\mathbb{N}}$ in $\Omega_{\rho(w)}$ (for fixed $\rho(w)$) which converges to $-it$, with $t \neq 0$.

Local dispersion for Schrödinger in $\mathbb{H}^d$

To end the proof of the local dispersion theorem we note that $u(t,\cdot) = u_0 \star S_t$ on $B_{\mathbb{H}}(0, R\sqrt{|t|})$, with $R < \sqrt{4d}$, and

$$\|S_t\|_{L^\infty(B_{\mathbb{H}}(0,R\sqrt{|t|}))} \leq \frac{1}{(4\pi|t|)^{\frac{Q}{2}}} \int_{\mathbb{R}} \Big(\frac{2\tau}{\sinh 2\tau}\Big)^d \exp\Big(\frac{R^2|\tau|}{2}\Big) d\tau \stackrel{\mathrm{def}}{=} \frac{M_R}{|t|^{\frac{Q}{2}}}.$$

But for any w in $B_{\mathbb{H}}(0, R\sqrt{|t|})$ and any v in $B_{\mathbb{H}}(0, R_0)$

$$\rho(v^{-1}\cdot w) \leq \rho(w) + \rho(v) \leq R\sqrt{|t|} + R_0 < \sqrt{4d|t|}\,,$$

provided that

$$|t| > T_R = \Big(\frac{R_0}{\sqrt{4d} - R}\Big)^2$$

which completes the proof by Young's inequality.

Conclusion

- Fourier restriction gives Strichartz estimates for Schrödinger, despite the absence of dispersion (but with no integrability in s / no integrability in t).
- Similar results hold for the wave equation.
- Local dispersion holds for Schrödinger.
- Possible applications : local smoothing (in progress), NLS.