Hautus-Yamamoto criteria for approximate and exact controllability of linear difference delay equations

Y. Chitour,
Laboratoire des signaux et systèmes
Université Paris- Saclay, CentraleSupélec

Joint Work with S. Fueyo, G. Mazanti and M. Sigalotti

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GT Contrôle, LJLL, Sorbonne Université

Linear Difference Delay Systems

- Systems of the form:

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), \ \ x \in \mathbb{R}^d, \ \ u \in \mathbb{R}^m, \ \ t \geq 0. \qquad (1)$$

- d, m, N are positive integers, $A_j \in \mathbb{R}^{d \times d}$ for $1 \leq j \leq N$, $B \in \mathbb{R}^{d \times m}$, $0 < \Lambda_1 < \cdots < \Lambda_N$.

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Here, given $q \in [1, +\infty]$, L^q controllability (in time T)=
Targets $x(\cdot) \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$; inputs $u(\cdot) \in L^q([0, T], \mathbb{R}^m)$.

1-D network of transport equations

$$\begin{cases} \partial_t R(t,x) + \Lambda \partial_x R(t,x) = 0, (t,x) \in \Omega_{\mathrm{hyp}}, \\ R(t,0) = KR(t,1) + Bu(t), \qquad t \geq 0, \end{cases}$$

with

- Diagonal matrix Λ:

$$\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_d\},\ \lambda_i > 0 \text{ for all } i = 1, \ldots, d. \tag{2}$$

- Domain :

$$\Omega_{\mathrm{hyp}} \ = \ \{(t,x) \in \mathbb{R}^2,\ 0 \leq x \leq 1 \text{ and } t \geq 0\}; \tag{3}$$

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Characteristic method implies (after setting $y(t) = R(t,0)$):

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = K \begin{pmatrix} y_1(t - \Lambda_1) \\ \vdots \\ y_n(t - \Lambda_n) \end{pmatrix} + Bu(t), \; \text{for } t \geq 0, \; \Lambda_i = 1/\lambda_i. \quad (4)$$

Hautus criteria in finite dimension

Linear control system in $\mathbb{R}^d$, $d \in \mathbb{N}$:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^d, \quad u \in \mathbb{R}^m. \tag{5}$$

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- *(iii)* $\mathrm{Rank}\,[pI_d - A, B] = d, \quad \forall p \in \mathbb{C}$ (Hautus-Fattorini).

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Remark:

- (A, B) stabilizable ($\exists F$ s.t. $A + BF$ Hurwitz) $\Leftrightarrow$ $\mathrm{Rank}\,[pI_d - A, B] = d$ $\forall p \in \mathbb{C}$ and $\Re(p) \geq 0$.
- For discrete system $x(t) = Ax(t-1) + Bu(t)$ "identical" result.

(Some) Hautus criteria in infinite dimention - Linear PDEs

$D(A) \subset X, U$ Hilbert spaces
$A : D(A) \to X$ skew adjoint, $B \in L(U, X)$.

Theorem (Ramdani and al. 05, Miller 06-12)

The following statements are equivalent

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(*ii*) $\exists \delta > 0$ *s.t.* $\forall \omega \in \mathbb{R}$, $\forall z \in D(A)$,
$$\|(i\omega Id - A)z\|_X + \|B^* z\|_U \geq \delta \|z\|_X.$$

(Some) Hautus criteria in infinite dimention - Delay control systems

Retarded (Neutral) linear control system

$$\frac{d}{dt}\Big(x(t) - \sum_{i=1}^{N} A_{-i}x(t-\Lambda_l)\Big) = A_0x(t) + \sum_{i=1}^{N} A_ix(t-\Lambda_l) + Bu(t). \tag{6}$$

$x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$, $0 < \Lambda_1 < \cdots < \Lambda_N$

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Theorem (Manitius 1982, Yamamoto 89, ...)

Assume (6) L^2 *APPROX. CONT.* $\Longleftrightarrow$

(i) $\mathrm{Rank}\,[\Delta(p), B] = d, \quad \forall p \in \mathbb{C};$

(ii) $\mathrm{Rank}\,[A_N + pA_{-N}, B] = d$ *for some* $p \in \mathbb{C}$.

Here $\Delta(p) = pId - \sum_{i=1}^{N} e^{-p\Lambda_i} A_{-i} - \sum_{i=1}^{N} e^{-p\Lambda_i} A_i$.

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Remark: In all these results controllability not in finite time!!!!

Hautus Criteria for Linear Difference delay system

Taking Laplace transform in $\dot{x} = Ax + Bu$ yields:

$$(pI_d - A)\,\hat{x}(p) = B\hat{u}(p), \quad p \in \mathbb{C}, \tag{7}$$

where $\hat{x}(p) = \int_{-\infty}^{+\infty} x(t)e^{-pt}dt$ and $\hat{u}(p) = \int_{-\infty}^{+\infty} u(t)e^{-pt}dt$.

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$$\widehat{Q}_0(p)\widehat{x}(p) = B\widehat{u}(p), \quad p \in \mathbb{C}, \tag{8}$$

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GOAL: L^q controllability criteria in terms of $\widehat{Q}_0(p)$ and B

Case $N = d = 2$, $m = 1$

(Σ) $x(t) = A_1 x(t - \Lambda_1) + A_2 x(t - \Lambda_2) + Bu(t)$,
$x \in \mathbb{R}^2, u \in \mathbb{R}$, $\frac{\Lambda_1}{\Lambda_2} \in (0,1) \setminus \mathbb{Q}$, (A_i, B) controllable, $i = 1, 2$.

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For $p \in \mathbb{C}$ set $\widehat{Q}_0(p) = Id_2 - e^{-\Lambda_1 p} A_1 - e^{-\Lambda_2 p} A_2$.

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- $\mathrm{Rank}\,[M, B] = 2, \quad \forall M \in \mathrm{cl}(\widehat{Q}_0(\mathbb{C}))$,
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(iii) (Σ) L^2 CONT. in time $T \geq 2\Lambda_2 \Leftrightarrow (\Sigma)$ CONT. in time $T \geq \Lambda_2$ from zero to constant functions.

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Kalman condition for (*ii*) =
$\exists$ (explicit) 2D circle C s.t. EX. CONT. $\Leftrightarrow 0 \notin C$.

General Case: First Conjectures

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), \ x \in \mathbb{R}^d, \ u \in \mathbb{R}^m, \ t \geq 0. \qquad (9)$$

$\widehat{Q}_0(p) = Id_2 - e^{-Lp}A_1 - e^{-p}A_2$, $p \in \mathbb{C}$. $0 < \Lambda_1 < \cdots < \Lambda_N$.

Conjecture

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Realization theory (Yamamoto 78 − 20)

Space of inputs u = subspace of $L^q(\mathbb{R}, \mathbb{R}^m)$ with compact support in $\mathbb{R}_-$. Initial state = Origin.

$$\begin{cases} x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), & \text{for } t \geq \inf \operatorname{supp}(u), \\ x(t) = 0, & \text{for } t < \inf \operatorname{supp}(u), \\ y(t) = x(t - \Lambda_N), & \text{for } t \in [0, +\infty), \end{cases} \tag{10}$$

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Write (10) as convolution operator with kernel in the space of Radon measures $R(\mathbb{R}_+)$ supported in $\mathbb{R}_+$, i.e. find $A \in M_+(\mathbb{R})$ s. t. input-output system (10) can be represented as

$$y(t) = \int_{-\infty}^{+\infty} A(t-\tau)u(\tau)d\tau = (A*u)(t), \qquad t \in \mathbb{R}^+, \tag{11}$$

where $*$ is the convolution product.

Realization theory (Yamamoto 78 – 20)

$$y(t) = (A * u)(t), \qquad t \in \mathbb{R}^+ \quad \text{and} \quad A = Q^{-1} * P, \qquad (12)$$

where

$$Q := \delta_{-\Lambda_N} I_d - \sum_{j=1}^{N} \delta_{-\Lambda_N + \Lambda_j} A_j, , \quad P := B\delta_0,$$

and Q^{-1}invertible over $R(\mathbb{R})$ (space of Radon measures).

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Remark: $\widehat{Q}(p) = \widehat{Q}_0(p) e^{p\Lambda_N}$.

Realization theory (Yamamoto $78-20$)

Let $\pi : \phi \to \phi|_{\mathbb{R}_+}$ truncation operator on $L^q(\mathbb{R}, \mathbb{R}^d)$.
Original Diff. Delay System (10) can be realized (input-output)

$$(\Sigma)^Q : \quad y = \pi(A * u). \tag{13}$$

State space of $(\Sigma)^Q$ in terms of distribution Q

$$(L^q)^Q := \left\{ y \in L^q \left(\mathbb{R}_+, \mathbb{R}^d \right) \mid \pi(Q * y) = 0 \right\}. \tag{14}$$

Realization theory (Yamamoto $78-20$)

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Original Diff. Delay System (10) can be realized (input-output)

$$(\Sigma)^Q : \quad y = \pi(A * u). \tag{13}$$

State space of $(\Sigma)^Q$ in terms of distribution Q

$$(L^q)^Q := \left\{ y \in L^q \left(\mathbb{R}_+, \mathbb{R}^d \right) \mid \pi(Q * y) = 0 \right\}. \tag{14}$$

Similarly state space in Radon measures:

$$(R)^Q := \left\{ \pi\phi \mid \phi \in (R(\mathbb{R}_+))^d \text{ and } \pi(Q * \pi\phi) = 0 \right\}. \tag{15}$$

State space in distributions:

$$(D)^Q := \left\{ \pi\phi \mid \phi \in \left(D'(\mathbb{R}_+)\right)^d \text{ and } \pi(Q * \pi\phi) = 0 \right\}. \tag{16}$$

Classical Definitions of Controllability

Set $X \in \{L^q, R, D'\}$ (R for Radon and D' for distrib.)

Difference Delay System (10) is said to be:

1) X approximately controllable (from the origin) if $\forall \phi \in X([-\Lambda_N, 0], \mathbb{R}^d)$, $\exists n \in \mathbb{N}$, $T_n > 0$ and $u_n \in X([0, T_n], \mathbb{R}^m)$ s. t.

$$y(T_n + \cdot) \underset{n \to +\infty}{\longrightarrow} \phi(\cdot), \quad \text{in } X \text{ sense.}$$

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Remark: Similar definitions with uniform controllability time.

Controllability in terms of realization theory

$X \in \{L^q, R, D\}$. (What follows is a tautology!)

Realization $(\Sigma)^Q$ is

1) X-approximately controllable if, for every $\forall \pi\phi \in (X)^Q$, $\exists$ a sequence of inputs $(u_n)_{n\in\mathbb{N}}$ (in "X") s. t.:

$$\pi(A * u_n) \underset{n\to+\infty}{\longrightarrow} \pi\phi \quad \text{in} \quad X\left(\mathbb{R}_+, \mathbb{R}^d\right);$$

2) X-exactly controllable if, $\forall \pi\phi \in (X)^Q$ $\exists$ u (in "X") s. t. its associated output through System (10) satisfies

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Approximate controllability

Answer to Conjecture (*App.*). (except finite controllability time)

Theorem (Yamamoto 88,89)

The following are equivalent:

a) L^q-*approximate controllability* ($q \in [1, \infty)$)

b) *Radon approximate controllability*

c) *Distributional approximate controllability*

d) $\exists$ *two sequences of distributions* $(S_n)_{n\in\mathbb{N}}$ *and* $(R_n)_{n\in\mathbb{N}}$ *compactly supported in* $\mathbb{R}_-$ *s.t.:*

$$Q * R_n + P * S_n \underset{n\to+\infty}{\longrightarrow} \delta_0 I_d, \quad \text{in distributional sense.} \tag{17}$$

e) (*Hautus-Yamamoto criteria*) *The two conditions hold true:*

1) $\operatorname{rank}\left[\widehat{Q}(p), B\right] = d$ *for every* $p \in \mathbb{C}$,

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Fundamental ingredients: (a) algebraic approach with distributions; (b) Approximate Bézout identity (17).

Sketch of Proof

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$\mathcal{A}$ = generator of semigroup of right translations on $(L^q)^Q$.
$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C}, \det \widehat{\mathcal{Q}}(\lambda) = 0\}$, uniform finite multiplicity.
(Salamon 84) $(\Sigma)^Q$ approx. cont. $\Leftrightarrow$ $(\Sigma)^Q$ spectrally controllable and spectrally complete
$a), \cdots, d) \Rightarrow e)$: $\mathrm{rank}\left[\widehat{Q}(p), B\right] = d \ \forall p \in \mathbb{C}$ $\Leftrightarrow$ spectral cont.;
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<u>$e) \Rightarrow d)$</u>:

- Notice *e*) also holds for any $(Q + BK, B)$.
- (Hard part) $\mathrm{rank}\,[A_N, B] = d$ implies $(\Sigma)^{Q+BK}$ spectrally complete for some K.
- $(\Sigma)^{Q+BK}$ approx. cont. $\Rightarrow$ $(\Sigma)^Q$ approx. cont.

Exact controllability

Theorem (Yamamoto 11)

Distributional exact controllability $\Leftrightarrow \exists$ *two distributions* S *and* R *compactly supported in* $\mathbb{R}_-$ *s.t. the following* *Bézout Identity* *holds*

$$\text{(Bézout-Dist.)} \qquad Q * R + P * S = \delta_0 I_d.$$

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Theorem (Yamamoto and Willems 08)

Let $d = 1$ *(single output) and* $Z := \{\lambda \in \mathbb{C} | \, \widehat{Q}(\lambda) = 0\}$.
The following are equivalent

1. *Distributional exact controllability holds.*
2. (a) *Algebraic multiplicity of* $\lambda \in Z$ *uniformly bounded;*
   (b) $\exists \; k \geq 0, \; c > 0$ *s.t.* $|\lambda^k \widehat{P}(\lambda)| \geq c \; \forall \lambda \in Z$.

Bézout and exact controllability

$(\Sigma)^Q$ distrib. exact controllable $\Rightarrow$
since $\pi Q^{-1} \in (D)^Q$, $\exists$ distrib. S s.t.

$$\pi(A * S) = \pi(Q^{-1} * P * S) = \pi Q^{-1}.$$

Then $R := Q^{-1} * P * S - Q^{-1}$ has compact support in $\mathbb{R}_-$, hence Bézout-Dist. (i.e. Bézout identity in $(D'(\mathbb{R}_+))^d$).

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Compute $Q^{-1} *$ Bézout-Dist. $* Q * \phi$.

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It yields

$$\begin{aligned}
\phi &= Q^{-1} * (\delta_0 Id) * Q * \phi \\
&= Q^{-1} * (P * S + Q * R) * Q * \phi \\
&= Q^{-1} * P * S * Q * \phi + R * Q * \phi \\
&= A * S * Q * \phi + R * Q * \phi.
\end{aligned}$$

Bézout and exact controllability

Since (!!) $\pi(R * Q * \phi) = \pi(R * \pi(Q * (\pi\phi))) = 0$, we get:

$$\pi\phi = \pi\left(A * S * Q * \phi\right) = \pi\left(A * \omega\right).$$

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Strategy to get Conjecture (*Ex.*) (L^q ex. cont $\Leftrightarrow$ $cl(HY)$)
Prove that $cl(HY) \Rightarrow$ Bézout-Radon.

$cl(HY) \Rightarrow$ Bézout-Radon
A Corona problem

Recall $cl(HY)$

- $\mathrm{Rank}\,[M, B] = d, \quad \forall M \in \mathrm{cl}(\widehat{Q}_0(\mathbb{C})),$
- $\mathrm{Rank}\,[A_N, B] = d.$

On the other hand, Laplace transform of Bézout-Radon yields

$$\widehat{Q}(p)\widehat{R}(p) + B\widehat{S}(p) = I_d, \quad p \in \mathbb{C}. \tag{19}$$

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For simplicity, assume $m = 1$ (single input). Take $\lambda \in \mathbb{C}$, $0 \neq g \in \mathbb{C}^d$ of unit s.t. $\det \widehat{Q}(\lambda) = 0, \;\; g^T\widehat{Q}(p) = 0$. Then

$$\widehat{S}(p) = \frac{g^T}{(g^T B)} \text{ is imposed.}$$

$\Rightarrow$ solving Bézout-Radon = interpolation problem for $\widehat{S}(.)$ on zeros of $\det \widehat{Q}(\cdot)$, within a class of holomorphic function.

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For the moment (?), such Corona problem has no known solution

Representation formula

- Systems of the form:

$$x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), \ x \in \mathbb{R}^d, \ u \in \mathbb{R}^m, \ t \geq 0. \quad (20)$$

- d, m, N are positive integers, $A_j \in \mathbb{R}^{d \times d}$ for $1 \leq j \leq N$, $B \in \mathbb{R}^{d \times m}$, $0 < \Lambda_1 < \cdots < \Lambda_N$.

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Theorem (Variation-of-constants formula)

For $q \in [1, \infty]$, $T \in [0, +\infty)$, $u \in L^q([0, T], \mathbb{R}^m)$, $x_0 \in L^q([-\Lambda_N, 0], \mathbb{R}^d)$, and $t \in [0, T]$, we have

$$x_t = \Upsilon_q(t) x_0 + E_q(t) u, \quad (21)$$

Representation formula

1) (FLOW)
$\Upsilon_q(T) : L^q([-\Lambda_N, 0], \mathbb{R}^d) \longrightarrow L^q([-\Lambda_N, 0], \mathbb{R}^d)$

$$(\Upsilon_q(T)x_0)(s) = \sum_{\substack{(n,j)\in\mathbb{N}^N\times[\![1,N]\!] \\ -\Lambda_j\leq T+s-\Lambda\cdot n<0}} \Xi_{n-e_j} A_j x_0(T+s-\Lambda\cdot n),$$

2) (End-Point MAP)
$E_q(T) : L^q([0,T], \mathbb{R}^m) \longrightarrow L^q([-\Lambda_N, 0], \mathbb{R}^d)$

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Definition

Consider family of matrices $\Xi_n \in \mathcal{M}_{d,d}(\mathbb{R})$, $n \in \mathbb{Z}^N$,defined by

$$\Xi_n = \begin{cases} 0 & \text{if } n \in \mathbb{Z}^N \setminus \mathbb{N}^N, \\ I_d & \text{if } n = 0, \\ \sum_{k=1}^N A_k \Xi_{n-e_k} & \text{if } n \in \mathbb{N}^N \text{ and } |n| > 0, \end{cases} \tag{22}$$

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Definition: System (20) L^q-exactly controllable in time $T > 0$
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For $q \in [1, \infty)$, System (20) L^q-exactly controllable in time $T > 0$
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Remark: Back to Case $N = d = 2$, $m = 1$, $q = 2$.

- Case of 1 delay ($N = 1$): All controllabilities equivalent to Kalman condition, with explicit expression of $E_q^*(T)$.
- Case of 2 rational delays: Back to one delay with matrix size equal to l.c.m. of delays
- Case of 2 irrational delays: approximate $\frac{\Lambda_1}{\Lambda_2}$ by sequences of rationals $(r_l)_{l\geq 0}$ and prove $(c_2^l)_{l\geq 0}$ (from (23)) uniformly lower bdd.

Controllability in finite time

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Lemma (Generalized Cayley-Hamilton)

$\exists$ *real coefficients* α_k, $k \in \mathbb{N}^N$, $0 < |k| \leq d$, *s.t.* $\forall n \in \mathbb{N}^N$, $|n| \geq d$,

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Theorem (Range saturation for End-Point Map E_q)

For $q \in [1, +\infty]$,

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Theorem

Let $q \in [1, +\infty]$. *For System* (20),
L^q *exact cont. from the origin* $\Leftrightarrow$ L^q *exact cont. in time* $T = d\Lambda_N$.

Radon Bézout's identity characterization

Theorem

System (10) L^1 *exactly controllable from the origin* $\iff$ *Bézout-Radon, i.e.,* $\exists\, R, S$ *with entries in* $R(\mathbb{R}_-)$ *s.t.*

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From Open Mapping Theorem, $\|S_n\|_1 \leq C$ for $C > 0$ independent of n. Weak compactness in $R(\mathbb{R}_-) \Rightarrow \exists\, S \in R(\mathbb{R}_-)$ s.t. $S_n \underset{n\to+\infty}{\rightharpoonup} S$.

Radon Bézout's identity characterization

Theorem

System (10) L^1 exactly controllable from the origin $\iff$ Bézout-Radon, i.e., $\exists\, R, S$ with entries in $R(\mathbb{R}_-)$ s.t.

$$Q * R + P * S = \delta_0 I_d. \qquad (25)$$

Sketch of proof: ($\implies$) If L^1 exactly controllable, $\exists$ sequence (S_n) in $L^1(\mathbb{R}, \mathbb{R}^{m\times d})$, compactly supported in $[-d\Lambda_N, 0]$ s.t.

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$S_n \underset{n\to+\infty}{\rightharpoonup} S$.
Conclude with $\pi(Q^{-1} * P * S) = \pi Q^{-1}$.

A necessary L^1 exact controllability condition

Theorem

System (10) L^1-exactly controllable in time $T \geq d\Lambda_N$ if and only if $\exists\, R, S$ with entries in $M(\mathbb{R}_-)$ s.t.

$$Q * R + P * S = \delta_0 I_d. \tag{27}$$

A necessary L^1 exact controllability condition

Theorem

System (10) L^1-exactly controllable in time $T \geq d\Lambda_N$ if and only if $\exists\, R, S$ with entries in $M(\mathbb{R}_-)$ s.t.

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L^1 Bézout characterization implies the following corollary.

Corollary

If System (10) is L^1 exact controllable, then

1) $\mathrm{rank}\ [M, B] = d\ \forall M \in \mathrm{cl}(\widehat{Q}(\mathbb{C}))$,
2) $\mathrm{rank}\,[A_N, B] = d$.

Corona problem to be proved

Conjecture

Let $k \in \mathbb{N}$ and $q_j \in M(\mathbb{R}_-)$ for $j = 1, \ldots, k$. Assume $\exists c > 0$ s.t.

$$\sum_{j=1}^{k} |\hat{q}_j(p)| \geq c > 0, \quad \forall p \in \mathbb{C}. \tag{28}$$

Then $\exists\, p_j \in M(\mathbb{R}_-)$ for $j = 1, \ldots, k$ s.t.

$$\sum_{j=1}^{k} \hat{q}_j(p)\hat{p}_j(p) = 1, \quad \forall p \in \mathbb{C}. \tag{29}$$