

Inverse Design for scalar conservation laws

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Joint works with Rinaldo Colombo and Abraham Sylla.

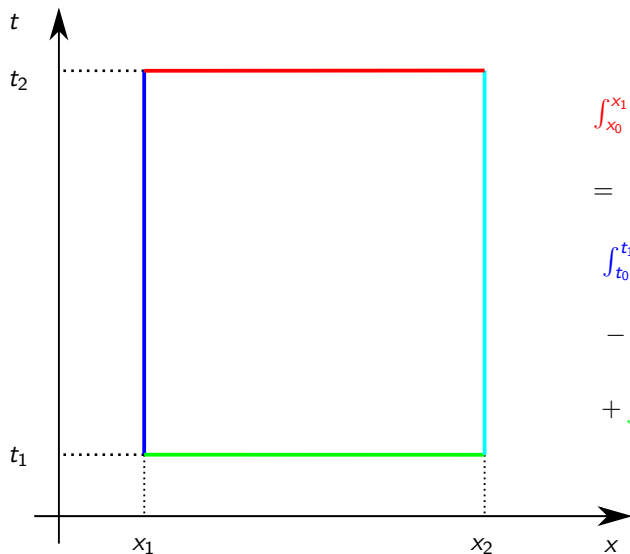
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Outline of the talk

- 1 Conservation laws: Origin and Cauchy Problem
 - Origins
 - Characteristics: paradise lost
 - Entropy solutions
- 2 Inverse Design for homogeneous laws: going back in time
- 3 The non-homogeneous case

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Integral Form



$$\int_{x_0}^{x_1} \rho(t_2, x) dx$$

$$=$$

$$\int_{t_0}^{t_1} F(t, x_1) dx$$

$$- \int_{t_0}^{t_1} F(t, x_2) dx$$

$$+ \int_{x_0}^{x_1} \rho(t_1, x) dx$$

Differential Form

- Integral Form: ρ and F just L^1_{loc} .
- Differential Form:

$$\rho, F \in C^1$$

$$\implies \int_{x_1}^{x_2} \int_{t_1}^{t_2} \partial_t \rho(t, x) dt dx = - \int_{t_1}^{t_2} \int_{x_1}^{x_2} \partial_x F(t, x) dt dx$$

$$\implies \frac{\partial \rho}{\partial t}(t, x) + \frac{\partial F}{\partial x}(t, x) = 0.$$

Closure

- $F(t, x) = -\kappa \partial_x \rho(t, x)$ Heat Equation.

$$\partial_t \rho - \kappa \partial_{xx}^2 \rho = 0.$$

- $F(t, x) = \frac{\rho^2(t, x)}{2}$ Burgers' equation (inspired by Euler's equation)

$$\partial_t \rho + \partial_x \left(\frac{\rho^2}{2} \right) = 0$$

- $F(t, x) = \rho(t, x) v_{\max} \left(1 - \frac{\rho(t, x)}{\rho_{\max}} \right)$ LWR equation

$$\partial_t \rho + \partial_x \left(\rho(t, x) v_{\max} \left(1 - \frac{\rho(t, x)}{\rho_{\max}} \right) \right) = 0$$

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Characteristics' method

$$\bullet \begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0, x) = \rho_0(x) \\ \rho \in \mathcal{C}^1 \end{cases}$$

$$\bullet \partial_t \rho + \partial_x f(\rho) = 0$$

$$\implies \partial_t \rho + f'(\rho) \partial_x \rho = 0,$$

$$\bullet \begin{cases} q \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{R}) \\ p(t) := \rho(t, q(t)) \end{cases}$$

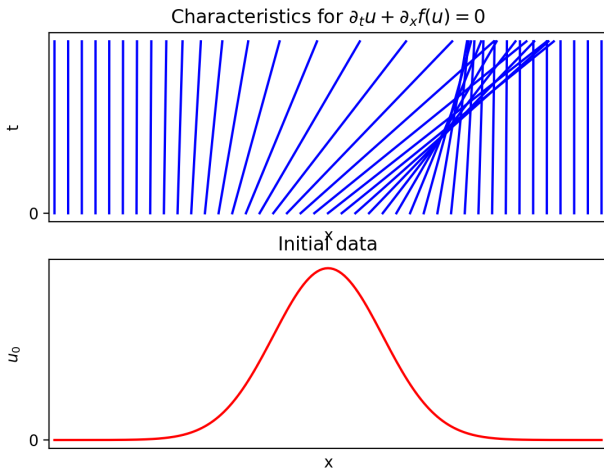
$$\implies \dot{p}(t) = \partial_t \rho + \dot{q}(t) \partial_x \rho,$$

$$\text{"} \implies \text{"} \begin{cases} \dot{q}(t) = f'(\rho(t)) \\ \dot{p}(t) = 0 \end{cases}$$

$$\implies \begin{cases} p(t) = p(0) \\ \dot{q}(t) = f'(\rho(0)) \end{cases}$$

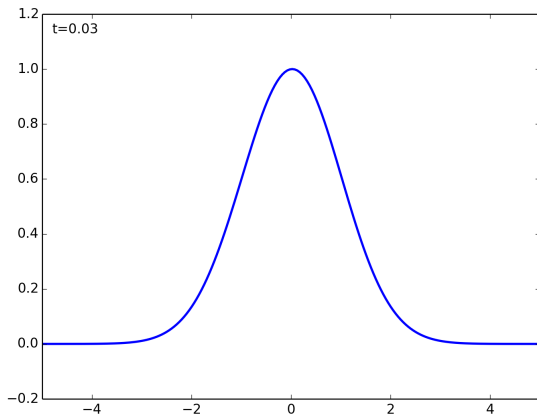
$$\implies \rho(t, x) = \rho_0(x - tf'(\rho(t, x)))$$

In a picture with Burgers $f'(q) = q$



Animated

- Let the plane evolve according to $\dot{p} = 0$ and $\dot{q} = f'(p)$.
- Look at the evolution of the graph of $p = \rho(t, q)$.



Generic Blowup

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & t > 0, \quad x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x). \end{cases} \quad (1)$$

Theorem

For $f(\rho) = \frac{\rho^2}{2}$ (in fact convex or concave) and any $\rho_0 \in C_c^\infty(\mathbb{R})$

$$\rho_0 \not\equiv 0 \Rightarrow \exists T > 0, \quad \exists X \in \mathbb{R}, \quad \partial_x \rho(t, X) \xrightarrow{t \rightarrow T^-} -\infty.$$

But

$$\|\rho(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|\rho_0\|_{L^\infty(\mathbb{R})}.$$

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Weak/Integral solutions

Three formulations, different regularity.

- Differential: $\partial_t u(t, x) + \partial_x f(u(t, x)) = 0,$
 $\forall t > 0, \forall x \in \mathbb{R}$
- Integral: $\frac{d}{dt} \int_a^b u(t, x) dx = f(u(t, a)) - f(u(t, b))$
 $\forall t > 0, \quad \forall a < b$
- Weak: $\int_0^{+\infty} \int_{-\infty}^{+\infty} u(t, x) \partial_t \phi(t, x) + f(u(t, x)) \partial_x \phi(t, x) dx dt = 0$
 $\forall \phi \in C_c^\infty((0, +\infty) \times \mathbb{R}).$

Riemann initial data: one discontinuity!

- Simplest discontinuity and invariance by $x \mapsto x + \eta$:

$$\rho_0(x) := \begin{cases} \rho_l & \text{if } x < 0 \\ \rho_r & \text{if } x > 0 \end{cases}$$

- Invariance by $(t, x) \mapsto (\lambda t, \lambda x) \Rightarrow$

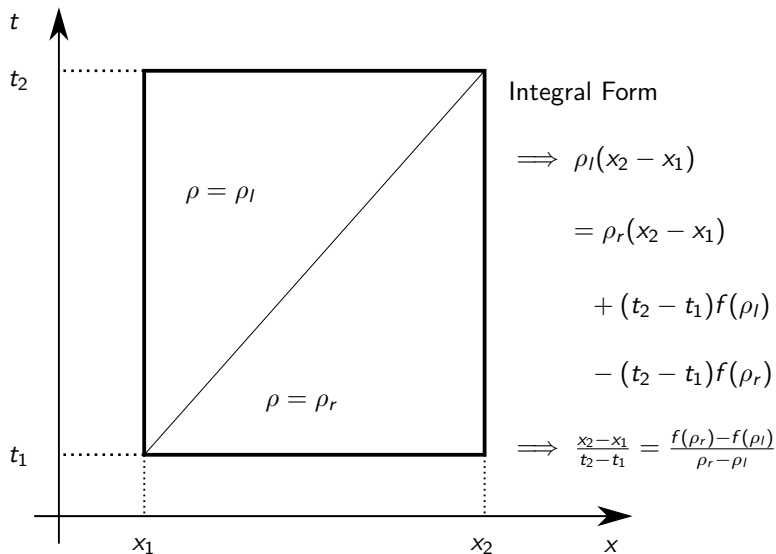
$$\rho(t, x) = v\left(\frac{x}{t}\right)$$

- Simplest case :

$$\rho(t, x) = \begin{cases} \rho_l & \text{if } x < \lambda t \\ \rho_r & \text{if } x > \lambda t \end{cases}$$

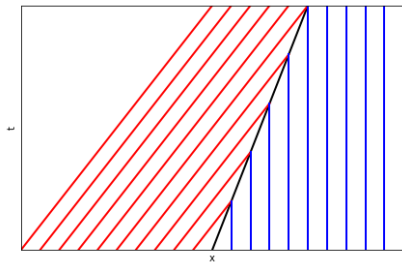
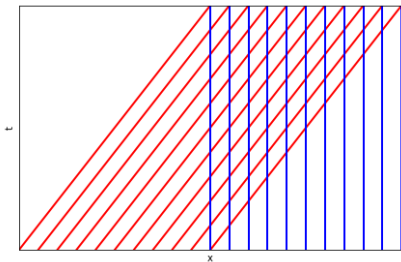
- What is λ ?

Rankine Hugoniot condition for weak solution



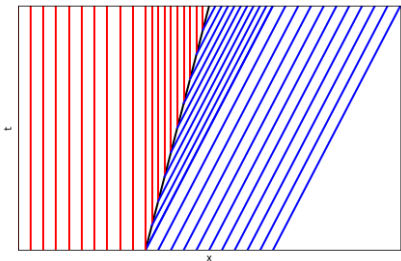
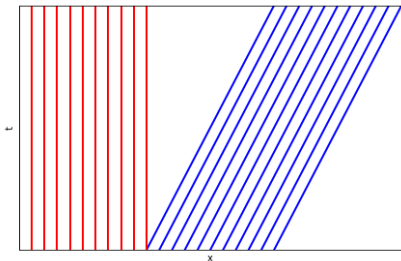
Burgers' characteristics : shock wave

$$\rho_0 = \begin{cases} 1.0 & \text{if } x < 0 \\ 0.0 & \text{if } x > 0. \end{cases} \Rightarrow \rho(t, x) = \begin{cases} 1.0 & \text{if } x < \frac{t}{2} \\ 0 & \text{if } x > \frac{t}{2} \end{cases}$$



Burgers' characteristics : rarefaction wave

$$\rho_0 = \begin{cases} 0.0 & \text{if } x < 0 \\ 1.0 & \text{if } x > 0. \end{cases} \quad \Rightarrow \quad \rho(t, x) = \begin{cases} 0 & \text{if } x < \frac{t}{2} \\ 1.0 & \text{if } x > \frac{t}{2} \end{cases}$$



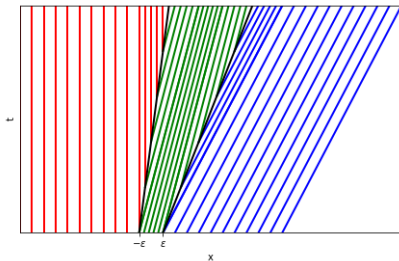
Stability condition

- $\delta > 0$, consider

$$\rho_0^\delta(x) = \begin{cases} \rho_l & \text{if } x < -\delta \\ \rho_m & \text{if } -\delta < x < \delta \\ \rho_r & \text{if } \delta < x \end{cases}$$

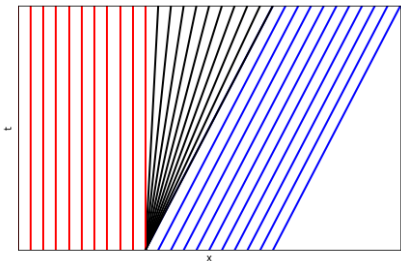
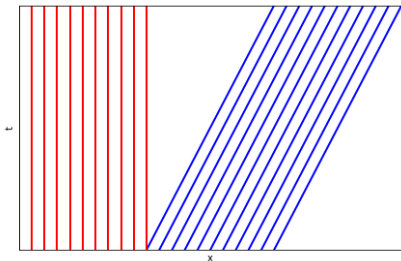
- $\delta \rightarrow 0$, admissibility condition on discontinuities (for continuous semigroup)

$$\forall \rho_m, \quad \frac{f(\rho_l) - f(\rho_m)}{\rho_l - \rho_m} \geq \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} \geq \frac{f(\rho_m) - f(\rho_r)}{\rho_m - \rho_r}$$



Burgers' characteristics : rarefaction wave the return

$$\rho_0 = \begin{cases} 0.0 & \text{if } x < 0 \\ 1.0 & \text{if } x > 0. \end{cases} \Rightarrow \rho(t, x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < t \\ 1.0 & \text{if } x > t \end{cases}$$



Vanishing Viscosity and entropy solution.

- Analogy with gas dynamics

$$\partial_t \rho^\epsilon + \partial_x f(\rho^\epsilon) = \epsilon \partial_{xx}^2 \rho^\epsilon. \quad (2)$$

- For E convex, $Q' = E' f'$.

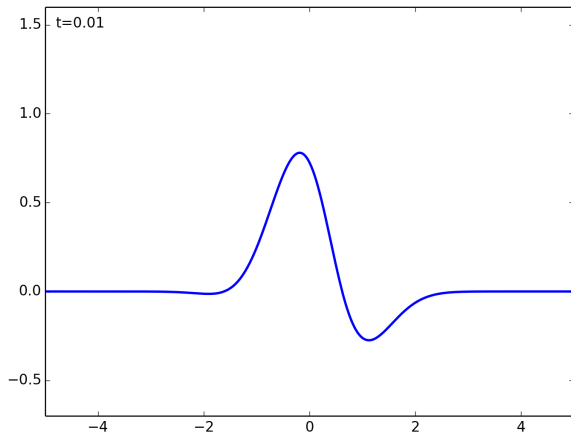
$$\partial_t E(\rho) + \partial_x Q(\rho) = \epsilon \partial_{xx}^2 E(\rho) - E''(\rho) (\partial_x \rho)^2.$$

- $\epsilon \rightarrow 0^+$ (**FORMALLY**):

$$\partial_t E(\rho) + \partial_x Q(\rho) \leq 0, \quad \mathcal{D}((0, +\infty) \times \mathbb{R})$$

- $\rho = (t, x) \mapsto (S_t \rho_0)(x)$, $(S_t)_{t \geq 0}$ semigroup of entropy solutions.

Simulations



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What and Why!

Questions:

- ① What is $S_t(L^\infty(\mathbb{R}))$?
- ② Characterize $I_T(\omega) := \{\rho_0 \in L^\infty(\mathbb{R}) : S_T \rho_0 = \omega\}$ for $\omega \in L^\infty(\mathbb{R})$?

Reasons:

- ① Irreversible dynamics for entropy solutions.
- ② Entropy semigroup compactifying.
- ③ Sonic boom minimization. (Gosse Zuazua 17)
- ④ Accident localization through tollgate estimates.
- ⑤ Control theory through Russell's extension method (Ancona-Marson 98, Horsin 98).

Characterization of reachable states

Going back to Oleinik 56!

Definition

$$T > 0, \quad \omega \in L^\infty(\mathbb{R}) \quad r_\omega^T(x) := x - Tf'(\omega(x)).$$

Theorem

For f convex,

$$I_T(\omega) \neq \emptyset \iff \omega \in S_T(L^\infty(\mathbb{R})) \iff r_\omega^T \text{ nondecreasing a.e.}$$

- 1 $\omega \in S_T(L^\infty(\mathbb{R})) \implies \omega \in BV(\mathbb{R})$,
- 2 Not better: take r_ω^T Cantor's staircase.

Reversing space and time

1 ω, T such that r_ω^T increasing

2 χ solution of

$$\begin{cases} \partial_s \chi + \partial_y f(\chi) = 0, \\ \chi(0, y) = \omega(-y) \end{cases}$$

3 χ isentropic: $\partial_s E(\chi) + \partial_y Q(\chi) = 0$

4 $\rho(t, x) := \chi(T - t, -x)$ entropy solution

$$\partial_t \rho + \partial_x f(\rho) = 0.$$

5 $\rho_0^* := \rho(0) \in I_T(\omega)$

Characterization of initial data

Theorem (Colombo-P.)

$I_T(\omega) \neq \emptyset \implies$

- ① $I_T(\omega)$ convex,
- ② $I_T(\omega)$ is a cone of vertex ρ_0^* ,
- ③ $I_T(\omega)$ is a F_σ set for the L^1_{loc} topology.

Furthermore

- ① $I_T(\omega)$ singleton iff additionally $\omega \in \mathcal{C}^0$
- ② otherwise unbounded L^∞ , but locally L^∞ closed L^1_{loc} ,
- ③ and there is no extremal facet of finite dimension besides ρ_0^*

In fact complete characterization of $I_T(\omega)$.

Tool : the Hopf-Lax formula

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0 \\ \rho(0) = \rho_0 \end{cases}$$

f strongly convex $\iff f^*$ Legendre transform

$$\begin{cases} r(t, x) := \arg \min_{y \in \mathbb{R}} \left(t f^* \left(\frac{x-y}{t} \right) + \int_0^y \rho_0(z) dz \right) \\ \rho(t, x) = f' \left(\frac{x-r(t, x)}{t} \right) \end{cases}$$



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Why

- ① LWR: different speed limit, different number of lanes:

$$\partial_t \rho(t, x) + \partial_x \left(\rho(t, x) v_{\max}(x) \left(1 - \frac{\rho(t, x)}{\rho_{\max}(x)} \right) \right) = 0.$$

- ② Nonhomogeneous conservation laws:

- ① Richer geometry!
- ②  Litterature 

- ③ Homogeneity in space too special:

- ① Vanishing viscosity for LWR?
- ② Infinitely many contracting semigroups of weak solutions!

Characteristics equation

$$\bullet \begin{cases} \partial_t \rho + \partial_x (H(x, \rho)) = 0 \\ \rho(0, x) = \rho_0(x) \\ \rho \in \mathcal{C}^1 \end{cases}$$

$$\bullet \partial_t \rho + \partial_x (H(x, \rho)) = 0$$

$$\implies \partial_t \rho + \partial_2 H(x, \rho) \partial_x \rho = -\partial_1 H(x, \rho),$$

$$\bullet \begin{cases} q \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{R}) \\ \rho(t) := \rho(t, q(t)) \end{cases}$$

$$\implies \dot{\rho}(t) = \partial_t \rho + \dot{q}(t) \partial_x \rho,$$

$$" \implies " \begin{cases} \dot{q}(t) = \partial_2 H(q(t), \rho(t)) \\ \dot{\rho}(t) = -\partial_1 H(q(t), \rho(t)) \end{cases}$$

Inverse design results: The Same

$$T > 0, \omega \in L^\infty(\mathbb{R}), \quad r_\omega^T(x) := q(0) \text{ where } \begin{cases} \dot{q}(t) = \partial_2 H(q(t), p(t)) \\ \dot{p}(t) = -\partial_1 H(q(t), p(t)) \\ q(T) = x, \quad p(T) = w(x). \end{cases}$$

Theorem (Colombo-P.-Sylla)

When $\forall x, \quad q \mapsto H(x, q)$ strongly convex

$$I_T(\omega) \neq \emptyset \iff \omega \in S_T(L^\infty(\mathbb{R})) \iff r_\omega^T \text{ nondecreasing a.e.}$$

And $I_T(\omega) \neq \emptyset \implies$

- ① $I_T(\omega)$ convex,
- ② $I_T(\omega)$ is a cone,
- ③ $I_T(\omega)$ is a F_σ set for the L_{loc}^1 topology.

...

Inverse design results: but not exactly

Theorem (Colombo-P.-Sylla)

There exists, $T > 0$, H and ω such that

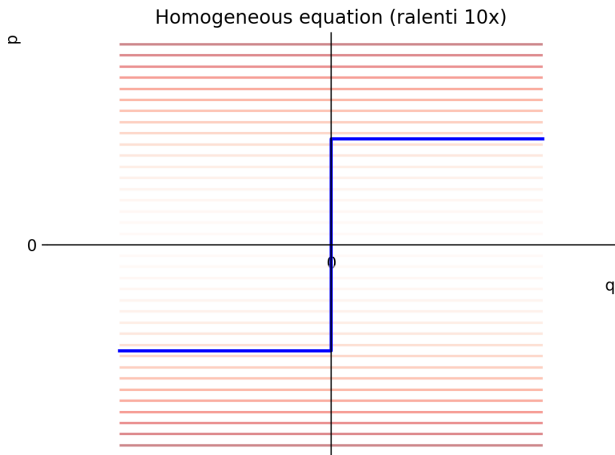
- ① $I_T(\omega) \neq \emptyset$,
- ② $\forall \rho_0 \in I_T(\omega), \exists (E, Q)$ entropy-entropy flux pair, such that

$$\partial_t E(\rho) + \partial_x (Q(x, \rho)) - E'(\rho) \partial_1 H(x, \rho) - \partial_1 Q(x, \rho) \neq 0$$

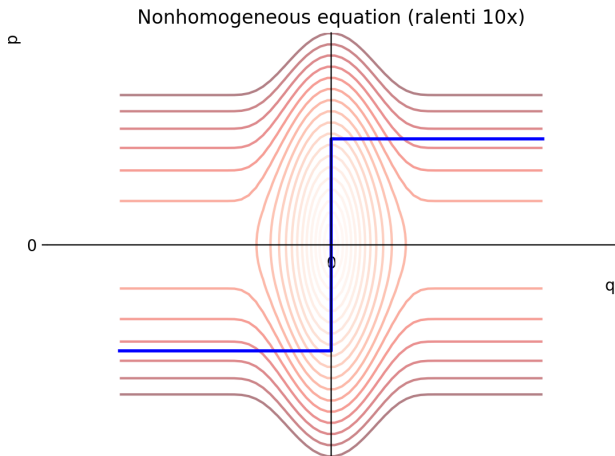
Remark

- ① *Isentropic solutions* " = " *closure of classical solutions.*
- ② *Range of S_t homogeneous: just look at classical solutions.*
- ③ *Range of S_t nonhomogeneous: larger.*

Rarefaction in the homogeneous case



Rarefaction in the nonhomogeneous case



Tool: Hamilton-Jacobi connection and optimal control

If $L(x, q) = \sup_{p \in \mathbb{R}} (pq - H(x, p))$ (Legendre transform)

$$\begin{cases} V(t, x) := \inf_{c \in L^\infty(0, T)} \left(\int_0^t L(y(s), c(s)) ds + P(y(0)) \right) \\ \dot{y}(s) = c(s) \quad 0 < s < t \\ y(t) = x \end{cases}$$

$$\iff \begin{cases} \partial_t V + H(x, \partial_x V) = 0, \\ V(0) = P \\ V \text{ viscosity solution} \end{cases}$$

$$\stackrel{\rho = \partial_x V}{\iff} \begin{cases} \partial_t \rho + \partial_x (H(x, \rho)) = 0, \\ \rho(0) = P' \\ \rho \text{ entropy solution} \end{cases}$$

Open Problems

- Numerics.
- Flux not convex:
 - ① generic flux: no more convexity of I_T .
 - ② still HJB but differential games, Isaacs condition?
- Multi dimensional conservation laws.
 - ① No more HJB.
 - ② Still entropy measure.
- Systems of conservation laws.

THANK YOU FOR YOUR ATTENTION