Minimal time issues for the observability of Grushin-type equations

Karine Beauchard[(1)] Jérémi Dardé[(2)]
Sylvain Ervedoza[(2)]

[(1)] ENS Rennes
[(2)] Institut de Mathématiques de Toulouse

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Outline

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   - Context
   - Results
2. Proofs
   - Strategy
   - Control on the two lateral boundaries
   - Boundary condition at $x = 0$
   - Observation at one end
3. Further comments

Outline

Context

Null-controllability problem at T for the heat equation through Γ

Let $\Omega \subset \mathbb{R}^d$, $\Gamma \subset \partial\Omega$.
The heat equation with control function $v \in L^2(0, T; L^2(\Gamma))$:

$$\begin{cases} \partial_t u - \Delta_x u = 0, & \text{in } (0, T) \times \Omega, \\ u(t, x) = v(t, x)\mathbf{1}_\Gamma(x), & \text{in } (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Given $u_0 \in L^2(\Omega)$, can we find a control function $v \in L^2(0, T; L^2(\Gamma))$ s.t. $u(T, \cdot) = 0$?

$\longrightarrow$ YES [Fursikov Imanuvilov '96, Lebeau Robbiano '95]
For any time $T > 0$ and any non-empty open subset $\Gamma \subset \partial\Omega$.

By duality [Fattorini-Russell '71],

Null-controllability $\Leftrightarrow$ Observability.

Observability of the heat equation through $(0, T) \times \Gamma$

Let $\Omega \subset \mathbb{R}^d$, $\Gamma \subset \partial\Omega$.
We consider the following heat equation:

$$\begin{cases} \partial_t z - \Delta_x z = 0, & \text{in } (0, T) \times \Omega, \\ z(t, x) = 0, & \text{in } (0, T) \times \partial\Omega, \\ z(0, x) = z_0(x) & \text{in } \Omega. \end{cases}$$

Does there exist a constant $C > 0$ such that for all $z_0 \in H^1_0(\Omega)$

$$\|z(T, \cdot)\|_{L^2(\Omega)} \leq C \, \|\partial_\nu z\|_{L^2((0,T)\times\Gamma)} \quad ?$$

YES Proof by Carleman estimates [Fursikov Imanuvilov '96].

General motivation

Understand what happens for degenerate parabolic equations.

Typical examples:

- Degeneracies on the boundary of the domain:

$$\begin{cases} \partial_t z - \partial_x(x^{2\alpha}\partial_x z) = 0, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = z(t,L) = 0, & t \in (0,T). \end{cases}$$

  See [Cannarsa Martinez Vancostenoble '16] for latest developments.

- Degeneracies inside the domain $\Omega = (-L, L) \times (0, \pi)$:

$$\begin{cases} \partial_t z - \partial_{xx} z - |x|^{2\alpha}\partial_{yy} z = 0, & (t,x,y) \in (0,T) \times \Omega, \\ z(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega. \end{cases}$$

Focus on the case of interior degeneracies

In $\Omega = (-L, L) \times (0, \pi)$, Grushin type operators:

$$\begin{cases} \partial_t z - \partial_{xx} z - |x|^{2\alpha} \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\ z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial\Omega. \end{cases}$$

- Boundary observation through $\Gamma \subset \partial\Omega$:

$$\|z(T)\|_{L^2(\Omega)} \leq C \|\partial_\nu z\|_{L^2((0,T)\times\Gamma)} .$$

- Internal/distributed Observation through $\omega \subset \Omega$:

$$\|z(T)\|_{L^2(\Omega)} \leq C \|z\|_{L^2((0,T)\times\omega)} .$$

Known results

$\Omega = (-L, L) \times (0, \pi)$, Grushin type operators:

$$\begin{cases} \partial_t z - \partial_{xx} z - |x|^{2\alpha} \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\ z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial\Omega. \end{cases}$$

[Beauchard Cannarsa Guglielmi '14]

- $\alpha < 1$: Observable in any time $T > 0$ from any open subset $\omega \subset \Omega$ or $\Gamma \subset \partial\Omega$.
  $\rightsquigarrow$ *Like the usual heat equation.*
- $\alpha > 1$: Not observable from ω, whatever $T > 0$, when $\overline{\omega} \cap \{x = 0\} = \emptyset$.
  $\rightsquigarrow$ *The strong degeneracy prevents from any observability result.*

The case $\alpha = 1$

$\Omega = (-L, L) \times (0, \pi)$, Grushin type operators:

$$\begin{cases} \partial_t z - \partial_{xx} z - |x|^2 \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\ z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial\Omega. \end{cases}$$

If $\omega = \omega_x \times (0, \pi)$, and $\overline{\omega_x} \cap \{0\} \times (0, \pi) = \emptyset$, there exists a critical time $T_\omega^* > 0$ such that

- The equation is not observable in any time $T < T_\omega^*$.
- For any $T > T_\omega^*$, the equation is observable:

$$\|z(T)\|_{L^2(\Omega)} \leq C \, \|z\|_{L^2((0,T)\times\omega)} .$$

cf [Beauchard Cannarsa Guglielmi '14].

If there is an horizontal strip which does not meet ω, there is never observability whatever $T > 0$ is [Koenig '17].

The case $\alpha = 1$

$\Omega = (-L, L) \times (0, \pi)$, Grushin type operators:

$$\begin{cases} \partial_t z - \partial_{xx} z - |x|^2 \partial_{yy} z = 0, & (t, x, y) \in (0, T) \times \Omega, \\ z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial\Omega. \end{cases}$$

If $\Gamma = \Gamma_x \times (0, \pi)$, with $\Gamma_x = \{L\}, \{-L\}$ or $\{-L, L\}$, there exists a critical time $T_\Gamma^* > 0$ such that

- The equations are not observable in any time $T < T_\Gamma^*$.
- For any $T > T_\Gamma^*$, the equations are observable:

$$\|z(T)\|_{L^2(\Omega)} \leq C \|\partial_x z\|_{L^2((0,T)\times\Gamma)} .$$

cf [Beauchard Cannarsa Guglielmi '14]: $T_\Gamma^* \geq L^2/2$.

Outline

Observations from two ends

$\Omega = (-L, L) \times (0, \pi)$:

$$\begin{cases} \partial_t z - \partial_{xx} z - x^2 \partial_{yy} z = 0 & \text{in } (0,T) \times \Omega, \\ z(t,x,y) = 0 & \text{in } (0,T) \times \partial\Omega, \\ z(0,x,y) = z_0(x,y) & \text{in } \Omega, \end{cases}$$

Theorem [K. Beauchard, J. Dardé & S.E. 2018]

Let $T > L^2/2$ and $\Gamma = \{-L, L\} \times (0, \pi)$. Then there exists $C > 0$ such that for all smooth solutions z,

$$\|z(T)\|_{L^2(\Omega)} \leq C \|\partial_x z\|_{L^2((0,T)\times\Gamma)} .$$

Sharp time: No observability if $T < L^2/2$.
$\rightsquigarrow$ See [Beauchard Cannarsa Guglielmi 2014].

Same as in [Beauchard Miller Morancey 2015], but $\neq$ proof.

Observation from one end

$\Omega = (-L, L) \times (0, \pi)$:

$$\begin{cases} \partial_t z - \partial_{xx} z - x^2 \partial_{yy} z = 0 & \text{in } (0,T) \times \Omega, \\ z(t,x,y) = 0 & \text{in } (0,T) \times \partial\Omega, \\ z(0,x,y) = z_0(x,y) & \text{in } \Omega, \end{cases}$$

Theorem [K. Beauchard, J. Dardé & S.E. 2018]

Let $T > L^2/2$ and $\Gamma = \{L\} \times (0, \pi)$. Then there exists $C > 0$ such that for all smooth solutions z,

$$\|z(T)\|_{L^2(\Omega)} \leq C \|\partial_x z\|_{L^2((0,T)\times\Gamma)} .$$

$\rightsquigarrow$ Same time as for $\Gamma = \{-L, L\} \times (0, \pi)$.

Sharp time: No controllability if $T < L^2/2$.

$\rightsquigarrow$ See [Beauchard Cannarsa Guglielmi 2014].

Grushin equations in non-symmetric domains

$\Omega = (-L_-, L_+) \times (0, \pi)$:

$$\begin{cases} \partial_t z - \partial_{xx} z - x^2 \partial_{yy} z = 0 & \text{in } (0, T) \times \Omega, \\ z(t, x, y) = 0, & (t, x, y) \in (0, T) \times \partial\Omega. \end{cases}$$

Theorem [K. Beauchard, J. Dardé & S.E. 2018]

Let $T > L_+^2/2$ and $\Gamma = \{L_+\} \times (0, \pi)$. Then there exists $C > 0$ such that for all smooth solutions z,

$$\|z(T)\|_{L^2(\Omega)} \leq C \|\partial_x z\|_{L^2((0,T)\times\Gamma)} .$$

Sharp time $\rightsquigarrow$ Agmon estimates.

The role of boundary conditions

$\Omega = (0, L) \times (0, \pi)$:

$$\begin{cases} \partial_t z - \partial_{xx} z - x^2 \partial_{yy} z = 0 & \text{in } (0, T) \times \Omega, \\ z(t, x, y) = 0, & (t, x, y) \in (0, T) \times (\partial\Omega \setminus \{0\} \times (0, \pi)). \end{cases}$$

+ Dirichlet conditions $z(t, 0, y) = 0$ in $(0, T) \times (0, \pi)$
or Neumann conditions $\partial_x z(t, 0, y) = 0$ in $(0, T) \times (0, \pi)$.

Theorem [K. Beauchard, J. Dardé & S.E. 2018]

When $\Gamma = \{L\} \times (0, 1)$, the critical time for observability is

- $T^* = L^2/6$ in the Dirichlet case;
- $T^* = L^2/2$ in the Neumann case.

$\rightarrow$ The BC at $x = 0$ plays an important role.

Comments

Our results also apply to

- Heisenberg equations in tensorized domains:

$$\partial_t - \partial_x^2 - (x\partial_y + \partial_z)^2.$$

  see [Beauchard Cannarsa '17] for previous results.
- Some inverse problems similar to the ones of [Beauchard Cannarsa Yamamoto '14].
- Slightly more general settings for Grushin equations:
  - $\Omega = \Omega_x \times \Omega_y$, with Ω_y of any dimension,
  - Operators of the form $\partial_t - \partial_x^2 - q(x)^2 \partial_y^2$ with

$$q \in C^3(-L_-, L_+), \quad q(0) = 0, \quad \inf \partial_x q > 0.$$

$\rightsquigarrow$ Critical time $T^* = \frac{1}{q'(0)} \int_0^{L_+} q(x)\, dx$ when observed from L_+.

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- Each result concerns observability properties for the equation:

$$\begin{cases} (\partial_t - \partial_x^2 - x^2\partial_y^2) z(t,x,y) = 0\,, & (t,x,y) \in (0,T) \times \Omega\,, \\ z(t,x,y) = 0\,, & (t,x,y) \in (0,T) \times \partial\Omega\,, \\ z(0,.,.) = z_0 \in L^2(\Omega)\,. & \end{cases}$$

We shall take advantage of the tensorized form of the problem:
$\rightsquigarrow$ We use Fourier series.

$$z(t,x,y) = \sum_n z_n(t,x) \sin(ny).$$

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) z_n(t,x) = 0\,, & (t,x) \in (0,T) \times (-L,L)\,, \\ z_n(t,-L) = z_n(t,L) = 0\,, & t \in (0,T)\,, \\ z_n(0,.) = z_{0,n} \in H_0^1(-L,L)\,, & \end{cases}$$

- Use Fourier series.

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) z_n(t,x) = 0\,, & (t,x) \in (0,T) \times (-L,L)\,, \\ z_n(t,-L) = z_n(t,L) = 0\,, & t \in (0,T)\,, \\ z_n(0,.) = z_{0,n} \in L^2(-L,L)\,, & \end{cases}$$

? Uniform ? observability problems

⇝ If observed from $-L$ and L:

$$\int_{-L}^{L} |z_n(T,x)|^2 dx \leqslant C \int_0^T (|\partial_x z_n(t,-L)|^2 + |\partial_x z_n(t,L)|^2) dt.$$

⇝ If observed from L:

$$\int_{-L}^{L} |z_n(T,x)|^2 dx \leqslant C \int_0^T |\partial_x z_n(t,L)|^2 dt.$$

To prove a uniform observability result in time T, we use

- A careful analysis of the cost of observability of a family of 1-d heat equations as $n \to \infty$, for a given $T_0 > 0$ (small):

$$\|z_n(T_0)\|_{L^2(-L,L)} \leq Ce^{An} \|\text{observation}\|_{L^2(0,T_0)},$$

  with C and A independent of $n \in \mathbb{N}$.

- The dissipation of each semigroup:

$$\|z_n(T)\|_{L^2(-L,L)} \leq e^{-\mu n(T-T_0)} \|z_n(T_0)\|_{L^2(-L,L)},$$

  with μ independent of n.

$\Rightarrow$ Uniform observability provided

$$T \geq T_0 + \frac{A}{\mu}.$$

As T_0 is arbitrarily small, this gives $T > A/\mu$.

- Follows the same strategy as in [Beauchard Cannarsa Guglielmi '14].
- The dissipation rates of each semi-group are known ($\mu = 1$ in the case $x \in (-L, L)$)

$\rightarrow$ It mainly remains to analyze the cost of observability of a family of 1-d heat equations, in the asymptotics $n \to \infty$ and for a given $T_0 > 0$ (small):

$$\|z_n(T_0)\|_{L^2(-L,L)} \leq Ce^{An} \|\text{observation}\|_{L^2(0,T_0)} .$$

Remark

To get a sharp result, we should obtain sharp estimates on A.

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Observation from both boundaries $\pm L$

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) z_n(t,x) = 0\,, & (t,x) \in (0,T) \times (-L,L)\,, \\ z_n(t,-L) = z_n(t,L) = 0\,, & t \in (0,T)\,, \\ z_n(0,.) = z_{0,n} \in H_0^1(-L,L)\,, \end{cases}$$

Then

$$w_n(t,x) = z_n(t,x) \exp\left(- \frac{n \coth(2nt)}{2} (L^2 - x^2) \right)$$

satisfies

$$\begin{cases} \left(\partial_t - \partial_x^2 + 2\theta_n(t) x \partial_x + \theta_n'(t) \frac{L^2}{2} + \theta_n(t) \right) w_n(t,x) = 0 \\ w_n(t,-L) = w_n(t,L) = 0\,, \end{cases}$$

where we set $\theta_n(t) = n \coth(2nt)$. No terms in x^2 anymore !

(Carleman type) Energy estimates:

$$\int_{-L}^{L}\left(|\partial_x w_n(T_0)|^2 - \frac{n^2}{\sinh(2nT_0)^2}\frac{L^2}{2}|w_n(T_0)|^2\right)\,dx \\ \leq 2L\int_0^{T_0}(|\partial_x w_n(t,-L)|^2 + |\partial_x w_n(t,L)|^2)dt.$$

$\Rightarrow$ Given $T_0 > 0$, $\exists n_0 \in \mathbb{N}$, s.t. for all $n \geq n_0$,

$$\int_{-L}^{L}|w_n(T_0)|^2 = \int_{-L}^{L}|z_n(T_0,x)|^2 \exp\left(-n\coth(2nT_0)(L^2-x^2)\right) \\ \leq C\int_0^{T_0}(|\partial_x z_n(t,-L)|^2 + |\partial_x z_n(t,L)|^2)dt$$

$\rightsquigarrow$ Observability cost in $\exp(nL^2/2)$ at time T_0.

To be combined with the dissipation in $\exp(-n(T-T_0))$.

Why this choice? see [Dardé Ervedoza, '16, '17].

The fundamental solution of

$$(\partial_t - \partial_{xx} + x^2)K(t,x,y) = \delta_{t=0}\delta_{x=y}$$

is given by the Mehler kernel

$$K(t,x,y) = \frac{1}{(2\pi\sinh(2t))^{1/2}} e^{-\coth(2t)\left(\frac{|x|^2+|y|^2}{2}\right) - \frac{2x\cdot y}{\sinh(2t)}}.$$

Scaling $(t,x,y) \to (nt, \sqrt{n}x, \sqrt{n}y)$ and $y \rightsquigarrow -iL$, gives

$$|K_n(t,x,iL)| = \frac{1}{(2\pi\sinh(2nt))^{1/2}} e^{-n\coth(2nt)\left(\frac{|x|^2-L^2}{2}\right)}.$$

$\rightsquigarrow$ The weight function is the exponential envelop of $1/K_n(t,x,iL)$.

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The role of the boundary condition at $x = 0$

Corresponding to Dirichlet boundary conditions:

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) z_n(t,x) = 0\,, & (t,x) \in (0,T)\times(0,L)\,, \\ z_n(t,0) = z_n(t,L) = 0\,, & t \in (0,T)\,, \end{cases}$$

Corresponding to Neumann boundary condition

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) z_n(t,x) = 0\,, & (t,x) \in (0,T)\times(0,L)\,, \\ \partial_x z_n(t,0) = z_n(t,L) = 0\,, & t \in (0,T)\,, \end{cases}$$

- Cost of observability in $\exp(nL^2/2)$ by symmetry arguments.
- Dissipation in $\exp(-3nt)$ for Dirichlet BC, $\exp(-nt)$ for Neumann BC.

$\Rightarrow$ $T^* = L^2/6$ for Dirichlet BC, $T^* = L^2/2$ for Neumann BC.

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Observation at one end

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) z_n(t,x) = 0\,, & (t,x) \in (0,T) \times (-L,L)\,, \\ z_n(t,-L) = z_n(t,L) = 0\,, & t \in (0,T)\,, \\ z_n(0,.) = z_{0,n} \in H_0^1(-L,L)\,, \end{cases}$$

Goal

Get a uniform observability inequality

$$\|z_n(T)\|_{L^2(-L,L)} \leq C \exp(A\, n) \, \|\partial_x z_n(t,L)\|_{L^2(0,T)}\,,$$

with a precise knowledge of A.

Proof done in several steps:

- Passing the information from the right $x = L$ to the left of the singularity $x = -\varepsilon$, $\varepsilon > 0$ small.
- Passing the information from the singularity $x = 0$ to the extreme left of the domain $-L$.
- A gluing argument.

The first two steps are done by Carleman estimates.

From $x = L$ to $x = 0$ (or $-\varepsilon$)

$$\varphi_{R,n}(t,x) = n\theta(t)\Psi_R(x) + \theta(t), \quad (t,x) \in (0,T) \times (-\varepsilon, L)$$

with θ and Ψ_R as follows

$$\theta \in \mathscr{C}^\infty(0,T), \quad \theta(t) = \begin{cases} 1/t & \text{for } t < T/4, \\ 1 & \text{for } t \in (T/3, 2T/3), \\ 1/(T-t) & \text{for } t > 3T/4, \\ \geq 1 & \text{for } t \in (0,T), \end{cases}$$

$$\Psi_R(x) = \frac{L^2 - x^2}{2} + 2\varepsilon(L - x), \quad x \in (-\varepsilon, L).$$

Remark

$$\varphi_{R,n}(t,x) \simeq n \coth(2nt) \frac{L^2 - x^2}{2}.$$

Then $\exists n_0 > 0$ and $C > 0$ s.t. for all $n \geq n_0$, for all u_n satisfying

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) u_n = 0\,, & (t,x) \in (0,T) \times (-\varepsilon, L)\,, \\ u_n(t,-\varepsilon) = u_n(t,L) = 0\,, & t \in (0,T), \\ u_n(0,.) = u_{0,n} \in H_0^1(-\varepsilon, L)\,, & \end{cases}$$

we have

$$n^{3/2} \left\| \theta^{3/2} u_n e^{-\varphi_{R,n}} \right\|_{L^2((0,T)\times(-\varepsilon,L))} \leqslant C n^{1/2} \left\| \theta^{1/2} \partial_x u_n(t, L_+) e^{-\theta(t)} \right\|_{L^2(0,}$$

A typical Carleman estimate:

- Weight function adapted to the potential x^2, inspired by the Mehler kernel.
- The Carleman parameter is chosen $= n$.
- Sources terms can be handled easily.

From $x = 0$ to $x = -L$

There, the potential $n^2 x^2$ improves the observability property.

We choose a weight function:

$$\varphi_{L,n}(t,x) = n\theta(t)A - \sqrt{n}\theta(t)\left(\frac{x^2}{2} + 2Lx\right), \quad (t,x) \in (0,T)\times(-L,0),$$

where A is a suitable positive constant.

$\exists n_0 > 0$ and $C > 0$ s.t. for all $n \geq n_0$, for all u_n satisfying

$$\begin{cases} (\partial_t - \partial_x^2 + n^2 x^2) u_n = 0\,, & (t,x) \in (0,T) \times (-L,0)\,, \\ u_n(t,-L) = u_n(t,0) = 0\,, & t \in (0,T), \\ u_n(0,.) = u_{0,n} \in H_0^1(-L,0)\,. \end{cases}$$

we have

$$n^{3/4} \left\| \theta^{3/2} u_n e^{-\varphi_{L,n}} \right\|_{L^2((0,T)\times(-L,0))} \leqslant C n^{1/4} \left\| \theta^{1/2} \partial_x u_n(t,0) e^{-n\theta(t)A} \right\|_{L^2(0,$$

The weight function

$$\varphi_{L,n}(t,x) = n\theta(t)A - \sqrt{n}\theta(t)\left(\frac{x^2}{2} + 2Lx\right),$$

- $\varphi_{L,n}$ is essentially constant in space $\simeq n\theta(t)A$.
- Variations (in x) of the weight function are of lower order.
- A will be chosen to match $\Psi_{R,n}(t,0)$ for the gluing argument.

Outline

Further comments

Our results also apply to

- Non-symmetric geometric settings.
- Heisenberg equations in tensorized domains:

$$\partial_t - \partial_x^2 - (x\partial_y + \partial_z)^2.$$

  see [Beauchard Cannarsa '17] for previous results.
- Some inverse problems similar to the ones of [Beauchard Cannarsa Yamamoto '14].
- Slightly more general settings for Grushin equations:
  - $\Omega = \Omega_x \times \Omega_y$, with Ω_y of any dimension,
  - Operators of the form $\partial_t - \partial_x^2 - q(x)^2\partial_y^2$ with

$$q \in C^3(-L_-, L_+), \quad q(0) = 0, \quad \inf \partial_x q > 0.$$

$\rightsquigarrow$ Critical time $T^* = \frac{1}{q'(0)} \int_0^{L_+} q(x)\, dx$ when observed from L_+.

Open problems

- Determine the correct geometric condition in non-tensorized settings for which observability holds for Grushin operators.
- Determine the time required for observability for Kolmogorov like operators

$$\partial_t - \partial_{vv} + v^2 \partial_x, \quad \text{or} \quad \partial_t - \partial_{vv} + v^2 (-\Delta_x)^{1/2}.$$

  See [Beauchard Helffer Henry Robbiano '15].
- Precisely describe the reachable sets in each of the above situations, and how it evolves in time.
  See [Dardé Ervedoza '16].

Merci pour votre attention !

Ref: Minimal time issues for the observability of Grushin-type equations,
Karine Beauchard, Jérémi Dardé, Sylvain Ervedoza, 2018.
Available on HaL.