ISS Lyapunov functional for a Korteweg-de Vries equation: applications in automatic control

Swann Marx (LS2N)

Joint work with Ismaïla Balogoun (LS2N), Eduardo Cerpa (UC Chile) and Daniele Astolfi (LAGEPP)

GT Contrôle, LJLL, 6/10/2023

First goal

System

$$\begin{cases} \frac{d}{dt}w = \mathcal{A}w + \mathcal{B}u, \\ w(0) = w_0. \end{cases}$$

1. H and U are Hilbert spaces.
2. $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ generates a strongly continuous semigroup.
3. $\mathcal{B} \in \mathcal{L}(U, D(\mathcal{A}^*)')$ is the control operator.

Goal

Our goal is to find a feedback operator $\mathcal{K} \in \mathcal{L}(D(\mathcal{A}^*), U)$ so that, when setting $u = \mathcal{K}w$, the equilibrium point of the resulting system is asymptotically stable.

Notions of stability

Closed-loop system

$$\begin{cases} \frac{d}{dt}w = (\mathcal{A} + \mathcal{B}\mathcal{K})w, \\ w(0) = w_0. \end{cases}$$

Definition : Equilibrium point

It is given by $(\mathcal{A} + \mathcal{B}\mathcal{K})w^\star = 0$.

Notions of stability

Closed-loop system

$$\begin{cases} \frac{d}{dt}w = (\mathcal{A} + \mathcal{BK})w, \\ w(0) = w_0. \end{cases}$$

Definition : Lyapunov stability

For every $\delta > 0$, there exists $\varepsilon := \varepsilon(\delta) > 0$ such that, for all $t \geq 0$

$$\|w_0 - w^*\|_H \leq \varepsilon \Rightarrow \|w(t) - w^*\|_H \leq \delta.$$

Notions of stability

Closed-loop system

$$\begin{cases} \frac{d}{dt}w = (\mathcal{A} + \mathcal{B}\mathcal{K})w, \\ w(0) = w_0. \end{cases}$$

Definition : (Global) Asymptotic stability

For every initial conditions $w_0 \in H$, one has

$$\lim_{t \to +\infty} \|w(t) - w^*\|_H = 0$$

Notions of stability

Closed-loop system

$$\begin{cases} \frac{d}{dt}w = (\mathcal{A} + \mathcal{B}\mathcal{K})w, \\ w(0) = w_0. \end{cases}$$

Definition : (Global) Exponential stability

There exist $C, \mu > 0$ such that, for every initial conditions $w_0 \in H$ and every $t \geq 0$:

$$\|w(t) - w^*\|_H \leq Ce^{-\mu t}\|w_0 - w^*\|_H.$$

Notions of stability

Closed-loop system

$$\begin{cases} \frac{d}{dt}w = (\mathcal{A} + \mathcal{B}\mathcal{K})w, \\ w(0) = w_0. \end{cases}$$

Question

How to find a feedback operator $\mathcal{K}$?

Answer

For linear systems :

- Frequency domain approach [Paunonen, Zwart, etc...]
- Backstepping [Krstic, Auriol, Coron, etc...]
- Lyapunov [Coron, Prieur, Andrieu, etc...]

For nonlinear systems... Lyapunov method seems to be the only one !

Automatic control goals

Stabilization is just the first step of automatic control.

Disturbed closed-loop system

$$\begin{cases} \frac{d}{dt}w = \mathcal{A}w + \mathcal{B}_1 u + \mathcal{B}_2 d, \\ w(0) = w_0. \end{cases}$$

Question

Is this possible to stabilize despite d ?

Some methods

- Internal model approach [Paunonen, Xu, Andrieu, etc...]
- Sliding mode control [Orlov, Balogoun, etc...]

Automatic control goals

Stabilization is just the first step of automatic control.

Disturbed closed-loop system with output

$$\begin{cases} \frac{d}{dt}w = \mathcal{A}w + \mathcal{B}_1 u + \mathcal{B}_2 d, \\ y = \mathcal{C}w \\ w(0) = w_0. \end{cases}$$

Question

Is this possible to make converge y to a given reference r ?

Problem statement

Consider the Korteweg-de Vries equation

$$\begin{cases} w_t + w_x + w_{xxx} + ww_x + d_1(x) = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+ \\ w_x(t,L) = u(t) + d_2, \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ x \in [0,L], \end{cases}$$

$u(t) \in \mathbb{R}$ is a **control** $(d_1, d_2) \in L^2(0,L) \times \mathbb{R}$ are unknown **disturbances**.

Question

Assume the **output** is $y(t) := w_x(t,0)$. Is it possible to design an **output feedback law** $u(t) := f(w_x(t,0))$ such that :

$$\lim_{t \to +\infty} |y(t) - r| = 0,$$

where r is a given **reference**, and despite the disturbances ?

A finite-dimensional example

A finite-dimensional example

$$\dot{w}(t) = u(t) + d,$$

$w(t) \in \mathbb{R}$, u is the **control**, d is a constant **disturbance**.

How can one design a feedback $u(t) = f(w(t))$ such that

$$\lim_{t\to+\infty} |w(t) - r| = 0, \text{ where } r \text{ is a given reference ?}$$

Obstruction

Static feedback-laws are **not enough**. Indeed, if $u(t) = -kw(t)$, the equilibrium point is $kw_\infty = d$.

$\Rightarrow$ Use instead a **dynamical** feedback-law.

The integral action principle

PI controller

$$\begin{cases} \dot{w}(t) = -\underbrace{k_p w(t)}_{\text{proportional action}} - \underbrace{k_i z(t)}_{\text{integral action}} + d, \\ \dot{z}(t) = w(t) - r \end{cases}$$

The equilibrium points are

$$\begin{cases} k_p w_\infty + k_i z_\infty = d, \\ w_\infty = r, \end{cases}$$

Principle

- The **proportional** action **stabilizes** w.
- The **integral** action modifies the **equilibrium points**.

Question

How does one **select** k_p and k_i ?

Gain design : a Lyapunov approach

We consider $\bar{w} = w - w_\infty$ and $\bar{z} = z - z_\infty$, then :

$$\begin{cases} \dot{\bar{w}}(t) = -k_p \bar{w}(t) - k_i \bar{z}(t) \\ \dot{\bar{z}}(t) = \bar{w}(t). \end{cases}$$

First step : pre-stabilization and robustness

We choose $k_p > 0$ such that $\dot{\bar{w}}(t) = -k_p \bar{w} + v(t)$ is **ISS**, i.e. :

$$\frac{1}{2}\frac{d}{dt}|\bar{w}(t)|^2 \leq -k_p|\bar{w}(t)|^2 + |v(t)|^2$$

Any $k_p > 0$ works.

Gain design : a Lyapunov approach

We consider $\bar{w} = w - w_\infty$ and $\bar{z} = z - z_\infty$, then :

$$\begin{cases} \dot{\bar{w}}(t) = -k_p\bar{w}(t) - k_i\bar{z}(t) \\ \dot{\bar{z}}(t) = \bar{w}(t). \end{cases}$$

Second step : forwarding method [Mazenc & Praly, 1996]

Idea : take advantage of the stability of $\bar{w}$!

$$W(\bar{w},\bar{z}) := \frac{1}{2}|\bar{w}(t)|^2 + b|\bar{z}(t) - M\bar{w}|^2.$$

How does one select M ? We pick : $M := -\frac{1}{k_p}$. Then :

$$\frac{d}{dt}|\bar{z}(t) - M\bar{w}(t)|^2 = -k_i\bar{z}(t)(\bar{z}(t) - M\bar{w}(t))$$

Gain design : a Lyapunov approach

We consider $\bar{w} = w - w_\infty$ and $\bar{z} = z - z_\infty$, then :

$$\begin{cases} \dot{\bar{w}}(t) = -k_p \bar{w}(t) - k_i \bar{z}(t) \\ \dot{\bar{z}}(t) = \bar{w}(t). \end{cases}$$

Third step : selecting the gain

$$\frac{d}{dt} W(\bar{w}, \bar{z}) = \underbrace{-k_p |\bar{w}(t)|^2 + k_i^2 |\bar{z}_i(t)|^2}_{\text{ISS-Lyapunov function}} \underbrace{-bk_i \bar{z}(t)^2 + bk_i z(t) M \bar{w}(t)}_{\text{choice of } M}$$

Applying **Young's inequality**, we find conditions on k_i and b such that

$$\frac{d}{dt} W(\bar{w}, \bar{z}) \leq -CW(\bar{w}, \bar{z}), \; C := C(k_i, b, k_p) > 0$$

Gain design : a Lyapunov approach

We consider $\bar{w} = w - w_\infty$ and $\bar{z} = z - z_\infty$, then :

$$\begin{cases} \dot{\bar{w}}(t) = -k_p \bar{w}(t) - k_i \bar{z}(t) \\ \dot{\bar{z}}(t) = \bar{w}(t). \end{cases}$$

Recap : three steps design

1. Find a **ISS-Lyapunov** function.
2. Use the **forwarding** method.
3. Select the **gains**.

This is the **same strategy** for the **KdV** equation !

Stability properties : linear case

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = w_x(t,L) = 0, \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ x \in [0,L], \end{cases}$$

the origin is **globally exponentially stable** as soon as $L \notin \mathcal{N}$

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} \, | k, l \in \mathbb{N} \right\}.$$

Example

The natural energy $E(w) := \frac{1}{2} \|w(t,\cdot)\|_{L^2}^2$ satisfies :

$$\frac{d}{dt} E(w) = -|w_x(t,0)|^2.$$

If $L = 2\pi$ and $w_0(x) = 1 - \cos(x)$, one has $w_0'(0) = 0 \Rightarrow$ Stability but **no attractivity** !

Observability properties : linear case

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = w_x(t,L) = 0, \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ x \in [0,L], \end{cases}$$

the output $y(t) = w_x(t,0)$ is **exactly observable** if and only if $L \notin \mathcal{N}$ [Rosier, 1997].

Example

If $L = 2\pi$, and $y_0(x) = 1 - \cos(x)$, the output may **vanish** !

We **focus** on the **linear KdV** equation.

Assumption

$$L \notin \mathcal{N}$$

Regulation for the KdV

The open-loop is stable $\Rightarrow$ **no need** of a **proportional** action.

$$\begin{cases} w_t + w_x + w_{xxx} + d_1(x) = 0, \ (t,x) \in \mathbb{R}_+, \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = kz(t) + d_2, \ t \in \mathbb{R}_+ \\ \dot{z}(t) = y(t) - r, \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ x \in [0,L]. \end{cases}$$

Existing literature

1. Semigroup approach [Paunonen, 2015] [Paunonen & Pohjolain, 2010] $\Rightarrow$ quite **general**, but **difficult to apply**.
2. **Lyapunov** approach [Terrand-Jeanne, et al., 2019] $\Rightarrow$ **stronger** assumptions, but **constructive** approach.

Well-posedness

Notation :
$D(\mathcal{A}) := \{z, w \in \mathbb{R} \times L^2(0, L) \mid w(0) = w(L) = 0,\ w'(L) = z\}$.

Theorem [Balogoun, Marx, Astolfi, 2022]

There exist $k^\star > 0$ such that for any $k \in (0, k^\star)$, for any $(d, r) \in L^2(0, L) \times \mathbb{R}$ and

- **Weak solution :** For any $(z_0, w_0) \in \mathbb{R} \times L^2(0, L)$, $\exists!$ $(z, w) \in C^0(\mathbb{R}_+; \mathbb{R} \times L^2(0, L))$
- **Strong solution :** For any $(z_0, w_0) \in D(\mathcal{A})$, $\exists!$ $(z, w) \in C^1(\mathbb{R}_+; L^2(0, L) \times \mathbb{R}) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$

The proof is based on a **semigroup approach**.

Main result

Notation : w_∞, z_∞ are the equilibrium points.

Theorem [Balogoun, Marx, Astolfi, 2022]

Let $k \in (0, k^\star)$. Then, for any $(d, r) \in L^2(0, L) \times \mathbb{R}$:

1. There exist $\nu, C > 0$ such that, for all $(z_0, w_0) \in \mathbb{R} \times L^2(0, L)$, and for all $t \geq 0$

$$\|(z, w) - (z_\infty, w_\infty)\|_{L^2 \times \mathbb{R}} \leq Ce^{-\nu t}\|(z_0, w_0) - (z_\infty, w_\infty)\|_{L^2 \times \mathbb{R}}$$

2. The output y is **regulated** towards the reference r. In other words, for any $(z_0, w_0) \in D(\mathcal{A})$

$$\lim_{t \to +\infty} |w_x(t, 0) - r| = 0.$$

for any **strong solution**.

Change of coordinates

Equilibrium points

$$\begin{cases} w'_\infty(x) + w'''_\infty(x) + d_1(x) = 0, \\ w_\infty(0) = w_\infty(L) + kz_\infty, \\ w'_\infty(L) = d_2, \\ w'_\infty(0) - r = 0. \end{cases}$$

Consider the change of variable $\bar{w} = w - w_\infty$ and $\bar{z} = z - z_\infty$:

$$\begin{cases} \bar{w}_t + \bar{w}_x + \bar{w}_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ \bar{w}(t,0) = \bar{w}(t,L) = 0, \ t \in \mathbb{R}_+, \\ \bar{w}_x(t,L) = k\bar{z}(t), \ t \in \mathbb{R}_+, \\ \dot{\bar{z}}(t) = \bar{y}(t). \end{cases}$$

Regulation = stabilization

Claim

Stabilization of $\bar{w}$ and $\bar{z}$ $\Rightarrow$ **Regulation** with **strong solutions**.

Multiplying by $\bar{w}$, integrating and performing integration by parts :

$$k^2\bar{z}^2(t) - \bar{w}_x^2(t,0) = \int_0^L \bar{w}_t(t,x)\bar{w}(t,x)dx.$$

Then :

$$|\bar{w}_x(t,0)|^2 \leq \underbrace{\|\bar{w}_t\|_{L^2}}_{\to 0 \text{ strong solution}} \|\bar{w}\|_{L^2} + k^2 \underbrace{\bar{z}(t)}_{\to 0}.$$

Then, by definition of $\bar{w}_x(t,0)$:

$$\lim_{t\to+\infty} |w_x(t,0) - r| = 0.$$

Focus on the undisturbed case $\Rightarrow$ $d_1 = d_2 = r = 0$.

General strategy : first step

$$\begin{cases} w_t + w_x + w_{xxx} = 0 \\ w(t,0) = w(t,L) = 0 \\ w_x(t,L) = u(t) \end{cases}$$

Goal

Build a **ISS-Lyapunov functional** for the KdV equation, i.e.

$$\frac{d}{dt}V(w) \leq -\lambda V(w) + \sigma |u(t)|^2, \ \lambda, \sigma > 0.$$

Tools

1. **Strictification**.
2. **Observer** design by **backstepping** method.

General strategy : choice of k

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+, \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = kz(t), \ t \in \mathbb{R}_+ \\ \dot{z}(t) = y(t) - r, \ t \in \mathbb{R}_+ \end{cases}$$

Goal

Condition on k s.t. the derivative of the Lyapunov functional :

$$W(w,z) := V(w) + b|z - \mathcal{M}w|^2$$

is **negative-definite**, where $\mathcal{M} : L^2(0,L) \to \mathbb{R}$ is a **linear operator**.

Tools

1. **Sylvester** equation.
2. **Forwarding** method.

Open-loop properties

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+ \\ w_x(t,L) = u(t), \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ x \in [0,L], \end{cases}$$

Recall

When $u = 0$ and $L \notin \mathcal{N}$, 0 is **exponentially stable**.

The proof does not **build a Lyapunov functional**.

$\Rightarrow$ **difficult** to prove an ISS property.

Open-loop properties

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+ \\ w_x(t,L) = u(t), \ t \in \mathbb{R}_+ \\ w(0,x) = w_0(x), \ x \in [0,L], \end{cases}$$

Recall that $E(w) := \frac{1}{2}\|w\|_{L^2}^2$ and that :

$$\frac{d}{dt}E(w) = -|w_x(t,0)|^2 + |u(t)|^2 = -|y(t)|^2 + |u(t)|^2.$$

When $u = 0$, **nonpositivity** is ensured but the right hand side **depends only on the output**.

$\Rightarrow$ It is a **weak Lyapunov functional**

Strictification principle

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+ \\ w_x(t,L) = u(t), \ t \in \mathbb{R}_+. \end{cases}$$

Strictification ?

Consists in **modifying** a weak Lyapunov functional to make it **strict** ([Malisoff & Mazenc, 2009], [Prieur & Mazenc, 2012]).

Following [Praly, 2019], we **strictify** E with an **observer**.

Strictifying with an ISS observer

Suppose that we have the **converging** observer $\hat{w}$

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + p(x)[y(t) - \hat{w}_x(t,0)] = 0 \\ \hat{w}(t,0) = \hat{w}(t,L) = \hat{w}_x(t,L) = 0. \end{cases}$$

Consider $\tilde{w} := w - \hat{w}$ which satisfies

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + p(x)\tilde{w}_x(t,0) = 0 \\ \tilde{w}(t,0) = \tilde{w}(t,L) = 0, \hat{w}_x(t,L) = u(t). \end{cases}$$

Suppose that there exists a ISS Lyapunov functional U, then

$$\frac{d}{dt}U(\tilde{w}) \leq -\lambda U(\tilde{w}) + \sigma |u(t)|^2$$

Strictifying with an ISS observer

$$\begin{cases} w_t + w_x + w_{xxx} + p(x)w_x(t,0) - p(x)w_x(t,0) = 0 \\ w(t,0) = w(t,L) = 0, \\ w_x(t,L) = u(t), \end{cases}$$

We have

$$\frac{d}{dt}U(w) \leq -\lambda U(w) + \sigma|u(t)|^2 + \|p\|_{L^2}^2 |w_x(t,0)|^2$$

Recall that

$$\frac{d}{dt}E(w) = -|w_x(t,0)|^2 + |u(t)|^2,$$

then choosing $V(w) := aE(w) + U(w)$ with a large enough :

$$\frac{d}{dt}V(w) \leq -\tilde{\lambda}U(w) + \tilde{\sigma}|u(t)|^2$$

Question

How can we **design** such an **observer** ? What is p ?

Backstepping method

Recall that we want

$$\begin{cases} w_t + w_x + w_{xxx} + p(x)w_x(t,0) = 0 \\ w(t,0) = w(t,L) = 0, \\ w_x(t,L) = u(t), \end{cases}$$

to be ISS-exponentially stable.

Backstepping principle [Smyshlyaev & Krstic, 2008]

It is based on a **linear transformation** (Volterra or **Fredholm**) :

$$w(t,x) = \Pi(\gamma(t,x)) := \gamma(t,x) - \int_0^L P(x,y)\gamma(t,y)dy,$$

where P has to be defined and γ is the solution of a PDE that is chosen to be **exponentially stable**.

Backstepping method

Recall that we want

$$\begin{cases} w_t + w_x + w_{xxx} + p(x)w_x(t,0) = 0 \\ w(t,0) = w(t,L) = 0, \\ w_x(t,L) = u(t), \end{cases}$$

to be ISS-exponentially stable.

Goal

- Selecting a suitable γ.
- Proving that the transformation is boundedly invertible.
- Selecting p.

What are γ and P ?

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = 0, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = u(t). \end{cases}$$

The natural energy $E(\gamma) := \frac{1}{2}\|\gamma\|_{L^2}^2$ satisfies :

$$\frac{d}{dt}E(\gamma) \leq -\lambda E(\gamma) + |u(t)|^2.$$

What are γ and P ?

The kernel P satisfies :

$$\begin{cases} -\lambda P + P_y + P_{yyy} + P_x + P_{xxx} = \lambda\delta(x-y)\,, \\ P(x,0) = P(x,L) = 0\,, \\ P(L,y) = P(0,y) = 0\,, \\ P_x(L,y) = P_x(0,y) = 0\,, \end{cases}$$

and we have

$$p(x) := P_y(x,0)$$

Lemma [Coron and Lu, 2014]

There exists a **unique solution** $P \in H_0^1((0,L)\times(0,L))$ if $L \notin \mathcal{N}$ and the transformation

$$w(t,x) = \Pi(\gamma(t,x)) := \gamma(t,x) - \int_0^L P(x,y)\gamma(t,y)dy,$$

is **boundedly invertible**.

Lyapunov functional for the observer

$$
\begin{aligned}
\|w(t,\cdot)\|_{L^2} \leq & \|\Pi^{-1}\| \|\gamma(t,\cdot)\|_{L^2(0,L)} \\
\leq & e^{-\lambda t}\|\Pi^{-1}\| \|\gamma_0\|_{L^2} + \|\Pi^{-1}\| \sup_{t\in\mathbb{R}_+} |u(t)| \\
\leq & e^{-\lambda t}\|\Pi\| \|\Pi^{-1}\| \|w_0\|_{L^2(0,L)} + \|\Pi^{-1}\| \sup_{t\in\mathbb{R}_+} |u(t)|
\end{aligned}
$$

Lyapunov functionals

For the **observer**, one has

$$
U(w) := \frac{1}{2}\|\Pi^{-1}w\|_{L^2}^2,
$$

and the **ISS-Lyapunov functional** is :

$$
V(w) := aE(w) + U(w).
$$

Forwarding approach principle

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+, \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = kz(t), \ t \in \mathbb{R}_+, \\ \dot{z}(t) = y(t). \end{cases}$$

Strategy [Mazenc & Praly, 1996]

Based on a **linear operator** $\mathcal{M} : L^2(0,L) \to \mathbb{R}$:

$$W(w,z) = V(w) + b|z - \mathcal{M}w|^2.$$

Idea : take advantage of the stability of w.

Definition of $\mathcal{M}$

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+, \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = kz(t), \ t \in \mathbb{R}_+, \\ \dot{z}(t) = y(t). \end{cases}$$

The operator $\mathcal{M} : L^2(0,L) \to \mathbb{R}$ solves a Sylvester equation

$$A\mathcal{M} - \mathcal{M}\mathcal{S} = -\mathcal{C}, \text{ where } \mathcal{S}w := -w' - w''' \text{ and } A = 0$$

with $D(\mathcal{S}) := \{w \in H^3(0,L) \mid w(0) = w(L) = w'(L) = 0\}$ and $\mathcal{C}w = w'(0)$.

Uniqueness of the solution

Invoking [Phóng, 1991], there exists a **unique solution**.

Question

Does there exist an **explicit form** for $\mathcal{M}$?

Explicit form

The operator $\mathcal{M}$ can be explicitly written :

$$\mathcal{M}w := \int_0^L M(x)w(x)dx,\ w \in D(\mathcal{S})$$

With the **Sylvester** equation :

$$\int_0^L M(x)(w'(x) + w'''(x))dx = w'(0).$$

After some integration by parts, M solves the **boundary value problem :**

$$\begin{cases} M'''(x) + M'(x) = 0\,, x \in [0, L], \\ M(0) = M(L) = 0\,, \\ M'(0) = -1\,, \end{cases}$$

whose (unique) explicit solution is :

$$M : x \in \mathbb{R} \mapsto \frac{-2\sin(\frac{x}{2})\sin(\frac{L-x}{2})}{\sin(\frac{L}{2})}.$$

Lyapunov computation

$$\frac{d}{dt}|z - \mathcal{M}w|^2 = 2\left(y(t) - \int_0^L M(x)w_t(t,x)dx\right)\left(z - \int_0^L M(x)w(t,x)dx\right)$$

But

$$\int_0^L M(x)w_t dx = \underbrace{\int_0^L M(x)[w_{xxx} + w_x]dx = w_x(t,0)}_{\text{Sylvester equation}} + kz(t)$$

Finally

$$\frac{d}{dt}|z - \mathcal{M}w|^2 = -kz\left(z - \int_0^L M(x)w(t,x)dx\right)$$
$$= -kz^2 + kz\int_0^L M(x)w(t,x)dx$$

Using the ISS property

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L] \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+ \\ w_x(t,L) = kz(t), \ t \in \mathbb{R}_+ \\ \dot{z}(t) = w_x(t,0), \ t \in \mathbb{R}_+ \end{cases}$$

Recall the Lyapunov functional :

$$W(w,z) = V(w) + b|z - \mathcal{M}w|^2.$$

Using the ISS property of V, one has :

$$\frac{d}{dt}V(w) \leq -\lambda V(w) + |kz(t)|^2$$

Using Young's inequalities, $\exists\, C > 0$ such that :

$$\frac{d}{dt}W(w,z) \leq -CW(w,z).$$

One also obtains a **condition on** k.

Systems with different time-scales

- Many natural phenomena feature interaction of processes on different times scales.
- A lot of difficulties appear. For instance, huge cost in numerical simulations since the fastest time scale sub-system must be fully solved over a timespan of the slowest scales' order.
- **Desirable** : we want instead solve a limit system, describing approximately the full behavior when some parameters (representing the scales) go to zero (or infinity)

Problem statement

Consider the following coupled system with different **time-scales**

$$\begin{cases} \varepsilon w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ w(0,x) = w_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cw_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0, \end{cases}$$

$a, b, c \in \mathbb{R}$, $\varepsilon > 0$ is supposed to be **small**.

Questions

What are the **conditions** on a, b, c such that the coupled system is **stable** ? Do these conditions change when ε is small ?

Solution : use (almost) the same techniques !

Main results

$$\begin{cases} \varepsilon w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ w(0,x) = w_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cw_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Proposition (Marx, Cerpa, 2022)

For any $\varepsilon > 0$, there exist a_*, k_1, k_2 such that if $a < a_*$ and b, c satisfy $k_1 < ac - b < k_2$, then the origin is globally exponentially stable.

This result can be seen as a generalization of the latter result where $b = 0$ and $c = 1$.

Main results

$$\begin{cases} \varepsilon w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ w(0,x) = w_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cw_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Question

What about the case of small ε ?

In this case, we use specific tools from singular perturbation techniques

The singular perturbation principle

The singular perturbation principle consists in decoupling the coupled system into two approximated systems :

1. The **reduced order system** $\simeq$ **slower system**
2. The **boundary layer system** $\simeq$ **faster system**

Question

How can one compute these two systems ?

Reduced order system

Suppose that $\varepsilon = 0$. Then,

$$\begin{cases} w_x + w_{xxx} = 0, \; (t,x) \in \mathbb{R}_+ \times [0,L], \\ w(t,0) = w(t,L) = 0, \; t \in \mathbb{R}_+, \\ w_x(t,L) = az(t), \; t \in \mathbb{R}_+. \end{cases}$$

There exists an explicit solution to this problem :

$$w(t,x) = -2az(t)f(x),$$

with $f(x) = \frac{1}{\sin\left(\frac{L}{2}\right)} \sin\left(\frac{x}{2}\right) \sin\left(\frac{L-x}{2}\right)$. Note that $w_x(t,0) = -az(t)$, then replacing in $\dot{z} = bz(t,0) + cw_x(t,0)$ $w_x(t,0)$ by $-az$, one obtains

Reduced order system

$$\dot{\bar{z}}(t) = (b - ac)\bar{z}(t).$$

If $(b - ac) < 0$, then this system is **exponentially stable** !

Boundary layer system

Set $\tau = \frac{t}{\varepsilon}$. After some computations, and using the fact that $\varepsilon = 0$, one obtains :

Boundary layer system

$$\begin{cases} w_\tau + w_x + w_{xxx} = 0, \\ w(\tau, 0) = w(\tau, L) = 0, \\ w_x(\tau, L) = 0. \end{cases}$$

If $L \notin \mathcal{N}$, the system is **exponentially stable** !

Singular perturbation result

$$\begin{cases} \varepsilon w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ w(0,x) = w_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cw_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Theorem (Marx, Cerpa, 2022)

For any $a, b, c \in \mathbb{R}$ such that $(b - ac) < 0$, there exists ε^* such that, for any $\varepsilon \in (0, \varepsilon^*)$, the origin is globally exponentially stable.

In other words, for a small ε, studying the stability of the subsytems is sufficient.

Singular perturbation result

$$\begin{cases} \varepsilon w_t + w_x + w_{xxx} = 0, \ (t,x) \in \mathbb{R}_+ \times [0,L], \\ w(t,0) = w(t,L) = 0, \ t \in \mathbb{R}_+, \\ w_x(t,L) = az(t), \ t \in \mathbb{R}_+, \\ w(0,x) = w_0(x), \ x \in [0,L], \\ \dot{z}(t) = bz(t) + cw_x(t,0), \ t \in \mathbb{R}_+, \\ z(0) = z_0. \end{cases}$$

Remark

While this result holds in the finite-dimensional setting, this is no longer the case for some examples in the infinite-dimensional case, see e.g., [Tang & Mazanti, 2017], [Cerpa & Prieur, 2019].

Sketch of the proof : change of coordinates

We consider

$$\tilde{w}(t,x) = w(t,x) + 2f(x)az(t)$$

One obtains

$$\begin{cases} \varepsilon\tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} = -\varepsilon((b-ac)z(t) + c\tilde{w}_x(t,0))f(x) \\ \tilde{w}(t,0) = \tilde{w}(t,L) = 0 \\ \tilde{w}_x(t,L) = 0 \\ \dot{z} = (b-ac)z + c\tilde{w}_x(t,0). \end{cases}$$

Using the same Lyapunov functional than in the PI case, one obtains the desired result !

Achievements and open problems

Achievements

1. We have built an **ISS-Lyapunov for KDV**.
2. We have applied it for some **output regulation** task, and for the case where there exist **two different time-scales**.

Open problems

1. What about the **nonlinear** case ? What about $L \in \mathcal{N}$? Follow [Coron & Rivas, 2016] ?
2. What about the case of **time-varying** disturbances ? Follow [Astolfi & Praly, 2017] ?
3. Is it possible to use singular perturbation techniques for regulation purposes as in [Lorenzetti & Weiss, 2021] ?

Thank you for your attention

Any question ?