Exact controllability in projections of the bilinear Schrödinger equation

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Control of the Schrödinger equation

$$i\frac{d\psi}{dt}(t) = \left(H_0 + \sum_{k=1}^{m} u_k(t) H_k\right)\psi(t)$$

with $H_0, \ldots, H_m$ selfadjoint on some infinite-dimensional Hilbert space $\mathcal{H}$, $(u_1, \ldots, u_m) \in U$ with U neighborhood of 0.
Assume that H_0 has discrete spectrum.

Example: harmonic oscillator

$\psi(t) \in L^2(\mathbb{R}, \mathbb{C})$, $m = 1$, $H_0 = -\partial_x^2 + x^2$, $H_1 = x$

Example: rotating linear molecule

$\psi(t) \in L^2(S^1, \mathbb{C})$, $m = 2$, $H_0 = -\partial_\theta^2$, $H_1 = \cos\theta$, $H_2 = \sin\theta$

Goal: give (sufficient) conditions for exact controllability of finitely many modes

Finite-dimensional controllability criteria

The drift $-iH_0 : \mathbb{C}^n \to \mathbb{C}^n$ generates unitary transformations of S^{2n-1}

$\Longrightarrow$ all points in S^{2n-1} are Poisson stable for iH_0

$\Longrightarrow$ trajectories of iH_0 are in closure of attainable sets for $\dot{\psi} = -iH_0\psi - i\sum_{j=1}^{m} u_j H_j \psi$, $u \in U$

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$\Longrightarrow$ trajectories of $\pm iH_j$, $j = 0, \ldots, m$, are in closure of attainable sets for $\dot{\psi} = -iH_0\psi - i\sum_{j=1}^m u_j H_j \psi$, $u \in U$ (convexification)

$\Longrightarrow$ exact controllability is equivalent to

$$\mathrm{Lie}_\psi(iH_0, \ldots, iH_m) = T_\psi S^{2n-1}, \quad \forall \psi \in S^{2n-1}.$$

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Since trajectories on S^{2n-1} are $\psi(t) = g(t)\psi_0$ where

$$i\frac{dg}{dt}(t) = \left(H_0 + \sum_{k=1}^m u_k(t)H_k\right) g(t), \quad g \in U(n),\ g(0) = \mathrm{Id}$$

we have exact controllability if

$$\mathrm{Lie}(iH_0, \ldots, iH_m) \supseteq \mathrm{su}(n).$$

The infinite-dimensional setting

For simplicity of notations $A = -iH_0$, $B_1 = -iH_1$, $\dots$, $B_m = -iH_m$, $U = [-\delta, \delta]^m$, i.e.,

$$\frac{d\psi}{dt} = A\psi + \sum_{k=1}^{m} u_k B_k \psi, \quad \psi \in \mathcal{H},\ |u_1|, \dots, |u_m| \leq \delta$$

with

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with

- $\mathcal{H}$ complex Hilbert space
- A with discrete spectrum $(\lambda_n)_{n\in\mathbb{N}}$ and corresponding orthonormal basis of eigenvectors $(\phi_n)_{n\in\mathbb{N}}$
- $\phi_n \in D(B_k)$ for every $n \in \mathbb{N}$, $k = 1, \dots, m$
- $\forall u \in U$, $A + \sum_{k=1}^{m} u_k B_k$ essentially skew-adjoint on $\mathrm{span}(\phi_n)_{n\in\mathbb{N}}$

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Examples for A:

- $i\Delta$ on compact manifolds (rotating molecule)
- $i\Delta$ on bounded domains with Dirichlet boundary conditions (potential well)
- $i(\Delta - V)$ with $\lim_{|x|\to\infty} V(x) = +\infty$ (harmonic oscillator)

Admissible controls and associated solutions

Definition

$u \in L^\infty([0,T],U)$ is admissible if for every $\psi_0 \in \mathcal{H}$ there exists a solution $\psi : [0,T] \to \mathcal{H}$ such that $\psi(0) = \psi_0$, $t \mapsto \langle \phi_n, \psi(t) \rangle$ is absolutely continuous for every $n \in \mathbb{N}$ and satisfies

$$\frac{d}{dt}\langle \phi_n, \psi(t)\rangle = -\langle (A + u_1(t)B_1 + \cdots + u_m(t)B_m)\phi_n, \psi(t)\rangle,$$

for almost every $t \in [0,T]$.

Skew-adjointness of $A + u_1 B_1 + \cdots + u_m B_m$ implies that the norm of the solutions is constant along the evolution ($\Longrightarrow$ uniqueness of solutions).
The propagator $\Upsilon^u_t : \psi_0 \mapsto \psi(t)$ is unitary.

Remark

Piecewise constant controls are admissible

Kato-smallness and admissibility of smooth controls

Let $u \in C^1([0,T],U)$ and $B_1,\dots,B_m$ be A-bounded with A-bound smaller than $1/\delta$, namely

- $D(B_j) \supset D(A)$,
- there exists $a < 1/\delta$ and $b \in \mathbb{R}$ such that for all $\phi \in D(A)$ one has

$$\|B_j\phi\| \le a\|A\phi\| + b\|\phi\|, \quad j = 1,\dots,m.$$

Then by Kato–Rellich theorem, u is admissible. If $\psi_0 \in D(A)$ then $t \mapsto \Upsilon_t^u \psi_0$ is a classical solution of the Cauchy problem.

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Definition

A class of controls $\mathcal{U}$ is said to be admissible if all controls in $\mathcal{U}$ are admissible and, moreover, if $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ L^1-converges to $u \in \mathcal{U}$ then, for every $\phi \in \mathcal{H}$, $\Upsilon^{u_n}_t\phi$ tends to $\Upsilon^u_t\phi$ in $\mathcal{H}$ uniformly w.r.t. t

In the case in which $B_1,\dots,B_m$ are bounded operators, then the assumption is satisfied thank to [Ball-Marsden-Slemrod, 1982] for $\mathcal{U} = L^\infty(U)$. More general conditions on $B_1,\dots,B_m$ can be found for [Boussaïd-Caponigro-Chambrion, 2014].

Notions of controllability ($\mathcal{S}$ unit sphere of $\mathcal{H}$)

Definition

The system is exactly controllable in $\tilde{\mathcal{H}}$, subspace of $\mathcal{H}$, if for every $\psi_0, \psi_1 \in \tilde{\mathcal{H}} \cap \mathcal{S}$ there exists an admissible control $u : [0, T] \to U$ such that $\Upsilon^u_t \psi_0 \in \tilde{\mathcal{H}}$ and $\Upsilon^u_T \psi_0 = \psi_1$.

Definition

The system is approximately controllable if for every $\psi_0, \psi_1 \in \mathcal{S}$ and every $\varepsilon > 0$ there exists an admissible control $u : [0, T] \to U$ such that $\|\psi_1 - \Upsilon^u_T \psi_0\| < \varepsilon$.

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Let $\pi_n : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection on $\mathrm{span}\{\phi_1, \dots, \phi_n\}$.

Definition

The system is exactly controllable in projections if for every $\psi_0, \psi_1 \in \mathcal{S}$, for every $n \in \mathbb{N}$ such that $\pi_n(\psi_1) \neq \psi_1$, there exists an admissible control $u : [0, T] \to U$ such that $\pi_n \Upsilon_T^u \psi_0 = \pi_n \psi_1$.

Exact controllability in projections $\Longrightarrow$ approximate controllability

Controllability results

Negative results

- non-exact controllability in the unit sphere of $L^2(\Omega)$ for $\mathcal{U}$ admissible (Ball-Marsden-Slemrod [1982], Turinici [2000], Boussaid-Chambrion-Caponigro [preprint])
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Positive results

- exact controllability in $H^3(\Omega)$ by linearization for potential well (Beauchard ['05], Beauchard-Coron ['06], Beauchard-Laurent ['10])
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  - Noncommutativity and filtration of invariant finite-dimensional spaces (Law–Eberly ['96], Ervedoza–Puel ['09], Bloch–Brockett–Rangan ['10], Bliss–Burgarth ['14], Keyl–Zeier–Schulte-Herbrueggen ['14])

Approximate controllability by the Lie–Galerkin method

In [Chambrion–Mason–S–Boscain, Ann. IHP, 2009], [Boscain–Caponigro–Chambrion–S, CMP, 2012], [Boscain–Caponigro–S, JDE, 2014] we obtained sufficient conditions for approximate controllability based on the following scheme.

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- Approximate controllability $\Longleftrightarrow \pi_n \Upsilon^u_T \psi_0 \approx \pi_n \psi_1$ for $n \gg 1$
- Control of Galerkin approximations

$$\dot{x} = \pi_n(A + u_1 B_1 + \dots u_m B_m)\pi_n x, \quad x \in \mathbb{C}^n \simeq \pi_n \mathcal{H}$$

  is not enough.
  We should control simultaneously
  - the Galerkin approximation
  - the transfer of population to the remaining components by selecting spectral gaps

General approximate controllability result

For $M \in \mathbb{C}^{n\times n}$, $\sigma > 0$, let $\mathcal{E}_\sigma(M) = (M_{lj}\delta_{\sigma,|\lambda_l-\lambda_j|})_{l,j=1}^n$.
$\mathcal{E}_\sigma(\cdot)$ corresponds to the selection of the spectral gap σ.

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Define $A^{(n)} = \pi_n A \pi_n \in \mathbb{C}^{n\times n}$, $B_j^{(n)} = \pi_n B_j \pi_n \in \mathbb{C}^{n\times n}$ and

$$\Xi_n = \Big\{(\sigma, k) \mid \sigma = |\lambda_l - \lambda_j|,\ 1 \le l,j \le n,\ 1 \le k \le m \\ \text{s.t. } \mathcal{E}_\sigma(B_k^{(N)}) = \left(\begin{array}{c|c} \mathcal{E}_\sigma(B_k^{(n)}) & 0 \\ \hline 0 & * \end{array}\right) \text{ for every } N > n\Big\}.$$

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Let

$$\mathcal{V}_n = \left\{A^{(n)}\right\} \cup \left\{\mathcal{E}_\sigma(B_k^{(n)}) \mid (\sigma, k) \in \Xi_n\right\}$$

collection of all compatible dynamics for the n-dimensional Galerkin approximation.

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collection of all compatible dynamics for the n-dimensional Galerkin approximation.

Theorem (Boscain–Caponigro–S, JDE, 2014)

If for every $n_0 \in \mathbb{N}$ there exists $n > n_0$ such that $\mathrm{Lie}\mathcal{V}_n \supseteq \mathfrak{su}(n)$, then $\dot\psi = (A + u_1 B_1 + \cdots + u_m B_m)\psi$ is approximately controllable

Example: potential well

Potential well: $m = 1$ and

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - u(t)x\psi(x,t), \quad x \in (-\pi/2, \pi/2)$$

with $\psi(-\pi/2, \cdot) \equiv 0 \equiv \psi(\pi/2, \cdot)$.
Here $\mathcal{H} = L^2\left((-\pi/2, \pi/2), \mathcal{C}\right)$, A has eigenvalues $\lambda_k = ik^2$ and

$$\phi_k(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\cos(kx) & \text{when } k \text{ is odd} \\ \sqrt{\frac{2}{\pi}}\sin(kx) & \text{when } k \text{ is even} \end{cases}$$

$$\langle \phi_j, B\phi_k \rangle \neq 0, \text{ if and only if } j + k \text{ is odd}$$

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For $k = 2,\dots,n$, $|\lambda_k - \lambda_{k-1}| = 2k-1$ and $[\mathcal{E}_{2k-1}(B^{(N)})]_{j,l} = 0$ for $j \leq n$ and $l \geq n+1$, since

$$l^2 - j^2 \geq (n+1)^2 - n^2 = 2n+1 > 2k-1.$$

Hence $\mathcal{E}_{2k-1}(B^{(n)}) \in \mathcal{V}_n$ for $k = 2,\dots,n$

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Hence $\mathcal{E}_{2k-1}(B^{(n)}) \in \mathcal{V}_n$ for $k = 2, \ldots, n$ and

$$\mathrm{Lie}\mathcal{V}_n \supset \mathrm{Lie}\left(\{A^{(n)}\} \cup \{\mathcal{E}_{2k-1}(B^{(n)}) \mid k = 2, \ldots, n\}\right) \supset \mathfrak{su}(n)$$

Example: rotating bipolar molecule

$$i\frac{\partial\psi(\theta,\varphi,t)}{\partial t} = -\Delta\psi(\theta,\varphi,t) + \\ + (u_1(t)\sin\theta\cos\varphi + u_2(t)\sin\theta\sin\varphi + u_3(t)\cos\theta)\psi(\theta,\varphi,t),$$

where θ, φ are the spherical coordinates and

$$\Delta = \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\varphi^2 + \frac{\cos\theta}{\sin\theta}\partial_\theta$$

is the Laplace–Beltrami operator on the sphere S^2.

Exact controllability in projections (with smooth controls) can be recovered despite severe degeneracy of the spectrum: the ℓ-th eigenvalue $-i\ell(\ell+1)$ has multiplicity $2\ell+1$.

Exact controllability in projections

Theorem (Caponigro–S, in preparation)

Let $\mathcal{U}$ be either the class of piecewise constant or the class of smooth controls and assume that $\mathcal{U}$ is admissible.
If for every $n_0 \in \mathbb{N}$ there exists $n > n_0$ such that $\mathrm{Lie}\mathcal{V}_n \supseteq \mathfrak{su}(n)$, then $\dot{\psi} = (A + u_1 B_1 + \cdots + u_m B_m)\psi$ is exactly controllable in projections with controls in $\mathcal{U}$.

Idea of the proof

- let $n \in \mathbb{N}$ and $\psi_1 \in \mathcal{S}$ such that $\pi_n \psi_1 \neq \psi_1$
- take $\varepsilon \in (0, \|\psi_1 - \pi_n \psi_1\|)$
- for every point in $\{\phi \in \mathbb{C}^n \mid \|\phi - \pi_n \psi_1\| \leq \varepsilon\}$ construct u_ϕ steering ψ_0 to a $\varepsilon/2$-neighborhood of $\hat{\phi} \in \mathcal{S}$ with $\pi_n \hat{\phi} = \phi$
- provided that $\phi \mapsto \Upsilon_T^{u_\phi} \psi_0$ is continuous, by topological arguments (degree theory), then there exists $\phi_\star$ such that $\pi_n \Upsilon_T^{u_{\phi_\star}} \psi_0 = \pi_n \psi_1$

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Lemma

Let $X \subset \mathbb{R}^n$ be open and bounded. Assume that $y_0 \in X$ and consider $\varepsilon = \mathrm{dist}(y_0, \partial X) > 0$. If $G \in C(\overline{X}, \mathbb{R}^n)$ satisfies

$$\max_{x \in \overline{X}} |x - G(x)| \leq \frac{\varepsilon}{2}$$

then $y_0 \in \mathrm{int}(G(X))$

Spectral gap selection and infinite-dimensional approximation (cf. [Chambrion, Automatica, 2012])

Lemma

Let $n \in \mathbb{N}$ and $(\sigma, j) \in \Xi_n$ with $\sigma > 0$. Let $u^*_{\sigma,j} = (0, \dots, 0, v, 0, \dots, 0) : \mathbb{R} \to \mathbb{R}^m$ be periodic of period $T = 2\pi/\sigma$ and such that

$$\int_0^T v(t) e^{ip\sigma t} dt = 0$$

for every integer $p \neq \pm 1$. Set $\nu = \int_0^T v(t) e^{i\sigma t} dt$. Assume that $u^*_{\sigma,j}/K$ is admissible for every $K \in \mathbb{N}$ large enough. Then, for every $\tau \in \mathbb{R}$ and $\psi \in \pi_n \mathcal{H}$,

$$\|\Upsilon^{u^*_{\sigma,j}/K}_{KT}(e^{\tau A}\psi) - e^{(\tau+KT)A} e^{\nu \mathcal{E}_\sigma(B^{(n)}_j)} \psi\| \to 0$$

as $K \to \infty$, where $e^{\nu \mathcal{E}_\sigma(B^{(n)}_j)} : \mathbb{C}^n \to \mathbb{C}^n$ is identified with an operator on $\pi_n \mathcal{H}$.

Normal controllability for the adapted Galerkin approximations

Let $p > n$ be such that $\mathrm{Lie}\mathcal{V}_p \supseteq \mathfrak{su}(p)$.
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Let $\tilde{\psi}_1 \in S^{2p-1}$ be such that $\pi_n \psi_1 = \pi_n^{\mathbb{C}^p} \tilde{\psi}_1$.
By classical results in control ([Jurdjevic–Sussmann, 1972, Sussmann, 1976]) there exist $M_1, \dots, M_k \in \mathcal{V}_p$ and $t_1, \dots, t_k > 0$ such that the map

$$E : (s_1, \dots, s_k) \mapsto e^{s_k M_k} \circ \cdots \circ e^{s_1 M_1}(\phi_1)$$

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Moreover, there exist $j_1, \ldots, j_{2n} \in \{1, \ldots, k\}$ such that the map

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has rank $2n$ at $(t_{j_1}, \ldots, t_{j_{2n}})$ and

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Then

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Degree theory $\Rightarrow$ $\pi_n\psi_1$ in the image of G^K for K large enough