Stabilisation rapide d'equations de Burgers et de Korteweg-de Vries

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PDEs theory

Let Ω be a smooth, compact, Riemannian manifold.

$$\text{u\_t-Delta u=f.}$$

People study the following properties for PDEs:

- Well-posedness.
- Stability: solutions converge to equilibrium points at some rate

  Asymptotically? Exponentially? In finite time?
- Regularity, blow up, soliton, limiting problem, stochastic etc.

Everything is fixed.
No acting on the dynamics!

PDEs' control theory: controllabilty

$$y_t - \Delta y = \chi_\omega u, \;\; \omega \subset \Omega.$$

Controllability: change dynamics of solutions by control terms, $u = u(t)$.
$Example$: let $z \in \mathbb{R}$ be the state and $v \in \mathbb{R}$ be the control.

$$\dot{z} = z + v.$$

Let $T > 0$. Then,

$$\forall z_0, z_1 \in \mathbb{R}, \; \exists \, v(t) : [0, T] \to \mathbb{R},$$

such that

$$z(t=0) = z_0 \;\text{ and }\; z(t=T) = z_1.$$

Theorem (Fattorini-Russell, Lebeau-Robbiano, Fursikov-Imanuvilov, ..., Coron-Nguyên, 1971-2016)

Let $T > 0$. The heat equation is null controllable:
$$\forall y_0 \in L^2, \exists u \text{ such that the solution satisfies } y(T) = 0.$$

$$y_t - \Delta y = \chi_\omega u, \ \ \omega \subset \Omega.$$

Stabilization: make systems stable with feedback laws, $u = u(t, y(t))$.
$Example$:

$$\dot{z} = z + v.$$

Let

$$v(t, z) := -2z,$$

then (exponential stabilization),

$$\dot{z} = -z.$$

Rapid stabilization: Barbu, Coron, Trélat, Tucsnak, Raymond...

Theorem (Small-time stabilization: Coron-Nguyên, 2016)

Let $T > 0$. For the one dimensional heat equation with boundary control, there exists a T-periodic feedback law f such that the flow of the solution becomes 0 after time T.

Stabilization: classical perturbation method

$$\dot{y} = y + y^2 + u, \text{ with } y, u \in \mathbb{R}.$$

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Linearization

(1) Linearization:

$$\dot{y} = y + u.$$

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Linearization $\Rightarrow$ Stabilization

(1) Linearization:

$$\dot{y} = y + u.$$

(2) Stabilization: let $u := -2y$,

$$\dot{y} = y + u = -y.$$

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Linearization ⇒ Stabilization ⇒ Perturbation

(1) Linearization:

$$\dot{y} = y + u.$$

(2) Stabilization: let $u := -2y$,

$$\dot{y} = y + u = -y.$$

(3) Perturbation: still let $u := -2y$, even for the nonlinear system

$$\dot{y} = y + y^2 + u = -y + y^2.$$

Perturbation: three critical cases

Linearization $\Rightarrow$ Stabilization $\Rightarrow$ Perturbation

Perturbation: three critical cases

Linearization ⇒ Stabilization ⇒ Perturbation

EXCEPT

(1) Stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable.

$$\dot{y} = uy \xrightarrow{\text{linearization}} \dot{y} = 0. \quad ✘$$

Perturbation: three critical cases

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(1) Stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable.

$$\dot{y} = uy \xrightarrow{\text{linearization}} \dot{y} = 0. \quad ✘$$

(2) Small-time local stabilization of linear controllable systems by nonlinear feedbacks.

$$\dot{y} = y + u \xrightarrow{u := -(1+\lambda)y} \dot{y} = -\lambda y. \quad ✘$$

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(3) Small-time global stabilization of nonlinear controllable systems.

$$\dot{y} = y^2 + u \xrightarrow{u:=-\lambda y} \dot{y} = -\lambda y + y^2. \quad ✘$$

Three critical cases: nonlinear feedback

Linearization ⇒ Stabilization ⇒ Perturbation

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(1) Stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable.

$$\dot{y} = uy \xrightarrow{u:=-y^2} \dot{y} = -y^3. \ ✔$$

(2) Small-time local stabilization of linear controllable systems by nonlinear feedbacks.

$$\dot{y} = y + u \xrightarrow{u:=-y-y^{1/3}} \dot{y} = -y^{1/3}. \ ✔$$

(3) Small-time global stabilization of nonlinear controllable systems.

$$\dot{y} = y^2 + u \xrightarrow{u:=-y^2-y^3-y^{1/3}} \dot{y} = -y^3 - y^{1/3}. \ ✔$$

Outline of the defense

(1) Stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable.

(2) Small-time local stabilization of linear controllable systems by nonlinear feedbacks.

(3) Small-time global stabilization of nonlinear controllable systems.

Problem 1: exponential stabilization by nonlinear feedback

(1) Stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable.

- *Local exponential stabilization of the KdV equation with a Neumann boundary control.*

(2) Small-time local stabilization of linear controllable systems by nonlinear feedbacks.

(3) Small-time global stabilization of nonlinear controllable systems.

KdV equations

Let $L \in (0, +\infty)$. We consider the controllability of the Korteweg-de Vries (KdV) system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = a(t) & \text{for } t \in (s,+\infty), \\ y(t,L) = b(t) & \text{for } t \in (s,+\infty), \\ y_x(t,L) = u(t) & \text{for } t \in (s,+\infty), \end{cases}$$

where $s \in \mathbb{R}$ and where, at time $t \in [s, +\infty)$, the state is $y(t,\cdot) \in L^2(0,L)$ (also the space of controllability) and the controls are $a(t), b(t), u(t) \in \mathbb{R}$.

Many tools: Hilbert Uniqueness Method, Linear Test, Power Series Expansion, Return Method, Boundary Layer, Dispersive, Soliton...

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Linearized system is not controllable

We consider the linearized KdV system

Theorem (Rosier 1997, H.U.M.)

For $\forall L > 0$, *we can decompose* L^2 *by*

$$L^2 = H \oplus M,$$

where

$$\begin{aligned} H &:= \textit{controllable states}, \\ M &:= \textit{uncontrollable states, with dim } M < +\infty. \end{aligned}$$

Remark

Dim $M \neq 0$, *if and only if*

$$L \in \mathcal{N} := \left\{ 2\pi \sqrt{\frac{l^2 + lk + k^2}{3}};\, l, k \in \mathbb{N}^* \right\}.$$

Nonlinear system is controllable

Even if there are uncontrollable states, KdV system is controllable:

Theorem

KdV system is (locally) controllable, if

- *(Coron-Crépeau, 2004) dim* $M = 1, \forall T > 0$*;*
- *(Cerpa, 2007) dim* $M = 2$*, for* T *large enough;*
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Power Series Method

$$y = \varepsilon^1 y^1 + \varepsilon^2 y^2 + \varepsilon^3 y^3 + \ldots$$

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$$y = \varepsilon^1 y^1$$

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Power Series Method

$$y = \varepsilon^1 y^1 + \varepsilon^2 y^2 + \varepsilon^3 y^3$$

Linear system: not stabilizable!

Projection: $y = y_1 + y_2 = P_H(y) + P_M(y)$.

Semi-group: $S(t)$.

$$\begin{cases} y_{1t} + y_{1x} + y_{1xxx} = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ y_{1x}(t,L) = u(t), \end{cases}$$

$$\begin{cases} y_{2t} + y_{2x} + y_{2xxx} = 0, \\ y_2(t,0) = y_2(t,L) = 0, \\ y_{2x}(t,L) = 0, \end{cases}$$

- No feedback

$$\|S(t)y_1\|_{L^2} \leq Ce^{-ct}\|y_1\|_{L^2}.$$

- With feedback

$$\|S(t)y_1\|_{L^2} \leq C_\lambda e^{-\lambda t}\|y_1\|_{L^2}.$$

- Any feedback

$$\|S(t)y_2\|_{L^2} = \|y_2\|_{L^2}.$$

Nonlinearity=Stabilizability

Theorem (Coron-Rivas-Xiang, 17)

Dim $M = 2n$*. The KdV system is* ***locally exponentially stabilizable*** *for the functional:*

$$\|P_H y\|^2_{L^2} + \|P_M y\|_{L^2}.$$

More precisely, there exist $T > 0$*,* $\lambda > 0$*,* $r > 0$*,* $C > 0$*, and a* T*-periodic feedback law such that, for every* $y_0 \in L^2(0, L)$ *satisfying* $\|y_0\|_{L^2(0,L)} \leqslant r$ *and for every initial time* $s \in \mathbb{R}$*, there exist solutions to the Cauchy problem in* $C^0([s, +\infty); L^2(0, L)) \cap L^2_{loc}([s, +\infty); H^1(0, L))$ *. Moreover, the solutions satisfy*

$$\|P_H(y(t))\|_{L^2} + \|P_M(y(t))\|^{\frac{1}{2}}_{L^2} \leqslant Ce^{-\lambda(t-s)} \left(\|P_H(y_0)\|_{L^2} + \|P_M(y_0)\|^{\frac{1}{2}}_{L^2}\right), \forall t \geqslant s.$$

Idea: projection

$$
\left\{
\begin{array}{l}
\left\{
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y_{1t} + y_{1x} + y_{1xxx} = -P_H\big((y_1 + y_2)(y_1 + y_2)_x\big), \\
y_1(t,0) = y_1(t,L) = 0, \\
y_{1x}(t,L) = u(t, y_1 + y_2),
\end{array}
\right. \\
\left\{
\begin{array}{l}
y_{2t} + y_{2x} + y_{2xxx} = -P_M\big((y_1 + y_2)(y_1 + y_2)_x\big), \\
y_2(t,0) = y_2(t,L) = 0, \\
y_{2x}(t,L) = 0.
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$$

Idea: $y_1^2 \sim y_2$

$$\begin{cases} \begin{cases} y_{1t} + y_{1x} + y_{1xxx} = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ y_{1x}(t,L) = u(t, y_1 + y_2), \end{cases} \\ \begin{cases} y_{2t} + y_{2x} + y_{2xxx} = -P_M\big(y_1 y_{1x}\big), \\ y_2(t,0) = y_2(t,L) = 0, \\ y_{2x}(t,L) = 0. \end{cases} \end{cases}$$

Idea: $y_1^2(t)$ vs $y_2(t)$

$$
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$$

- ✎ $\|y_1(t)\|_{L^2}^2 > \varepsilon^\beta \|y_2(t)\|_{L^2}$: y_1 dominates y_2.
- ✎ $\|y_1(t)\|_{L^2}^2 \leq \varepsilon^\beta \|y_2(t)\|_{L^2}$

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- ✎ $\|y_1(t)\|_{L^2}^2 > \varepsilon^{\beta}\|y_2(t)\|_{L^2}$: y_1 dominates y_2.
- ✎ $\|y_1(t)\|_{L^2}^2 \leq \varepsilon^{\beta}\|y_2(t)\|_{L^2}$: y_1 controls y_2.

Lemma (Coron-Rivas-Xiang, 2017)

There exist $v := [0,T] \times (\mathbb{S} \cap M) \to \mathbb{R}$ *and* $C_0 > 0$ *satisfy*

$$|v(t,f) - v(t,g)| \leqslant C_0 \|f - g\|_{L^2(0,L)}, \quad \forall t \in [0,T],\ \forall f, g \in \mathbb{S} \cap M,$$

such that, there exists $\delta > 0$ *such that, for every* $z \in \mathbb{S} \cap M$*, the solution* (y_1, y_2) *to the following equation*

$$\begin{cases} y_{1t} + y_{1x} + y_{1xxx} = 0, \\ y_1(t,0) = y_1(t,L) = 0, \\ y_{1x}(t,L) = v(t,f), \\ y_1(0,\cdot) = 0, \\ y_{2t} + y_{2x} + y_{2xxx} + P_M\left(y_1 y_{1x}\right) = 0, \\ y_2(t,0) = y_2(t,L) = 0, \\ y_{2x}(t,L) = 0, \\ y_2(0,\cdot) = 0 \end{cases}$$

satisfies

$$y_1(T) = 0 \quad \textit{and} \quad \left\langle y_2(T), S(T)f \right\rangle_{L^2(0,L)} < -2\delta.$$

Feedback law

For $\varepsilon > 0$, we define the feedback law u_ε by

$$u_\varepsilon\big|_{[0,T)\times L^2}(t,y) := \begin{cases} 0 & \text{if } \|y^M\|_{L^2} = 0, \\ \varepsilon\sqrt{\|y^M\|_{L^2}}\, v\Big(t, \frac{S(-t)y^M}{\|y^M\|_{L^2}}\Big) & \text{if } 0 < \|y^M\|_{L^2} \leqslant 1, \\ \varepsilon v\Big(t, \frac{S(-t)y^M}{\|y^M\|_{L^2}}\Big) & \text{if } \|y^M\|_{L^2} > 1, \end{cases}$$

with $y^M := P_M(y)$, and

$$u_\varepsilon(t,y) := u_\varepsilon\big|_{[0,T)\times L^2}(t - [t/T]T, y), \quad \forall t \in \mathbb{R}, \quad \forall y \in L^2(0,L).$$

Further questions

- ✍ Rapid exponential stabilization?
- ✍ Dim M= 2n+1?
- ✍ Controllability in small time?
- ✍ Global...

Problem 2: small-time local stabilization

(1) Stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable.

(2) Small-time local stabilization of linear controllable systems by nonlinear feedbacks.

- *Small-time local stabilization of the KdV equation with a Dirichlet boundary control.*

(3) Small-time global stabilization of nonlinear controllable systems.

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where $s \in \mathbb{R}$ and where, at time $t \in [s, +\infty)$, the state is $y(t,\cdot) \in L^2(0,L)$ (also the space of controllability) and the control is $a(t)$ or $b(t) \in \mathbb{R}$.

Many tools: Hilbert Uniqueness Method, Linear Test, Power Series Expansion, Return Method, Dispersive, Boundary Layer, Soliton...

Only one control

If there is only one control on $y(L) = b(t)$

Theorem (Glass-Guerrero, 2010)

The linearized KdV system is uncontrollable if and only if L belongs to a countable critical length set.

If there is only one control on $y(0) = a(t)$

Theorem (Rosier, 2004)

The linearized KdV system is null controllable, but not exactly controllable.

- No critical length with the left Dirichlet control;
- The proof relies on some Carleman estimates;
- There are similar results on Burgers equations (Barbu, Chapouly, Coron, Diaz, Fernández-Cara, Glass, Guerrero, Horsin, Imanuvilov, Marbach, Shirikyan, Xiang etc.)

Two controls

Open Problem

What about the global controllability with two Dirichlet controls?

A negative result on the Burgers equation

Theorem (Guerrero-Imanuvilov, 2007)

For the Burgers equation with two Dirichlet controls, the system is not globally controllable in small time.

"$Proof$": Hopf-Cole transformation, the maximum principle.

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Not for KdV...

Back to the case with one control $u(0)$

Theorem (Rosier, 2004; Xiang, 2017)

The KdV system is (locally) null controllable.

Methods for linear control problems (observability inequalities):

- ✎ Multiplier method (Ho & Lions).
- ✎ Moment method (Russell, hyperbolic equations).
- ✎ Defect measure (Bardos-Lebeau-Rauch, Burq-Gérard, wave equations).
- ✎ Global carleman estimates (Fursikov-Imanuvilov, the heat equation).
- ✎ Lebeau-Robbiano strategy (The heat equation).

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- ✎ Global carleman estimates (Fursikov-Imanuvilov, the heat equation).
- ✎ Lebeau-Robbiano strategy (The heat equation).
- ✎ Backstepping approach (Coron-Nguyên, the heat equation).

Natural energy dissipation

For the linearized system

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = 0 & \text{for } t \in (s,+\infty), \\ y(t,L) = 0 & \text{for } t \in (s,+\infty), \\ y_x(t,L) = 0 & \text{for } t \in (s,+\infty). \end{cases}$$

The energy decays

$$\frac{d}{dt}\|y(t,\cdot)\|_{L^2}^2 \leqslant 0.$$

A better system

We further consider

$$\begin{cases} z_t + z_{xxx} + z_x + \lambda z = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ z(t,0) = 0 & \text{for } t \in (s,+\infty), \\ z(t,L) = 0 & \text{for } t \in (s,+\infty), \\ z_x(t,L) = 0 & \text{for } t \in (s,+\infty). \end{cases}$$

Then,

$$\frac{d}{dt}\|z(t,\cdot)\|_{L^2}^2 \leqslant -2\lambda\|z(t,\cdot)\|_{L^2}^2,$$

hence exponential decay with rate λ.

Rapid stabilization: backstepping

$$\text{Bad} \Leftarrow y \rightsquigarrow z \Rightarrow \text{Good?}$$

Rapid stabilization: backstepping

$$\text{Bad} \Leftarrow y \rightsquigarrow z \Rightarrow \text{Good?}$$

Idea: Find a feedback law and a bounded linear invertible transformation

$$\Pi_\lambda : L^2_y \to L^2_z,$$

such that the flow of y (the solution of KdV with feedback law) is mapped into a flow of z $(z = \Pi_\lambda y)$.
Therefore

$$y(t,\cdot) = \Pi_\lambda^{-1} S_\lambda(t) \Pi_\lambda z(0,\cdot),$$
$$\|y(t,\cdot)\|_{L^2} \leqslant e^{-\lambda t} \|\Pi_\lambda^{-1}\|_{L^2\to L^2} \|\Pi_\lambda\|_{L^2\to L^2} \|y(0,\cdot)\|_{L^2}.$$

Kernel function: Coron-Cerpa, 2013

Volterra transformation

$$z(x) = \Pi_\lambda(y) := y(x) - \int_x^L k(x,r)y(r)dr,$$

with feedback law

$$a(t) = \int_0^L k(0,x)y(t,x)dx.$$

Hence the kernel k should satisfy

$$\begin{cases} k_{xxx} + k_{yyy} + k_x + k_y + \lambda k = 0 & \text{in } \mathcal{T}, \\ k(x,L) = 0 & \text{on } [0,L], \\ k(x,x) = 0 & \text{on } [0,L], \\ k_x(x,x) = \frac{\lambda}{3}(L-x) & \text{on } [0,L]. \end{cases} \tag{1}$$

From rapid stabilization to null controllability

Since $\lambda_n > 0$ and

$$\|y(t,\cdot)\|_{L^2} \leqslant e^{-\lambda_n t} \|\Pi_{\lambda_n}^{-1}\|_{L^2 \to L^2} \|\Pi_{\lambda_n}\|_{L^2 \to L^2} \|y(0,\cdot)\|_{L^2},$$

there exists t_n such that

$$\|y(t_n,\cdot)\|_{L^2} \leqslant \frac{1}{2} \|y(0,\cdot)\|_{L^2}.$$

From rapid stabilization to null controllability

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If there exist $\{\lambda_n\}_n$ such that

$$\sum_n t_n < T < +\infty$$

then null controllability in time T.

From rapid stabilization to null controllability

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there exists t_n such that

$$\|y(t_n,\cdot)\|_{L^2} \leqslant \frac{1}{2} \|y(0,\cdot)\|_{L^2}.$$

If there exist $\{\lambda_n\}_n$ such that

$$\sum_n t_n < T < +\infty \Longleftarrow \|\Pi_\lambda\|, \|\Pi_\lambda^{-1}\| \text{ are “well” controlled},$$

then null controllability in time T.

“Good enough" estimates

Lemma (Xiang, 2017)

Let $\lambda > 2$. Equation (1) *has a unique solution $k_\lambda \in C^3(\mathcal{T})$ (respectively, a unique solution $l_\lambda \in C^3(\mathcal{T})$) which satisfies*

$$\|k_\lambda\|_{C^3(\mathcal{T})} \leqslant e^{(1+L)^2\sqrt{\lambda}} \text{ and } \|l_\lambda\|_{C^3(\mathcal{T})} \leqslant e^{(1+L)^2\sqrt{\lambda}}.$$

“*Proof*”:

(1) Construct a solution.

(2) Give estimates on this solution.

(3) Prove the uniqueness.

On the uniqueness of k

The uniqueness of the solution of (1) is necessary, since our estimates on k and l are based on constructed solutions.
It suffices to prove that the only solution $h \in H^3([0, L] \times [0, L])$ of

$$\begin{cases} h_{xxx} + h_{yyy} + h_x + h_y = 0 & \text{in } [0, L] \times [0, L], \\ h(x, 0) = 0 & \text{on } [0, L], \\ h(x, L) = h_y(x, L) = h_{yy}(x, L) = 0 & \text{on } [0, L], \\ h(0, y) = h_x(0, y) = h_{xx}(0, y) = 0 & \text{on } [0, L]. \end{cases}$$

is 0.

Uniqueness problem: some advice from Coron

It only remains to prove an uniqueness result, however, for a strange equation (we have never seen it before)...

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- ✎ Unique continuation...

Ask Coron for advice:

- You are excellent, you can solve it!
- I will be surprised if it is not true...
- Have you tried eigenfunctions?

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is 0.

Uniqueness: Wavelet, Riesz basis

As the equation looks like a wave equation, we consider

$$\mathcal{A}_y : \mathcal{D}(\mathcal{A}_y) \subset L^2(0,L) \to L^2(0,L),$$
$$\mathcal{D}(\mathcal{A}_y) := \{f \in H^3(0,L); f(0) = f(L) = f_y(L) = 0\},$$
$$\mathcal{A}_y f := -f_y - f_{yyy}, \forall f \in \mathcal{D}(\mathcal{A}_y).$$

If the eigenfunctions $\{\varphi_n(y)\}_n$ form a Riesz basis of $L^2(0,L)$, then the fourier series decomposition

$$h(x,y) = \sum_n \phi_n(x) \cdot \varphi_n(y)$$

easily yield the required uniqueness.

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Theorem (Papanicolaou, 2011)

Eigenfunctions $\{\varphi_n(y)\}_n$ do not form a Riesz basis.

Uniqueness: Wavelet, Completeness

Another idea is to investigate the completeness of eigenfunctions, $\{\psi(y)_n\}_n$, of the adjoint operator $\mathcal{A}_y^*$. One can write the equation as

$$(\partial_{xxx} + \partial_x - \lambda_n)\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L_y^2} = 0.$$

Since

$$\langle \psi_n(\cdot), h(0,\cdot)\rangle_{L_y^2} = \partial_x \langle \psi_n(\cdot), h(0,\cdot)\rangle_{L_y^2} = \partial_{xx} \langle \psi_n(\cdot), h(0,\cdot)\rangle_{L_y^2} = 0,$$

we obtain

$$\langle \psi_n(\cdot), h(x,\cdot)\rangle_{L_y^2} = 0, \quad \forall x \in [0, L].$$

If $\{\psi_n(y)\}_n$ is complete in L_y^2, then $h(x,\cdot)$ is 0. However, we don't know the completeness of the eigenfunctions $\{\psi_n(y)\}_n$.

Eigenfunctions

Eigenfunctions

$$\Downarrow$$

Generalized Eigenfunctions

Uniqueness: Generalized Eigenfunctions

One of the most natural generalized eigenfunction space

$$\bigcup \mathcal{N}((\lambda_i I - \mathcal{L})^{m_i}), \text{ union for all } m_i \in \mathbb{N}, \text{ and } \lambda_i \text{ eigenvalues},$$

where $\mathcal{L}$ denotes the operator, $\mathcal{N}$ denotes the kernel.

Theorem (Locker, 2008)

Let $L > 0$, let a be a constant. For differential operator $\mathcal{L}f := f_{xxx} + af_x$ with boundary conditions

$$f(0) = f(L) = 0,$$
$$f_x(0) + \beta f_x(L) = 0,$$

the generalized eigenfunction space $\mathcal{E}_{\mathcal{G}}$ is complete in $L^2(0, L)$ space iff $\beta \neq 0$.

Uniqueness: e.a.f. (Naimark, 1967-1968)

Eigenfunctions and associated functions ($e.a.f.$): more general than $\mathcal{E}_{\mathcal{G}}$ but less than augmented eigenfunctions.

Theorem (Shkalikov, 1976)

The eigenfunctions and associated functions of the boundary-value problem generated by an ordinary differential equation with linearly independent separated boundary conditions

$$l(y) - \lambda^n y = y^{(n)} + p_{n-2}(x) y^{(n-2)} + \ldots + p_0(x) y - \lambda^n y = 0,$$

$$U_j(y) = \sum_{k=0}^{n-1} \alpha_{jk} y^{(k)}(0) = 0, \text{ with } j = 1, 2, \ldots, l,$$

$$U_j(y) = \sum_{k=0}^{n-1} \beta_{jk} y^{(k)}(L) = 0, \text{ with } j = 1, 2, \ldots, n-l,$$

form a complete system in L^2.

Lemma (Xiang, 2017)

There is a unique solution for the kernel equation.

Advantages of backstepping

(1) Quasi-linear system: loss of derivatives.

(2) Nonlinear terms: perturbation.

(3) Small-time stabilization.

$$y_t + y_x + yy_x = 0, x \in [0, L], t \in [0, T],$$
$$y(t, 0) = u(t), t \in [0, T].$$

Local controllability of the linearized system (Characteristic line)

$$y_t + y_x = 0.$$

Hyperbolic systems

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Local controllability of the linearized system (Characteristic line)

$$y_t + y_x = 0.$$

But we lose one derivative for nonlinear system

$$y_t + y_x = -y y_x.$$

No simple fixed-point argument!

Controllability: iterative scheme.

Hyperbolic systems

Theorem (Bastin-Coron-Krstic-Vazquez, 2013)

Local rapid stabilization of 2×2 *quasilinear hyperbolic system using backstepping.*

$Idea$: backstepping for linearized system; the same feedback for the nonlinear system with some well-chosen Lyapunov function.

$$y_t + y_x + y y_x = 0,$$
$$V(y) := \|y\|_{L^2} + \|y_t\|_{L^2} + \|y_{tt}\|_{L^2} \simeq \|y\|_{H^2}.$$

Thanks to $y_x \simeq y_t$, we have

$$\frac{d}{dt}\|y\|_{L^2}^2 \lesssim \|y\|_{L^2}^2 \|y_x\|_{L^\infty} + R \lesssim \|y\|_{L^2}^2 \|y_t\|_{L^\infty} + R \lesssim \|y\|_{L^2}^2 \|y_{tt}\|_{L^2} + R,$$
$$\frac{d}{dt}\|y_t\|_{L^2}^2 \lesssim \|y_t\|_{L^2}^2 \|y_x\|_{L^\infty} + R \lesssim \|y_t\|_{L^2}^2 \|y_{tt}\|_{L^2} + R$$
$$\frac{d}{dt}\|y_{tt}\|_{L^2}^2 \lesssim \|y_{tt}\|_{L^2}^2 \|y_{tt}\|_{L^2} + R.$$

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Lebeau-Robbiano strategy

Theorem (Russell, 1973)

If wave equation is exactly controllable, then the heat equation is null controllable.

Geometric condition for wave equation: not small-time...

Theorem (Lebeau-Robbiano, 1995)

The heat equation is null controllable in small time.

$Idea$: Fourier decomposition ($\{\lambda_k\}$); drive low frequencies to zero ($\sqrt{\lambda_k} \leqslant \mu$): transform the heat equation into an elliptic equation and get a cost estimation $\exp(C\mu)$; natural dissipation.

Backstepping: works easier for nonlinear terms.

$$-\lambda Id \ \text{ vs } \ \mathcal{F}_{\{|\xi| \leq \lambda\}}$$

Advantages of backstepping

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A part of the small-time stabilization problem

Open Problem

Small-time (global) stabilization for the system which is small-time (global) controllable.

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☺ *Natural*

Small-time stabilization $\xrightarrow{\text{trivial}}$ Small-time controllable

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Small-time (global) stabilization for the system which is small-time (global) controllable.

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Small-time stabilization $\xrightarrow{\text{trivial}}$ Small-time controllable

☹ *Difficult*

Small-time stabilization $\xleftarrow{(???)}$ Small-time controllable

Null controllability to small-time local stabilization

Thanks to the estimates on k and l, we get the small-time stabilization of the KdV system:

Theorem (Xiang, 2018)

The KdV system is (locally) small-time stabilizable.

“$Proof$”:

(1) Perturbation: Kato-smoothing effect. (Local)

(2) Boundary regularity: add an integrator.

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Global controllability (Chapouly, 09): the return method, thanks to nonlinear terms.

Global stabilization?

Problem 3: small-time global stabilization

(1) Stabilization of nonlinear systems whose linearized systems are not asymptotically stabilizable.

(2) Small-time local stabilization of linear controllable systems by nonlinear feedbacks.

(3) Small-time global stabilization of nonlinear controllable systems.

- *Small-time global stabilization of the viscous Burgers equation.*

The first global result on PDE models

The controlled viscous Burgers system

$$\begin{cases} y_t - y_{xx} + yy_x = \alpha(t) & \text{for } (t,x) \in (s,+\infty) \times (0,1), \\ y(t,0) = \beta(t) & \text{for } t \in (s,+\infty), \\ y(t,1) = \gamma(t) & \text{for } t \in (s,+\infty), \\ a_t(t) = \alpha(t) & \text{for } t \in (s,+\infty), \end{cases}$$

where the state is $(y(t,\cdot), a(t)) \in L^2(0,1) \times \mathbb{R}$ and the control is $(\alpha(t), \beta(t), \gamma(t)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.
An analogue to the Navier-Stokes equation:

$$\begin{aligned} -y_{xx} &\Longrightarrow -\Delta y \\ yy_x &\Longrightarrow (y.\nabla)y \\ \alpha(t) &\Longrightarrow \nabla p \end{aligned}$$

Theorem (Coron-Xiang, 2018)

*The viscous Burgers equation is **small-time global stabilizable**. More precisely,* $\forall T>0, \exists$ *a* $T-$*periodic feedback law* (α,β,γ) *such that*

(i) $\Phi(t+T,t;y_0,a_0)=0, \forall t\in\mathbb{R}, \forall (y_0,a_0)\in L^2\times\mathbb{R}, \forall t\in\mathbb{R}.$

(ii) *(Uniform stability property.) For every* $\delta>0$*, there exists* $\eta>0$ *such that*

$$\left(\|(y_0,a_0)\|_V \leqslant \eta\right) \Rightarrow \left(\|\Phi(t,t';y_0,a_0)\|_V \leqslant \delta, \forall t\in(t',+\infty)\right),$$

where Φ *denotes the flow of the closed-loop system.*

A strategy for the global stabilization

★ Global approximate stabilization of "$y - a$"

$$(y_0, a_0) \rightsquigarrow (y_1, a_1) \text{ s.t. } |y_1 - a_1| < \varepsilon.$$

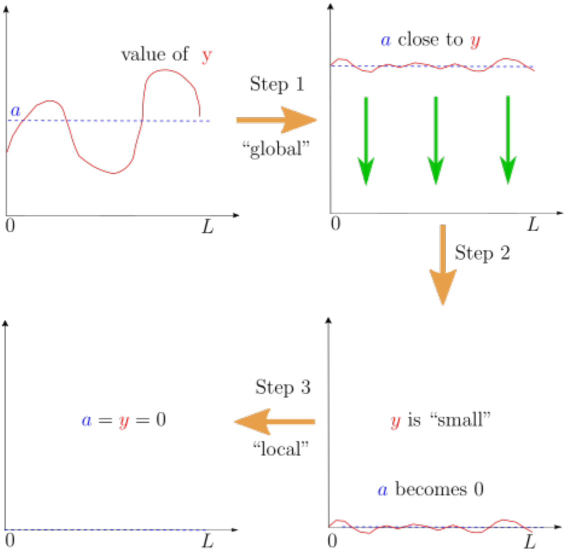
★ Global stabilization of "a"

$$(y_1, a_1) \rightsquigarrow (y_2, a_2) \text{ s.t. } |y_2 - a_2| < 2\varepsilon \text{ and } a_2 = 0.$$

★ Local stabilization of "y"

$$(y_2, 0) \rightsquigarrow (0, 0).$$

Our strategy



$$y_t - y_{xx} + yy_x$$

Idea 1: use nonlinear term and nonlinear control.

$$\dot{x} = -\lambda x \qquad \text{vs} \qquad \dot{x} = -x^3.$$

If $x > 0$

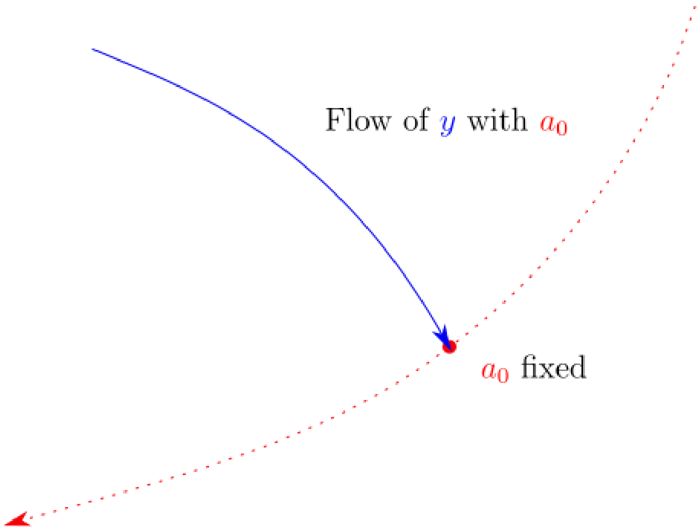
$$\frac{1}{2}\mathrm{d}x^{-2} = -x^{-3}\mathrm{d}x = \mathrm{d}t,$$

then

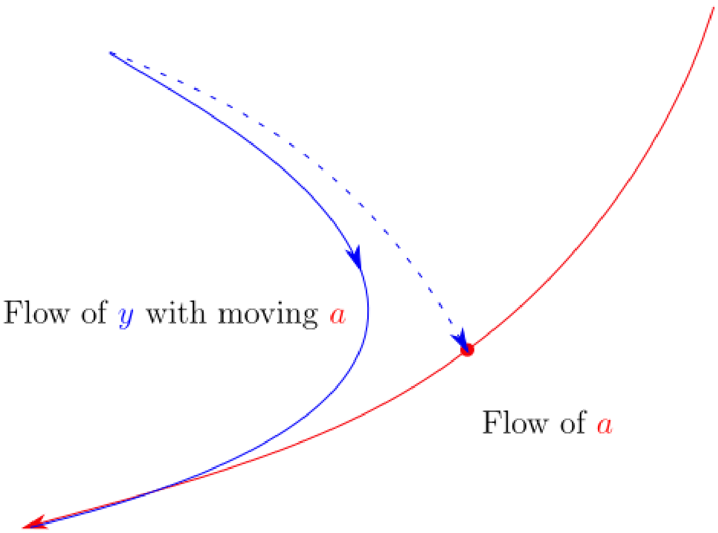
$$x(T) = (2T + x^{-2}(0))^{-1/2} \leqslant (2T)^{-1/2}.$$

Idea 2: use the dissipation of $-y_{xx}$.

Phantom stabilization



Global stabilization: Phantom Tracking



Small-time global stabilization

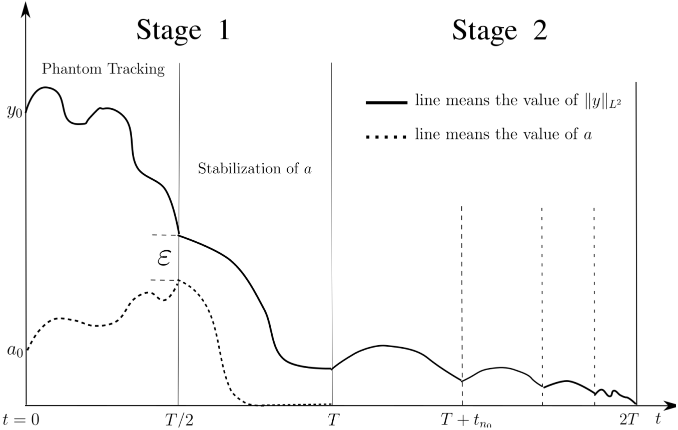


Figure: Evolution of the “flow"

Burgers is just a beginning

For KdV equations

- (Xiang, 2018) Small-time local controllability by backstepping approach. Small-time local stabilization.
- (Koudohode, 2019) Small-time global stabilization.

Other models:

- (Coron-Nguyên, 2016) Small-time semi-global stabilization of the one dimensional parabolic equation.
- (Coron-Hu-Olive, 2018) Finite time boundary stabilization of linear hyperbolic balance laws.
- (Christophe Zhang, 2018) Finite time (optimal) stabilization of the transport equation on torus with one (internal) scalar control.

ANR Finite4SoS, *Équipe CaGE*

Viscous limit and boundary layer

Lions' Problem for the Navier-Stokes equation:

- (Coron, Glass, 1992–2000) Small-time global controllability (STGC) of Euler with boundary control.
- (Coron-Fursikov, 1996) STGC of NS on manifold without boundary with internal control.
- (Fernández–Cara-Guerrero-Imanuvilov-Puel, 2004) STLC of NS with Dirichlet boundary control.
- (Coron-Marbach-Sueur, 2018) STGC of NS with Navier slip-with-friction boundary condition control.
- (Coron-Marbach-Sueur-Zhang, 2019) STGC of NS on a rectangular with boundary control and a little help of internal control.

- (Xiang, in preparation) Small-time global approximate stabilization of NS on $\mathbb{T}^2$ with internal control.
- To be continued...

About small-time local stabilization:

- ✍ Trace problem from the classical Lions-Magenes method.
- ✍ Backstepping in higher dimension.
- ✍ Stationary feedbacks.
- ✍ Relations between backstepping, Lebeau-Robbiano, and Carleman estimates...

About global approximate stabilization:

- ✍ Generalization of the phantom tracking method.
- ✍ Boundary layer difficulty when we use less boundary controls...

Thank you for your attention!