The Finite-codimensional Controllability Problem

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Joint work with Xu Liu and Xu Zhang

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- Consider the following nonlinear abstract evolution equation on a Hilbert space H:

$$\begin{cases} \dot{y}(t) = Ay(t) + f(t, y(t), u(t)) \qquad t \in (0, T], \\ \\ y(0) = y_0. \end{cases} \tag{1}$$

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- Here $u \in \mathcal{U}$ is the control variable, y is the state variable, $y_0 \in H$ and $A : \mathcal{D}(A) \subset H \to H$ is a linear operator generating a C_0-semigroup $\{S(t)\}_{t \geq 0}$.

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- Assume that $f : [0, T] \times H \times U \to H$ satisfies suitable conditions, so that for any $y_0 \in H$ and $u(\cdot) \in \mathcal{U}$, (1) admits a unique mild solution.

- Let $\tilde{U}$ be a nonempty subset of the Hilbert space U and S be a subset of $H \times H$.

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- Set

$$\mathcal{U}[0,T] = \left\{ u \in \mathcal{U} \;\middle|\; u : (0,T) \to \tilde{U} \text{ is measurable} \right\}$$

and

$$\mathcal{U}_{ad} = \Big\{ u \in \mathcal{U}[0,T] \;\Big|\; \text{the corresponding solution } y \text{ of (1) satisfies that } y(0) = y^0,\ y(T) = y^1,\ \text{for some } (y^0, y^1) \in S \Big\}.$$

- Set

$$\mathcal{J}(u(\cdot)) = \int_0^T f^0(t, y(t), u(t))dt,$$

where y is the solution of (1) associated to $y_0 \in H$ and $u \in \mathcal{U}$, and $f^0 : [0, T] \times H \times U \to \mathbb{R}$ satisfies suitable conditions.

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- Consider the following optimal control problem for the nonlinear system (1):

(P) Find a $\overline{u}(\cdot) \in \mathcal{U}_{ad}$, so that $\mathcal{J}(\overline{u}(\cdot)) = \inf_{u \in \mathcal{U}_{ad}} \mathcal{J}(u(\cdot))$.

- To solve Problem (P), people first study necessary conditions fulfilled by $(\bar{x}, \bar{u})$.

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- When $H = \mathbb{R}^n$, a very elegant necessary condition for optimal controls was established by Pontryagin and his group. Such kind of conditions are call Pontryagin type maximum principle (PMP for short).

- Consider the following control system:

$$\begin{cases} \dfrac{dx(t)}{dt} = f(t, x(t), u(t)), \quad \text{a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{2}$$

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- Here f is a suitable smooth function from $[0, T] \times \mathbb{R}^n \times U$ with U a separable metric space and

$$u(\cdot) \in \mathcal{V}[0, T] = \{u(\cdot) : [0, T] \to U \mid u(\cdot) \text{ measurable }, x(T) = x_1\}.$$

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- **(P)** Find a $\overline{u}(\cdot) \in \mathcal{U}_{ad}$, such that $\mathcal{J}(\overline{u}(\cdot)) = \inf_{u \in \mathcal{U}_{ad}} \mathcal{J}(u(\cdot))$.

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- **Theorem 1.** Assume that f and f^0 are smooth enough. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal pair of Problem **(P)**. There exists $\psi^0 \leq 0$ and $\psi : [0, T] \to \mathbb{R}^n$ such that

$$|\psi^0|^2 + |\psi(t)|^2 > 0, \quad \forall t \in [0, T],$$

$$\frac{d\psi(t)}{dt} = -f^0_x(t, \bar{x}(t), \bar{u}(t))\psi - f_x(t, \bar{x}(t), \bar{u}(t)),$$

and

$$H(t, \bar{x}(t), \bar{u}(t), \psi(t), \psi^0) = \max_{u \in U} H(t, \bar{x}(t), u, \psi(t), \psi^0), \text{ a.e. } t \in [0, T],$$

where

$$H(t, x, u, \psi, \psi^0) \triangleq \langle \psi, f(t, x, u) \rangle_{\mathbb{R}^n} + \psi^0 f^0(t, x, u),$$
$$(t, x, u, \psi, \psi^0) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}.$$

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- A.I. Egorov adopted the method of Pontryagin to derive the PMP for nonlinear evolution equations formally.

- **"Theorem":** Let $(\bar{y}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_{ad}$ be an optimal pair. Then there exists a pair $(\psi^0, \psi(\cdot)) \in \mathbb{R} \times C([0, T]; H)$ such that

$$\psi^0 \le 0, \qquad |\psi^0|^2 + |\psi(t)|_H^2 > 0, \quad \forall\, t \in [0, T],$$

$$\begin{aligned} \frac{d\psi(t)}{dt} = & -A^*\psi(t) - f_y(t, \bar{y}(t), \bar{u}(t))^*\psi(t) \\ & -\psi^0 f_y^0(t, \bar{y}(t), \bar{u}(t)), \text{ a.e. } t \in [0, T], \end{aligned}$$

$$\langle \psi(0), x_0 - \bar{y}(0)\rangle_H - \langle \psi(T), x_1 - \bar{y}(T)\rangle_H \le 0, \ \forall (x_0, x_1) \in S,$$

$$H(t, \bar{y}(t), \bar{u}(t), \psi^0, \psi(t)) = \max_{u \in U} H(t, \bar{y}(t), u, \psi^0, \psi(t)), \quad \text{a.e. } t \in [0, T],$$

where

$$H(t, y, u, \psi^0, \psi) = \psi^0 f^0(t, y, u) + \langle \psi, f(t, y, u)\rangle_H, \quad \forall (t, y, u, \psi^0, \psi) \in [0, T] \times X \times U \times \mathbb{R} \times H.$$

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- Let $H = L^2(0,1)$ and A be defined as follows:

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- Let

$$\begin{cases} b \in L^2(0,1), \quad b_k = \sqrt{2}\int_0^1 b(x)\sin(k\pi x)dx \neq 0, \quad k \geq 1; \\ a \in L^2(0,1), \quad c \in \mathbb{R}, \quad c \neq \langle b, A^{-1}a\rangle; \\ q = (e^A - I)A^{-1}b; \quad U = [-2,2]. \end{cases} \tag{4}$$

- Consider the following system:

$$\dot{y}(t) = Ay(t) + bu(t), \qquad t \in [0,1], \tag{5}$$

with $u(\cdot) \in \mathcal{U} \equiv \{u(\cdot) : [0,1] \to U \mid u(\cdot) \text{ measurable } \}$.

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- Our constraint for the terminal values of the state is

$$y(0) = 0, \qquad y(1) = q, \tag{6}$$

i.e., $S = \{0\} \times \{q\}$ and the cost functional is given by

$$\mathcal{J}(u(\cdot)) = \int_0^1 \Big[\langle a, y(t)\rangle_{L^2(0,1)} + cu(t)\Big] dt \tag{7}$$

- It is easy to see that $\bar{u}(t) \equiv 1$ is the optimal control.

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- Suppose the pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ satisfies the PMP. Then there exists a pair $(\psi^0, \psi(\cdot)) \neq 0$, with $\psi^0 \leq 0$ and

$$\psi(t) = e^{A(1-t)}\psi(1) + \psi^0(e^{A(1-t)} - I)A^{-1}a,$$

such that for a.e. $t \in [0, 1]$,

$$\left[\psi^0 c + \langle \psi(t), b \rangle_{L^2(0,1)}\right] = \max_{|u| \leq 2} \left[\psi^0 c + \langle \psi(t), b \rangle_{L^2(0,1)}\right] u. \quad (8)$$

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- (8) implies that $(\psi^0, \psi(\cdot)) = 0$.

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- In 1991, X. Li and J. Yong gave a general condition for obtaining the PMP.

- Let $(\bar{y}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_{ad}$ be optimal.

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- Consider the following equations:

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + f_y(t, \bar{y}, \bar{u})\xi + [f(t, \bar{y}, u) - f(t, \bar{y}, \bar{u})], \\ \xi(0) = 0. \end{cases}$$

$$\dot{\eta}(t) = A\eta(t) + f_y(t, \bar{y}(t), \bar{u}(t))\eta(t), t \in [0, T], \quad \eta(0) = y_0.$$

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- Let

$$\mathcal{R} = \Big\{ \xi(T) \in H \;\Big|\; u(\cdot) \in \mathcal{U}[0, T] \Big\},$$

and

$$\mathcal{Q} = \Big\{ y_1 - \eta(T) \;\Big|\; (y_0, y_1) \in S \Big\}.$$

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- Recall that a subset S of H is said to be *finite codimensional* in H, denoted by $\mathrm{codim}\,_Z S < \infty$, if there exists a point $z_0 \in \mathrm{co}S$, such that the closed subspace spanned by $\{z - z_0 \mid z \in S\}$ is a finite codimensional subspace of Z and $\mathrm{co}(S - z_0)$ has a nonempty interior in this subspace.

- **Theorem**(X. Li & J. Yong, 1991, SICON) Let (H1) hold and $(\bar{y}(\cdot), \bar{u}(\cdot)) \in \mathcal{A}_{ad}$ be an optimal pair. Then there exists a pair $(\psi^0, \psi(\cdot)) \in \mathbb{R} \times C([0,T]; H)$ such that

$$\psi^0 \le 0, \qquad |\psi^0|^2 + |\psi(t)|_H^2 > 0, \quad \forall\, t \in [0,T],$$

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$$H(t, \bar{y}(t), \bar{u}(t), \psi^0, \psi(t)) = \max_{u \in U} H(t, \bar{y}(t), u, \psi^0, \psi(t)),$$

$$\text{a.e. } t \in [0,T],$$

where

$$\begin{aligned}H(t, y, u, \psi^0, \psi) &= \psi^0 f^0(t, y, u) + \langle \psi, f(t, y, u)\rangle_H,\\ &\qquad \forall (t, y, u, \psi^0, \psi) \in [0,T] \times X \times U \times \mathbb{R} \times H.\end{aligned}$$

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- To see whether (H1) holds or not, one has to study the reachable set of a linear system with control constraint, which is a hard problem.

- Consider the following controlled wave equation:

$$\begin{cases} y_{tt} - \Delta y + f(x,t,y) = \chi_\omega u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \ y_t(0) = y_1 & \text{in } \Omega, \end{cases} \tag{9}$$

where u is the control variable and (y, y_t) is the state variable, $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ is any given initial value, and f is C^1 with respect to (t, x, y).

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- The admissible control set is

$$\mathcal{U} \triangleq \Big\{ u \in L^2(0,T; L^2(\Omega)) \Big| \, 0 \le u(t,x) \le 1 \quad \text{for a.e. } (t,x) \in (0,T) \times \Omega \Big\}.$$

- Let

$$\mathcal{J}(u(\cdot)) = \int_0^T \int_\Omega f^0(x,t,y(x,t),u(x,t))dxdt,$$

where $f^0 : Q \times \mathbb{R}^2 \to \mathbb{R}$ satisfies that

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where $f^0 : Q \times \mathbb{R}^2 \to \mathbb{R}$ satisfies that

- 1. f^0 is strongly measurable with respect to (x, t) in Q, and continuously differentiable with respect to (y, u) in $\mathbb{R}^2$ with $f^0(x, t, \cdot, \cdot), f^0_y(x, t, \cdot, \cdot)$ and $f^0_u(x, t, \cdot, \cdot)$ being continous, respectively.

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- 2. $f^0(\cdot,\cdot,0,0) \in L^1(Q)$, and there exists a constant $L > 0$, so that for any $(x,t,y,u) \in Q \times \mathbb{R}^2$,

$$|f^0_y(x,t,y,u)| + |f^0_u(x,t,y,u)| \le L.$$

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- How to check whether (H1) holds or not?

- Consider the following system:

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- Let

$$\mathcal{R}(T; y_0) = \left\{ y(T; y_0, u) \in H \,\middle|\, u(\cdot) \in L^2(0, T; \widetilde{U}) \right\}.$$

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- For any $T > 0$, (10) is called to be finite-codimensional exactly controllable at time T, if $\mathcal{R}(T; 0)$ is a finite-codimensional subspace of H.

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- Consider the following linear system:

$$\begin{cases} -\dot{\phi}(t) = A^*\phi(t) + F(t)^*\phi(t) & t \in (0, T], \\ \phi(T) = \phi_T, \end{cases} \tag{11}$$

where $\phi_T \in H$ and $F(t)^*$ denotes the adjoint operator of $F(t)$ for any $t \in [0, T]$.

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  where $\phi_T \in H$ and $F(t)^*$ denotes the adjoint operator of $F(t)$ for any $t \in [0, T]$.
- The following assertions are equivalent:
- **(1)** The system (10) is finite-codimensional exactly controllable in H.
- **(2)** There is a finite-codimensional subspace $\widetilde{H} \subseteq H$, so that any solution ϕ of (11) satisfies

$$|\phi_T|_H \leq C|B(\cdot)^*\phi|_{L^2(0,T;U)}, \qquad \forall\ \phi_T \in \widetilde{H}. \tag{12}$$

- **Theorem** Assume that for any $\phi_T \in H$, the solution ϕ of (11) satisfies that

$$|\phi_T|_H \le C\Big[|(B(\cdot))^*\phi|_{L^2(0,T;\widetilde{U})} + |G\phi_T|_{H^1}\Big], \qquad (13)$$

where G is a compact operator from H to a Banach space H^1. Then there is a finite-codimensional subspace $\widetilde{H} \subseteq H$, so that any solution ϕ of (11) satisfies

$$|\phi_T|_H \le C|B(\cdot)^*\phi|_{L^2(0,T;\widetilde{U})}, \qquad \forall\ \phi_T \in \widetilde{H}. \qquad (14)$$

- Consider the following equation:

$$\begin{cases} z_{tt} - \Delta z = f_y(t, x, \bar{y}(t))z & \text{in } (0, T) \times \Omega, \\ z = 0 & \text{in } (0, T) \times \partial\Omega, \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega. \end{cases} \tag{15}$$

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- As a byproduct of the result of Bardos-Lebeau-Rauch, inequality (16) holds if and only if (Ω, ω, T) fulfills the Geometric Control Condition.

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w = 0 & \text{on } (0, T) \times \partial\Omega, \\
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- The system (17) is not exactly controllable.

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$$\Upsilon(t,x) = w(t,x) + \int_0^t b(t-s)w(s,x)ds. \qquad (18)$$

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- From (18) and (19), we get that that

$$w_{tt} - \Delta w - \int_0^t b(t-s)\Delta w(s)ds$$

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- One can prove that the system for Υ is finite-codimensional exactly controllable. This, together with (19), implies that (17) is finite-codimensional exactly controllable.

Thank you!