Rough controls for the Schrödinger equation on the torus

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Groupe de travail contrôle 9 mars 2018

[1]Work supported in parts by ANR projects
ANAÉ ANR-13-BS01-0010-03 and ISDEEC ANR-16-CE40-0013

Plan of the talk

1. Control and stabilisation of waves
   - Control, Stabilisation
   - Geometric control
   - Results on the torus
   - A general geometric condition
2. The Schrödinger equation on the torus
   - Control, Stabilisation
   - A dispersion estimate
   - The one dimensional case
   - The two dimensional case

Control of waves

Consider the wave equation on a Riemanian manifold M_g, $a \in L^\infty(M), a \geq 0, \ T > 0$

$$(\partial_t^2 - \Delta)u = f \times 1_{(0,T)} \times a(x), \quad (u \mid_{t=0}, \partial_t u \mid_{t=0}) = (u_0, u_1)$$

Given $(u_0, u_1) \in \mathcal{H}^1 = H^1(M) \times L^2(M)$ initial data and $(v_0, v_1) \in \mathcal{H}^1$ target data in energy space, can we choose f in suitable space such that

$$(u \mid_{t=T}, \partial_t u \mid_{t=T}) = (v_0, v_1)?$$

Natural space for f is $L^2((0, T) \times M)$. If answer yes: exact controlability

Stabilisation for waves

$$(\partial_t^2 - \Delta + a(x)\partial_t)u = 0,$$
$$(u\mid_{t=0}, \partial_t u\mid_{t=0}) = (u_0, u_1) \in H^1 \times L^2 = \mathcal{H}^1$$

The natural energy is decaying ($a \geq 0$)

$$E(u)(t) = \int_M |\nabla_x u|^2 + |\partial_t u|^2 dx, \frac{d}{dt}E(t) = \int_M -a(x)|\partial_t u|^2 dx$$

Question: speed of decay of $E(u)(t)$?

- The energy of all solutions tend to 0 iff there exists no non trivial stationary equilibrium, i.e. $-\Delta e = \lambda^2 e, a \times e = 0 \Rightarrow e = 0$.
- Semi-group property: If there exists a uniform rate $f(t)$,

$$\forall (u_0, u_1) \in \mathcal{H}^1, E(u)(t) \leq f(t)E(u)(0), \lim_{t\to+\infty} f(t) = 0,$$

then can choose $f(t) = Ce^{-ct}$ (uniform) stabilisation.

Observation and HUM duality imply equivalence

- There exists a rate $f(t)$ such that $\lim_{t\to+\infty} f(t) = 0$ and
$$\forall (u_0, u_1) \in H^1(M) \times L^2(M), E(u)(t) \le f(t)E(u)(0).$$
(and then can choose $f(t) = Ce^{-ct}$)
- $\exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the damped wave equation, then
$$E(u)(0) \le C \int_0^T \int_M 2a(x)|\partial_t u|^2 dxdt.$$
- $\exists T > 0, c > 0; \forall (u_0, u_1) \in H^1(M) \times L^2(M)$, if u is the solution to the undamped wave equation then
$$E(u)(0) \le C \int_0^T \int_M 2a(x)|\partial_t u|^2 dxdt.$$
- There exists $T > 0$ such that The wave equation is exactly controlable in time T (and we can take the time given by observation)

The geometric control assumption for waves

($a \in C^0(M)$, T) controls geometrically (M, g) if every geodesic starting from any point $x_0 \in M$ in any direction ξ_0, $\gamma_{(x_0,\xi_0)}(s)$, encounters $\{a > 0\}$ in time smaller that T or

$$\inf_{(x_0,\rho_0)\in S^*M} \frac{1}{T} \int_0^T a(x(\Phi_s(x_0, \rho_0)))ds \geq \alpha > 0.$$

Theorem (Rauch-Taylor, Bardos-Lebeau-Rauch 88', N.B- P.G.)

$a \in C^0(M)$ *(GCC)* $\Leftrightarrow$ *control and stabilisation for waves.*
$a \in L^\infty(M)$ *(SGCC)* $\Rightarrow$ *control and stabilisation* $\Rightarrow$ *(WGCC).*

$$\begin{aligned} &\exists T, c > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T), \exists \delta > 0; \\ &a \geq c \text{ a.e. on } B(\gamma_{\rho_0}(s), \delta). \end{aligned} \qquad \text{(SGCC)}$$

$$\exists T > 0; \forall \rho_0 \in S^*M, \exists s \in (0, T); \gamma_{\rho_0}(s) \in \text{support}(a) \qquad \text{(WGCC)}$$

See also Humbert-Privat-Trelat 17-18 for related refined conditions.

Some examples on tori

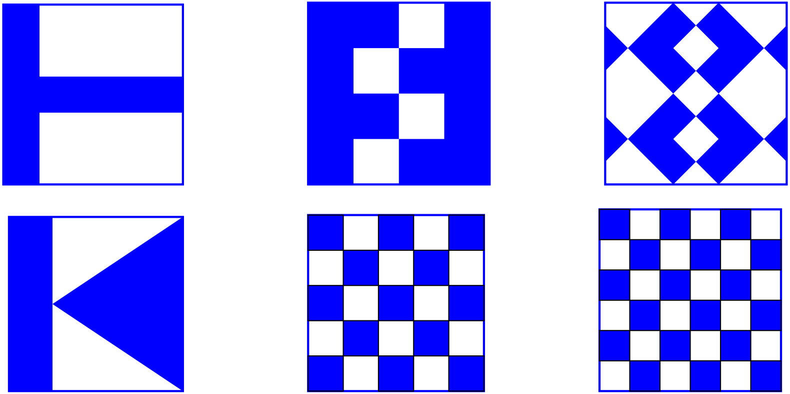
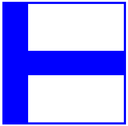


Figure: Checkerboards: the damping a is equal to 1 in the blue region, 0 elsewhere. The geodesics are (periodized) straight lines. The first example satisfies (SGCC) while all others satisfy (WGCC) but not (SGCC)

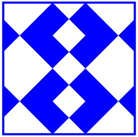
Stabilisation for wave equations: the result

Theorem (Stabilisation? $a = 1$ in blue region 0 otherwise)

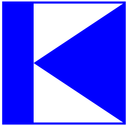


YES 80'
(Taylor-Rauch)

YES
(NB-PGérard 17)



NO
(NB-PGérard 17)



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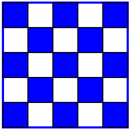
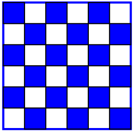


YES
(NB-PGérard 17)



NO
(NB-PGérard 17)

Rk Stabilisation $\Rightarrow \exists T > 0$ control holds in time T. The proof gives no geometric bound on T.

Another geometric condition

When the manifold is a two dimensional torus and the damping a is a linear combination of characteristic functions of polygons, i.e. there exists N polygons, $R_j, j = 1, \dots N$ (disjoint and non necessarily vertical), and $0 < a_j, j = 1, \dots, N$ such that

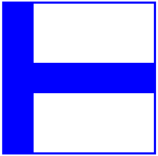
$$a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j}, \tag{1}$$

(piecewise smooth domains, no infinite contact with geodesics= much easier)

Theorem (NB–P. Gérard 15-17)

Stabilisation holds for the waves on $\mathbb{T}^2$ iff there exists $T > 0$ such that all geodesics (straight lines) of length T either encounters the interior of one of the rectangles or follows for some time one of the sides of a polygon R_{j_1} on the left and for some time one of the sides of another (possibly the same) polygon R_{j_2} on the right.

Stabilisation for wave equations: the result



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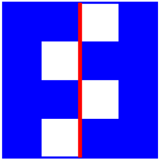

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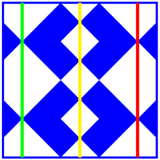

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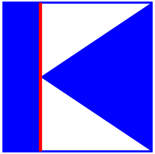

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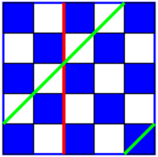
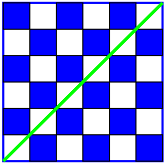


YES



NO

A geometric control condition for *control*

When the manifold is a two dimensional surface and a is a linear combination of characteristic functions of *geodesic polygons*, *i.e. there exists N polygons, $R_j, j = 1, \dots N$ (disjoint and non necessarily vertical), and $0 < a_j, j = 1, \dots, N$ such that* $a(x) = \sum_{j=1}^{N} a_j 1_{x \in R_j}$,

Theorem (NB 18)

Let $T > 0$. Then exact controlability holds for the waves on M^2 if and only if there exists $\alpha > 0$ such that,

$$\inf_{(x_0,\xi_0) \in S^* M} \text{ess} \left(\int_0^T a(x(s, x_0, \xi_0)) ds \geq \alpha > 0. \right.$$

here $(x(s, x_0, \xi_0), \xi(s, x_0, \xi_0)$ is the bicharacteristic starting from (x_0, ξ_0) at $s = 0$). On S^2 (endowed with its standard metric), we can translate this assumption into a checkable geometric condition, and recover (and generalise) Rauch-Lebeau's result on half sphere

End of first part

The previous approach could quite possibly give results for *ruled functions (fonctions réglées= uniform limits of finite sums of caracteristic functions of polygons). However, to go beyond, need other ideas.*
Problems still open

- *Other models (Schrödinger equations...)*
- *Understand general (L^∞) localizations*

Toward a general understanding. A model case

We shall now study the same question for Schrödinger equations on tori. For smooth (continuous) control/observation domains, situation still well understood (particularly on tory (N.B.-Zworski, Macia, Jacobson, Anantharaman-Macia, Bourgain-N.B-Zworski). Here want to use controls on sets of **positive measure** or more generally use control functions in L^4. The understanding is then much poorer and only partial results are available even for the simpler case of wave equations. Using previous works (Bourgain-NB-Zworski, NB-Zworski) we completely settle the question for Schrödinger equation on the two dimensional torus taking advantage of the particular simplicity of the **dynamical structure** and the **dispersive properties** of the equation.

In some sense this dynamics/dispersion mixing is bad. I'd prefer a model with only dispersive properties, but I know of none.

A model problem: The Schrödinger equation on the torus

- Control, Stabilisation
- A dispersion estimate
- Semiclassical reduction
- Study of irrational directions
- The one dimensional case
- Study of rational directions: microlocalization on a particular trajectory.
- Study of rational directions: reduction to the 1-d problem

Schrödinger equations two-tori

$$\begin{gathered}\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}\times\gamma\mathbb{Z}\,, \quad \gamma\in\mathbb{R}\setminus\{0\}\,, \quad a\in L^2(\mathbb{T}^2), \\ (i\partial_t+\Delta)u(t,z) = a(z)1_{(0,T)}f\,, \quad u(0,z)=u_0(z),\end{gathered} \tag{2}$$

a is a localisation function and f a control.

$$\begin{gathered}(i\partial_t+\Delta+ia)u(t,z)=0\,, \quad u(0,z)=u_0(z), \\ \frac{d}{dt}\|u\|^2_{L^2}(t) = -\int_M a(x)|u|^2(t,x)dx\end{gathered} \tag{3}$$

a is a damping function
Main question: **Under which assumptions** on a can we ensure

- Exact controlability in time T: for any $u_0, u_1 \in L^2(M)$ there exists a control f driving the system from u_0 to u_1 at time T
- Stabilisation : $\|u\|_{L^2(t)} \le Ce^{-ct}\|u_0\|_{L^2}$

Previous results

Theorem (Haraux 89, Jaffard 90, Komornik 92)

Let $a \in C^0(\mathbb{T}^d), a \not\equiv 0$ *and* $T > 0$. *Then for any* $u_0 \in L^2(\mathbb{T}^2)$ *there exists* $f \in L^2((0,T) \times \mathbb{T}^2)$ *such that the solution* u *of* (2) *satisfies* $u|_{t=T} = 0$ *(control problem).*
Let $a \geq 0 \in C^0(\mathbb{T}^d), a \not\equiv 0$. *Then there exists* $C, c > 0$ *such that*

$$\|u\|_{L^2(t)} \leq Ce^{-ct}\|u_0\|_{L^2}.$$

The results

Theorem (N.B. Zworski, 18)

Let $a \in L^2(\mathbb{T}^2), \|a\|_{L^2} > 0$ and $T > 0$. Then for any $u_0 \in L^2(\mathbb{T}^2)$ there exists $f \in L^4(\mathbb{T}^2; L^2(0, T))$ such that the solution u of (2) satisfies $u|_{t=T} = 0$. If in addition $a \in L^4(\mathbb{T}^2)$ then the same statement holds with $f \in L^2((0, T) \times \mathbb{T}^2)$. OK for caracteristic functions of any positive measure Lebesgue measurable set

Theorem (N.B. Zworski, 18)

For $a \in L^2(\mathbb{T}^2)$, $a \geq 0$, $\|a\|_{L^2} > 0$, there exist $C, c > 0$ such that for any $u_0 \in L^2(\mathbb{T}^2)$, the equation

$$(i\partial_t + \Delta + ia)u = 0, \qquad u|_{t=0} = u_0, \tag{4}$$

has a unique global solution $u \in L^\infty(\mathbb{R}; L^2(\mathbb{T}^2)) \cap L^4(\mathbb{T}^2; L^2_{\mathrm{loc}}(\mathbb{R}))$

$$\|u\|_{L^2(\mathbb{T}^2)}(t) \leq Ce^{-ct}\|u_0\|_{L^2(\mathbb{T}^2)}. \tag{5}$$

Rk $a \in L^2$ is minimal assumption to keep H^2 as domain of $-\Delta + a$

The geodesic flow on the (rational) 2-torus

Geodesics are straight lines:

$$(X, \Xi) \mapsto (X + s\Xi, \Xi).$$

two kinds of geodesics:

- Rational geodesics

$$\{\Xi = (\xi, \eta), \xi^2 + \eta^2 = 1, \xi/\eta \in \mathbb{Q} \Leftrightarrow (\xi, \eta) = \frac{p, q}{\sqrt{p^2 + q^2}}\},$$

  The geodesic is then **periodic**

- Irrational geodesics

$$\{\Xi = (\xi, \eta), \xi^2 + \eta^2 = 1, \xi/\eta \notin \mathbb{Q}\}$$

  geodesic is **dense**

A key point: Dispersive estimate

Theorem (Bourgain N.B. Zworski, 13)

Let $T > 0$. There exists $C = C_T$ such that for

$$u_0 \in L^2(\mathbb{T}^2), \quad f \in L^{\frac{4}{3}}(\mathbb{T}^2; L^2(0, T)),$$

the solution to $(i\partial_t + \Delta)u = f$, $u|_{t=0} = u_0$, satisfies

$$\|u\|_{L^\infty((0,T);L^2(\mathbb{T}^2)\cap L^4(\mathbb{T}^2;L^2((0,T)))} \leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)} + \|f\|_{L^1((0,T);L^2(\mathbb{T}^2))+L^{\frac{4}{3}}(\mathbb{T}^2;L^2(0,T))}\right). \quad (6)$$

Rk: This property implies that the Schrödinger semi-group generated by $\Delta + ia$ is continuous with respect to $a \in L^2$:

$$\exists C > 0; \forall t \in [0,1] \|e^{it(\Delta+ia)} - e^{it(\Delta+ib)}\|_{(L^2_x)} \leq C\|a - b\|_{L^2}.$$

Relaxing the assumption to $a \in L^p, p < 2$ would change dramatically the situation (change of domain for $\Delta + ia$)

Semi-classical reduction and contradiction argument

Prove observation estimates for spectrally localized initial data

$$\Pi_{h,\rho}(u_0) := \chi\left(\frac{-h^2\Delta - 1}{\rho}\right) u_0\,, \quad \rho > 0\,, \tag{7}$$

$\chi \in \mathcal{C}_c^\infty((-1,1))$ equal to 1 near 0.

Theorem

Suppose that $a \in L^2(\mathbb{T}^2)$, $a \geq 0$, $\|a\|_{L^2} > 0$. For any $T > 0$ there exist K, $\rho_0 > 0$ and $h_0 > 0$ such that for any $u_0 \in L^2(\mathbb{T}^2)$,

$$\|\Pi_{h,\rho} u_0\|_{L^2}^2 \leq K \int_0^T \int_{\mathbb{T}^2} a(z) |e^{it\Delta} \Pi_{h,\rho} u_0|^2 dz dt, \tag{8}$$

for $0 < \rho < \rho_0$ and $0 < h < h_0$.

Semiclassical observability

Argue by contradiction. Get sequences $(u_{0,n})$, $h_n \to 0$, $\rho_n \to 0$

$$1 = \|\Pi_{h_n,\rho_n} u_{0,n}\|^2_{L^2} \geq nK \int_0^T \int_{\mathbb{T}^2} a(z) |e^{it\Delta} \Pi_{h_n,\rho_n} u_{0,n}(z)|^2 dz dt,$$

$u_n(t) := e^{it\Delta} u_n$ bounded in $L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}^2)$. u_n's define a semiclassical defect measure μ on $\mathbb{R}_t \times T^*(\mathbb{T}^2_z)$ $\forall \varphi \in \mathcal{C}^0_{\mathrm{c}}(\mathbb{R}_t)$ and any $A \in \mathcal{C}^\infty_{\mathrm{c}}(T^*\mathbb{T}^2_z)$, we have

$$\langle \mu, \varphi(t) A(z,\zeta) \rangle = \lim_{n\to\infty} \int_{\mathbb{R}_t \times \mathbb{T}^2} \varphi(t) \langle A(z, h_n D_z) u_n(t), u_n(t) \rangle_{L^2(\mathbb{T}^2)} dt\,.$$

The measure μ enjoys the following properties:

$$\mu((t_0,t_1) \times T^*\mathbb{T}^2_z) = t_1 - t_0,$$
$$\operatorname{supp} \mu \subset \Sigma := \{(t,z,\zeta) \in \mathbb{R}_t \times \mathbb{T}^2_z \times \mathbb{R}^2_\zeta \;:\; |\zeta| = 1\},$$
$$\zeta \cdot \nabla_z \mu = 0,$$

Dispersive property of the measure

From the dispersion we get that the integral in (ξ, t) of the measure μ is absolutely continuous with respect to Lebesgue measure (and actually in L^2_x): n for any $\tau \geq 0$ there exists $m_\tau \in L^2(\mathbb{T}^2)$ such that for all $f \in \mathcal{C}(\mathbb{T}^2)$

$$\int_0^\tau \int_{T^*\mathbb{T}^2} f(z) d\mu(t, z, \zeta) = \int_{\mathbb{T}^2} f(z) m_\tau(z) dz. \qquad (9)$$

Smoothing out a

Dispersive properties of solution/measure imply

Lemma

Since the sequence (u_n) is bounded in $L^4_x; L^2_t$ (and $a \in L^2_x \Rightarrow a \in L^2_x L^\infty_t$) Hölder inequality implies

$$a, b \in L^2 \Rightarrow \Big| \int_0^T \int_{\mathbb{T}^2} (a-b)|u_n|^2(t,x)dxdt \Big| \leq C\|a-b\|_{L^2}$$

regularize assing to the limit $n \to +\infty$, get

$$\int_0^T \int_{\mathbb{T}^2} a d\mu = \int_{\mathbb{T}}^2 a(z)m_T(z)dz = 00$$

$$\Big| \int_0^T \int_{\mathbb{T}^2} (a_j - a)d\mu \Big| \leq C\|a_j - a\|_{L^2}$$

We shall later apply this estimate for sequences $a_j \in L^2 \cap C^\infty(\mathbb{T}^2)$ converging to a in L^2

The irrational directions (back to the microlocal proof of Jaffard's 90' result N.B.-Zworski 04)

$$W^m := \Big\{(z,\zeta) \in T^*\mathbb{T}^2; \zeta = \frac{(p,q)}{\sqrt{p^2+q^2}}, \max(|p|,|q|) \leq m, \quad (p,q) \in \mathbb{Z}^2, \ \gcd(p,q) = 1\Big\}, \quad (10)$$

its complement, $W_m := \complement W^m$, We note that

$$W_{m+1} \subset W_m, \quad W_\infty := \bigcap_{m=1}^{\infty} W_m = \{(z,\zeta) : |\zeta| = 1, \zeta \in \mathbb{R}^2 \setminus \mathbb{Q}^2\}. \quad (11)$$

and W_∞ is the set of irrational directions.

Lemma

Let $d\widetilde{\mu}_T = \int_0^T d\mu$. *Then*

$$\widetilde{\mu}_T(W_\infty) = 0 \Rightarrow \forall \epsilon > 0 \ \exists \, m \ \text{ such that } \widetilde{\mu}_T(W_m) < \epsilon.$$

Irrational directions continued

Let $\langle b\rangle_S(z,\zeta) := \frac{1}{S}\int_0^S b(z+s\zeta)ds$, the mean value of b along the flow. Flow invariance of the measure implies $\forall S>0$,

$$\int_{W_\infty} a(z,\zeta)d\widetilde{\mu}_T(z,\zeta) = \int_{W_\infty}\langle a\rangle_S(z,\zeta)d\widetilde{\mu}_T(z,\zeta).$$

a_j **smooth**. For $(z,\zeta)\in W_\infty$, unique ergodicity of flow $z\mapsto z+s\zeta$ givest $\langle a_j\rangle_S \to \langle a\rangle := \int_{\mathbb{T}^2} a_j(z)dz/(2\pi)^2$. Fatou's Lemma gives

$$\begin{aligned}\int_{W_\infty} a_j(z)d\widetilde{\mu}_T(z,\zeta) &= \liminf_{S\to\infty}\int_{W_\infty}\langle a_j\rangle_S(z,\zeta)d\widetilde{\mu}_T(z,\zeta)\\ &\geq \int_{W_\infty}\liminf_{S\to\infty}\langle a_j\rangle_S(z,\zeta)d\widetilde{\mu}_T(z,\zeta) = \widetilde{\mu}_T(W_\infty)\langle a_j\rangle.\end{aligned}$$

Now, since $\int_{W_\infty} a(z)d\widetilde{\mu}_T(z,\zeta)=0$, we can replace a_j by $a_j - a$ on left above and get

$$\widetilde{\mu}_T(W_\infty) \leq \frac{C\|a-a_j\|_{L^2}}{\langle a_j\rangle} \to 0, \quad j\to\infty,$$

which gives $\widetilde{\mu}_T(W_\infty)=0$.

A one dimensional result: a refinment from NB Zworski 04

Lemma

Suppose that $b \in L^1(\mathbb{T}^1)$, $b \geq 0$, $\|b\|_{L^1} > 0$ *and that* $T > 0$. *Then there exists* C *such that for* $w \in L^2(\mathbb{T}^1)$,

$$\|w\|^2_{L^2(\mathbb{T}^1)} \leq C \int_0^T b(x)[e^{it\partial_x^2} w](x)|^2 dxdt. \tag{12}$$

Semiclassical analog

Lemma

$\forall T > 0 \exists K, \rho_0$ *and* h_0 *such that for* $0 < h < h_0$ *and* $0 < \rho < \rho_0$

$$\|\pi_{h,\rho} u_0\|^2_{L^2(\mathbb{T}^1)} \leq K \int_0^T \int_{\mathbb{T}^1} b(z)|e^{it\Delta}\pi_{h,\rho} u_0(z)|^2 dzdt,$$

$$\pi_{h,\rho}(u_0) := \chi\left(\frac{h^2 D_x^2 - 1}{\rho}\right) u_0 \, .$$

Proof of one dimensional result

- In 1-d the dispersive $L^4_x L^2_t$ estimate is replaced by an $L^\infty_x L^2_t$ estimate (hence the condition $b \in L^1$).
- We proceed similarly by contradiction and get a measure $\nu(t,x,\xi)$, then $d\omega_T = \int_0^T d\nu$ on $T^*\mathbb{T}^1$ and satisfying:
  - $\operatorname{supp} \omega_T \subset \{\xi = \pm 1\} \Rightarrow \omega_T = \omega_T^+ + \omega_T^-$,
  - $\partial_x \omega_T = 0$,
  - $\int_M \sum_\pm b(x) b(x) d\omega_T(x,\pm) = 0$

We apply the same method as above (in the case of irrational directions) to the two directions ± 1 because now the flow $x \mapsto x \pm s$ is ergodic on $\mathbb{T}$! We get with

$$\langle b \rangle_S^\pm(z) = \frac{1}{S}\int_0^S b(z \pm s) ds$$

$$\int_{\mathbb{T}} \sum_\pm b(z) d\omega_T^\pm(x) = \liminf_{S\to\infty} \int_{\mathbb{T}} \sum_\pm \langle b \rangle_S^\pm(z) d\omega_T^\pm(z)$$

$$\geq \int_{\mathbb{T}} \sum_\pm \liminf_{S\to\infty} \langle b \rangle_S^\pm(z) d\omega_T^\pm(z) = |\omega_T| \langle b \rangle.$$

Reduction to the 1-d problem

From previous slides there is mass on some "isolated" rational directions. Assume for simplicity that all masss on $(0,1)$.
Based on Fourier expansion in y :

$$v_n(t)(x,y) := [e^{it\Delta}v_n](x,y) = \sum_{k\in\mathbb{Z}}[e^{it\partial_x^2}v_{n,k}](x)e^{-itk^2+iky} \quad (13)$$

Apply 1-d estimate to $b = \langle a\rangle_y = \frac{1}{2\pi}\int_{\mathbb{T}^1} a(x,y)dy$ $a_j \to a$ in L^2.

$$\begin{aligned}
0 < 1 = \|v_n\|^2_{L^2(\mathbb{T}^2)} &= 2\pi\sum_{k\in\mathbb{Z}}\|v_{n,k}\|^2_{L^2(\mathbb{T}^1)} \\
&\leq C'\int_0^T \langle a\rangle_y(x)\sum_{k\in\mathbb{Z}}|[e^{it\partial_x^2}v_{n,k}](x)|^2 dxdt \\
&= C\int_0^T\int_{\mathbb{T}^2}\langle a\rangle_y(x)|[e^{it\Delta}v_n](x,y)|^2 dxdydt \\
&= C\int_0^T\int_{\mathbb{T}^2}\langle a_j\rangle_y(x)|[e^{it\Delta}v_n](x,y)|^2 dxdydt + \mathcal{O}(\|a-a_j\|_{L^2(\mathbb{T}^2)}) \\
&\longrightarrow C\int_{T^*\mathbb{T}^2}\langle a_j\rangle_y(x)d\,\widetilde{\nu}_T(x,y,\xi,\eta) + \mathcal{O}(\|a-a_j\|_{L^2(\mathbb{T}^2)}), \quad n\longrightarrow\infty
\end{aligned}$$

End of proof

$$0 < 1/C \leq \int_{T^*\mathbb{T}^2} \langle a_j \rangle_y (x) d\,\widetilde{\nu}_T(x, y, \xi, \eta) + \mathcal{O}(\|a - a_j\|_{L^2(\mathbb{T}^2)}).$$

We now use that on $\Xi = (0, 1)$ the measure is invariant by geodesic flow i.e. $\partial_y \widetilde{\nu}_T = 0$. Hence (with $g \in L^2$.)

$$\widetilde{\nu}_T = g(x) dx dy,$$

$$\begin{aligned} I := \int_{\mathbb{T}^2 \times \{(0,1)\}} \langle a_j \rangle_y (x) d\,\widetilde{\nu}_T(x, y, \xi, \eta) &= \int_{\mathbb{T}^2 \times \{(0,1)\}} a_j(x, y) g(x) dx dy \\ &= \int_{\mathbb{T}^2} (a_j(x, y) - a(x, y) + a(x, y)) g(x) dx dy \\ &\leq \sqrt{2\pi} \|g\|_{L^2(\mathbb{T}^1)} \|a_j - a\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

When $j \to +\infty$, this provides a contradiction and proves the semi-classical estimate