A block moments method: minimal null controllability time for parabolic systems

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Abstract linear control problem

$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), \quad t \in (0,T), \\ y(0) = y_0. \end{cases} \tag{S}$$

Question :

Characterize the smallest time $T_0 \geq 0$ such that

for all $T > T_0$, for all y_0 there exists u such that $y(T) = 0$.

- $-\mathcal{A}$ generates a C^0-semigroup on the Hilbert space $(X, \|\cdot\|)$,
- The space of controls is the separable Hilbert space $(U, \|\cdot\|_U)$.
- The control operator $\mathcal{B} : U \to D(\mathcal{A}^*)'$. Assume (for simplicity) that

$$\int_0^T \left\| \mathcal{B}^* e^{-t\mathcal{A}^*} z \right\|_U^2 \mathrm{d}t \leq C\|z\|, \qquad \forall z \in D(\mathcal{A}^*).$$

$\longrightarrow$ Application to systems of coupled linear one dimensional parabolic equations.

The precise question under study

Wellposedness theorem

Let $T > 0$. For any $y_0 \in X$ and any $u \in L^2(0, T; U)$, there exists a unique solution $y \in C^0([0, T], X)$ characterized by

$$\langle y(t), z \rangle - \left\langle y_0, e^{-t\mathcal{A}^*} z \right\rangle = \int_0^t \left\langle u(\tau), \mathcal{B}^* e^{-(t-\tau)\mathcal{A}^*} z \right\rangle_U \mathrm{d}\tau,$$

for any $t \in [0, T]$, and any $z \in X$.
Moreover, there exists $C > 0$ such that for any such y_0, u, the solution satisfies

$$\|y(t)\| \leq C \left(\|y_0\| + \|u\|_{L^2(0,T;U)} \right), \quad \forall t \in [0, T].$$

Null controllability in time T: for any $y_0 \in X$, there exists $u \in L^2(0, T; U)$ such that $y(T) = 0$.

$\longrightarrow$ Minimal null control time $T_0 \in [0, +\infty]$:

- $T > T_0 \implies$ null controllability in time T holds;
- $T < T_0 \implies$ null controllability in time T does not hold.

Unexpected phenomenon in the parabolic setting.

A necessary condition for null controllability

- In finite dimension ($\mathcal{A}$ and $\mathcal{B}$ matrices), Fattorini-Hautus test states

$$\text{controllability} \quad \iff \quad \operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^* = \{0\}, \quad \forall \lambda.$$

Observation of eigenvectors of $\mathcal{A}^*$: $\mathcal{A}^* \phi_\lambda = \lambda \phi_\lambda \implies \mathcal{B}^* \phi_\lambda \neq 0.$

If $\dim U = 1$, the eigenvalues need to be simple.

$$\begin{cases} (\mathcal{A}^* - \lambda)\phi_\lambda = 0, \\ (\mathcal{A}^* - \lambda)\widetilde{\phi}_\lambda = 0, \end{cases} \implies \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} - \frac{\widetilde{\phi}_\lambda}{\mathcal{B}^*\widetilde{\phi}_\lambda} \in \operatorname{Ker}(\mathcal{A}^* - \lambda) \cap \operatorname{Ker} \mathcal{B}^*.$$

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- T. Duyckaerts & L. Miller (2012).

Null controllability in time $T \implies \exists C_T > 0; \quad \forall z \in D(\mathcal{A}^*), \forall \lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$,

$$\|z\|^2 \leq C_T e^{2\operatorname{Re}(\lambda) T} \left(\frac{\|(\mathcal{A}^* - \lambda) z\|^2}{\operatorname{Re}(\lambda)^2} + \frac{\|\mathcal{B}^* z\|_U^2}{\operatorname{Re}(\lambda)} \right).$$

Remark: a quantified Fattorini-Hautus test. Gives a lower bound on T_0.

Also a sufficient condition for null controllability in time T under certain conditions.
F. Ammar Khodja, A. Benabdallah, M. González Burgos & M. M. (2018)

Spectral consequences of quantified Fattorini-Hautus test

$$\|z\|^2 \leq C_T e^{2\,\mathrm{Re}(\lambda)T} \left(\frac{\|(\mathcal{A}^* - \lambda)z\|^2}{\mathrm{Re}(\lambda)^2} + \frac{\|\mathcal{B}^* z\|_U^2}{\mathrm{Re}(\lambda)} \right). \qquad (\mathrm{QFH}_T)$$

Necessary conditions for null controllability in time T:

$$\mathcal{A}^* \phi_\lambda = \lambda \phi_\lambda, \qquad \|\phi_\lambda\| = 1.$$

- Sufficient observation of eigenvectors (depending on T):

$$\begin{aligned} z = \phi_\lambda \quad &\Longrightarrow \quad (\mathcal{A}^* - \lambda)z = 0 \\ &\Longrightarrow \quad \|\mathcal{B}^* \phi_\lambda\|_U \geq C_T \sqrt{\mathrm{Re}(\lambda)} e^{-\mathrm{Re}(\lambda)T}. \end{aligned}$$

$$\|z\|^2 \leq C_T e^{2\,\mathrm{Re}(\lambda)T}\left(\frac{\|(\mathcal{A}^*-\lambda)z\|^2}{\mathrm{Re}(\lambda)^2}+\frac{\|\mathcal{B}^*z\|_U^2}{\mathrm{Re}(\lambda)}\right). \qquad (\mathrm{QFH}_T)$$

Necessary conditions for null controllability in time T:

$$\mathcal{A}^*\phi_\lambda = \lambda\phi_\lambda, \qquad \|\phi_\lambda\| = 1.$$

- Sufficient observation of eigenvectors : $\|B^*\phi_\lambda\|_U \geq C_T\sqrt{\mathrm{Re}(\lambda)}e^{-\mathrm{Re}(\lambda)T}$.

In the case of scalar controls $(\dim U = 1)$:

- Distance between eigenvalues:

$$z = \frac{\phi_\mu}{\mathcal{B}^*\phi_\mu} - \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} \implies (\mathcal{A}^*-\lambda)z = (\mu-\lambda)\frac{\phi_\mu}{\mathcal{B}^*\phi_\mu}, \qquad \mathcal{B}^*z = 0.$$

If $(\phi_\lambda)_\lambda$ is a Riesz basis of X

$$\|z\| = \left\|\frac{\phi_\mu}{\mathcal{B}^*\phi_\mu} - \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda}\right\| \geq \frac{c}{|\mathcal{B}^*\phi_\mu|}$$

then

$$|\lambda-\mu| \geq C_T\,\mathrm{Re}(\lambda)e^{-\mathrm{Re}(\lambda)T}.$$

Spectral consequences of quantified Fattorini-Hautus test

$$\|z\|^2 \leq C_T e^{2\,\mathrm{Re}(\lambda)T}\left(\frac{\|(\mathcal{A}^*-\lambda)z\|^2}{\mathrm{Re}(\lambda)^2}+\frac{\|\mathcal{B}^*z\|_U^2}{\mathrm{Re}(\lambda)}\right). \qquad (\mathrm{QFH}_T)$$

Necessary conditions for null controllability in time T:

$$\mathcal{A}^*\phi_\lambda = \lambda\phi_\lambda, \qquad \|\phi_\lambda\| = 1.$$

- Sufficient observation of eigenvectors : $\|B^*\phi_\lambda\|_U \geq C_T\sqrt{\mathrm{Re}(\lambda)}e^{-\,\mathrm{Re}(\lambda)T}$.

In the case of scalar controls $(\dim U = 1)$:

- Distance between eigenvalues: $|\lambda-\mu| \geq C_T\lambda e^{-\,\mathrm{Re}(\lambda)T}$ (if Riesz basis)
- Generalized eigenspaces:

$$\mathcal{A}^*\psi_\lambda = \lambda\psi_\lambda + \phi_\lambda, \qquad \mathcal{B}^*\psi_\lambda = 0.$$

$$z = \psi_\lambda \quad \Longrightarrow \quad (\mathcal{A}^*-\lambda)z = \phi_\lambda, \quad \mathcal{B}^*z = 0.$$

Then

$$\|\psi_\lambda\| \leq C_T\frac{e^{\mathrm{Re}(\lambda)T}}{\lambda}$$

Spectral consequences of quantified Fattorini-Hautus test

$$\|z\|^2 \leq C_T e^{2\,\mathrm{Re}(\lambda)T} \left(\frac{\|(\mathcal{A}^* - \lambda)z\|^2}{\mathrm{Re}(\lambda)^2} + \frac{\|\mathcal{B}^* z\|_U^2}{\mathrm{Re}(\lambda)} \right). \qquad (\mathrm{QFH}_T)$$

Necessary conditions for null controllability in time T:

$$\mathcal{A}^* \phi_\lambda = \lambda \phi_\lambda, \qquad \|\phi_\lambda\| = 1.$$

- Sufficient observation of eigenvectors : $\|B^* \phi_\lambda\|_U \geq C_T \sqrt{\mathrm{Re}(\lambda)} e^{-\,\mathrm{Re}(\lambda)T}$.

In the case of scalar controls $(\dim U = 1)$:

- Distance between eigenvalues: $|\lambda - \mu| \geq C_T \lambda e^{-\,\mathrm{Re}(\lambda)T}$ (if Riesz basis)
- Generalized eigenspaces: $\|\psi_\lambda\| \leq \frac{C_T}{\lambda} e^{\mathrm{Re}(\lambda)T}$

$$\mathcal{A}^* \psi_\lambda = \lambda \psi_\lambda + \phi_\lambda, \qquad \mathcal{B}^* \psi_\lambda = 0.$$

Existence of minimal null control time

Each of the previous necessary conditions can actually lead to a positive minimal null control time.

- Examples in the context of systems of coupled one dimensional parabolic equations
- Examples in the context of degenerate parabolic equations (in dimension 2 or 3)

F. Ammar Khodja, K. Beauchard, A. Benabdallah, F. Boyer, P. Cannarsa, J. Dardé, S. Dolecki, C. Dupaix, M. Duprez, S. Ervedoza, M. González Burgos, R. Guglielmi, B. Helffer, R. Henry, A. Koenig, I. Kostin, L. Miller, M. M., L. Ouaili, L. Robbiano, E.H. Samb, L. de Teresa . . .

Unsufficent observation of eigenvectors: a pointwise control

S. Dolecki (1973)

$$\begin{cases} \partial_t y - \partial_{xx} y = \delta_{x=x_0} u(t), & t \in (0,T), x \in (0,1), \\ y(t,0) = y(t,1) = 0. \end{cases}$$

Notations: $\phi_k(x) = \sqrt{2}\sin(k\pi x)$, $\lambda_k = k^2\pi^2$.

- $\mathcal{B}^*\phi_k = \phi_k(x_0) = \sqrt{2}\sin(k\pi x_0)$

Minimal time: $$T_0 = \limsup_{k\to+\infty} \frac{-\ln|\mathcal{B}^*\phi_k|}{\lambda_k}.$$

For any $\tau \in [0,+\infty]$, there exists $x_0 \in (0,1)$ such that $T_0 = \tau$.

Remark: This is the same phenomenon for the 2D Grushin equation.

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014), E.H. Samb (2018).
Assume that $\sqrt{d} \notin \mathbb{Q}$.

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -d\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y = 0, \quad t \in (0,T), x \in (0,1), \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

Notations : $\Lambda := \left\{ j^2\pi^2,\, dj^2\pi^2;\, j \in \mathbb{N}^* \right\} = \{\lambda_k \,;\, k \geq 1\}$.

$$\text{Minimal time:} \qquad T_0 = \limsup_{k \to +\infty} \frac{-\ln|\lambda_{k+1} - \lambda_k|}{\lambda_k}.$$

For any $\tau \in [0, +\infty]$, there exists $d \in (0, +\infty)$ such that $T_0 = \tau$.

With a zero-order coupling term supported outside the control domain

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2016).

$$\begin{cases} \partial_t y + \underbrace{\begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & q(x) \\ 0 & 0 \end{pmatrix} y}_{:=\mathcal{A}y} = 0, \quad t \in (0,T), x \in (0,1), \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

Notations: $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, $\lambda_k = k^2\pi^2$.

$$I_k(q) = \int_0^1 q(x)\varphi_k(x)^2 \mathrm{d}x \neq 0, \qquad \forall k \in \mathbb{N}^*.$$

$$\text{Minimal time:} \qquad T_0 = \limsup_{k\to+\infty} \frac{-\ln|I_k(q)|}{\lambda_k}.$$

Riesz basis of generalized eigenvectors with

$$(\mathcal{A}^* - \lambda_k)\begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} = 0, \qquad (\mathcal{A}^* - \lambda_k)\psi_k = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \qquad \text{and} \qquad \|\psi_k\| \approx \frac{1}{|I_k(q)|}.$$

Presentation of the moment problem when $\dim U = 1$

- The setting ($\dim U = 1$)
  - Assume that the operator $\mathcal{A}^*$ admits a sequence of positive simple eigenvalues Λ such that
  $$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty. \qquad \text{(Spectrum)}$$
  - We denote by $(\phi_\lambda)_{\lambda \in \Lambda}$ the associated sequence of normalized eigenvectors and we assume that it forms a complete family in X i.e. for $y \in X$,
  $$(\langle y, \phi_\lambda \rangle = 0,\ \forall \lambda \in \Lambda) \quad \Longrightarrow \quad y = 0 \qquad \text{(CFE)}$$
  - $\mathcal{B}^* \phi_\lambda \neq 0$ for all $\lambda \in \Lambda$.

- Comments
  - Though restrictive the moment method is quite successful in the analysis of the minimal null control time for systems of coupled parabolic equations.
  - The assumption 'positive eigenvalues' is a simplification for this talk. Relates to parabolic problems.
  - The condition (Spectrum) is necessary for null controllability when $\dim U = 1$. Otherwise it is a restrictive condition (mostly one dimensional problems).
  - The condition $\mathcal{B}^* \phi_\lambda \neq 0$ is necessary for approximate controllability.

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  $$\left(\langle y, \phi_\lambda \rangle = 0, \ \forall \lambda \in \Lambda \right) \implies y = 0 \qquad \text{(CFE)}$$
  - $\mathcal{B}^* \phi_\lambda \neq 0$ for all $\lambda \in \Lambda$.

- Definition of solutions
$$\langle y(T), \phi_\lambda \rangle - \left\langle y_0, e^{-\lambda T} \phi_\lambda \right\rangle = \int_0^T e^{-\lambda(T-t)} \langle u(t), \mathcal{B}^* \phi_\lambda \rangle_U \, \mathrm{d}t.$$

- Complete family of eigenvectors (CFE)
$$y(T) = 0 \iff \boxed{\int_0^T e^{-\lambda(T-t)} u(t) \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, \ \forall \lambda \in \Lambda.}$$

Resolution of the moment problem using a biorthogonal family

$$\text{Find } u \text{ such that } \int_0^T e^{-\lambda(T-t)} u(t)\mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} \right\rangle, \forall \lambda \in \Lambda$$

Under condition (Spectrum) there exists a biorthogonal family $(q_\lambda)_{\lambda\in\Lambda}$ to the exponentials associated with Λ i.e.

$$\begin{cases} \displaystyle\int_0^T e^{-\mu t} q_\lambda(t)\mathrm{d}t = 0, \quad \forall \mu \in \Lambda\backslash\{\lambda\}, \\ \displaystyle\int_0^T e^{-\lambda t} q_\lambda(t)\mathrm{d}t = 1. \end{cases}$$

Then,

$$u : t \in (0,T) \mapsto -\sum_{\lambda\in\Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} \right\rangle q_\lambda(T-t)$$

formally solves the moment problem.

- Estimate $\mathcal{B}^*\phi_\lambda$ and $\|q_\lambda\|_{L^2(0,T;\mathbb{R})}$ to make this rigorous.
- H.O. Fattorini & D.L Russell (1974). Nice estimates of $\|q_\lambda\|_{L^2(0,T;\mathbb{R})}$ under the gap condition: $|\lambda - \mu| > \rho, \quad \forall \lambda \neq \mu \in \Lambda$.

Perturbation of a Jordan-block: positive controllability result

$$\mathcal{A}y = \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + \exp(\partial_{xx}) \end{pmatrix} y, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \text{a nice scalar control operator} \end{pmatrix}.$$

Eigenvectors of $-\partial_{xx}$: $-\partial_{xx}\varphi_k = k^2\varphi_k$. Thus,

$$\Lambda = \left\{ \lambda_{k,1} := k^2,\ \lambda_{k,2} := k^2 + e^{-k^2}\ ;\ k \in \mathbb{N}^* \right\}$$

Complete family of associated eigenvectors of $\mathcal{A}^*$:

$$\phi_{k,1} = \begin{pmatrix} e^{-k^2} \\ 1 \end{pmatrix} \varphi_k, \qquad \phi_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k,$$

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014): there exists a biorthogonal family satisfying

$$\frac{1}{C_\varepsilon} e^{(\mathbf{1}-\varepsilon)\lambda} \leq \|q_\lambda\|_{L^2(0,T;U)} \leq C_\varepsilon e^{(\mathbf{1}+\varepsilon)\lambda}.$$

$\longrightarrow$ Direct application of moments method yields null controllability in time $T > 1$.

Perturbation of a Jordan-block: negative controllability result ?

Null controllability in time $T \leq 1$?
Back to the necessary condition (QFH_T) for null controllability in time T

$$e^{-k^2} = \lambda_{k,2} - \lambda_{k,1} \geq C_T k^2 e^{-k^2 T} \quad \Longrightarrow \quad T > 1.$$

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IF $(\phi_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis...
Here,

$$\phi_{k,1} = \begin{pmatrix} e^{-k^2} \\ 1 \end{pmatrix} \varphi_k, \qquad \phi_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k,$$

implies

$$\|\phi_{k,1} - \phi_{k,2}\| \underset{k\to+\infty}{\longrightarrow} 0 \quad \Longrightarrow \qquad \text{not a Riesz basis.}$$

Thus, no obstruction to null controllability in time $T < 1$ in the literature...

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Thus, no obstruction to null controllability in time $T < 1$ in the literature...
... we prove that this example is null controllable **in any time** $T > 0$.

Perturbation of a Jordan-block: comments

- Philosophy: perturbation of the system associated to

$$\mathcal{A}y = \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} \end{pmatrix} y, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \text{a nice scalar control operator} \end{pmatrix}$$

  which is null controllable in any time: Riesz basis of generalized eigenvectors and gap between eigenvalues.

- Limitation in the use of biorthogonal family. As $\lambda_{k,1} \approx \lambda_{k,2}$, the biorthogonal elements $q_{k,1}$ and $q_{k,2}$ should not be considered separately. Thus, using

$$\|\alpha_{k,1}q_{k,1} + \alpha_{k,2}q_{k,2}\| \leq |\alpha_{k,1}|\|q_{k,1}\| + |\alpha_{k,2}|\|q_{k,2}\| \quad \text{is not a good idea !}$$

  We need more informations on the biorthogonal family than just $\|q_\lambda\|$.

Assumptions

$\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions for the wellposedness.

- Scalar control $U = \mathbb{R}$.
- Eigenvalues of $\mathcal{A}^*$.
  - Λ: positive simple eigenvalues of $\mathcal{A}^*$ satisfying (Spectrum) i.e. $\displaystyle\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty$.
  - Weak gap condition: there exists $\rho > 0$ and $p \in \mathbb{N}^*$ such that
    $$\mathrm{Card}\,(\Lambda \cap [\mu, \mu+\rho]) \leq p, \qquad \forall \mu \geq 0.$$
- $(\phi_\lambda)_{\lambda\in\Lambda}$ associated eigenvectors.
  - complete family of eigenvectors (CFE) in X.
  - $\mathcal{B}^*\phi_\lambda \neq 0$, for all $\lambda \in \Lambda$.

Let $p \in \mathbb{N}^*$ and $\rho > 0$. The weak-gap condition ensures the existence of sets $(G_k)_{k\geq 1} \subset \mathcal{P}(\Lambda)$ such that

$$\Lambda = \bigcup_{k\geq 1} G_k, \qquad \sup(G_k) < \inf(G_{k+1}),$$

with the additional properties that for every $k \geq 1$,

$$g_k := \# G_k \leq p, \qquad \operatorname{dist}(G_k, G_{k+1}) \geq r, \qquad \operatorname{diam} G_k < \rho.$$

with $r = r_{p,\rho} > 0$.

- Labelling the eigenelements

$$G_k = \{\lambda_{k,1}, \dots, \lambda_{k,g_k}\} \qquad \text{with } \lambda_{k,1} < \cdots < \lambda_{k,g_k},$$

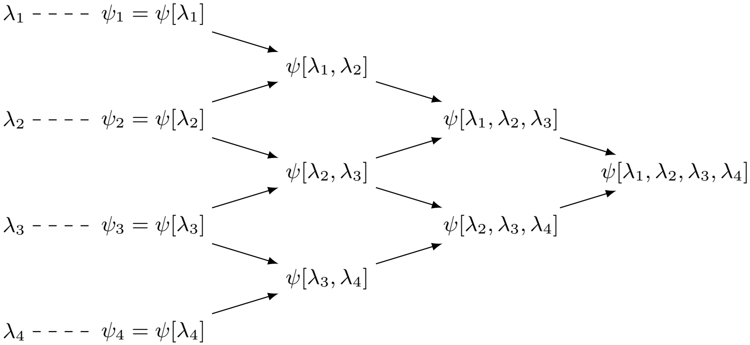
$$\phi_{k,j} := \phi_{\lambda_{k,j}}, \quad \forall k \geq 1, \forall 1 \leq j \leq g_k.$$

- The analysis is insensitive to the particular choice of such a grouping.

- For any $j \in \{1, ..., g_k\}$, we set $\psi[\lambda_j] := \psi_j = \frac{\phi_j}{\mathcal{B}^* \phi_j} \in X$.
- For any $i \neq j$ we set

$$\psi[\lambda_i, \lambda_j] := \frac{\psi[\lambda_j] - \psi[\lambda_i]}{\lambda_j - \lambda_i} \in X.$$

- and so on ... following the usual diagram



Characterization of the minimal null control time

$$T_0 = \limsup_{k\to\infty} \frac{\ln\left(\max\limits_{1\le l\le g_k} \|\psi[\lambda_{k,1},\dots,\lambda_{k,l}]\|\right)}{\lambda_{k,1}}.$$

A. Benabdallah, F. Boyer & M. M. (2018)

Let $T > 0$ and T_0 be defined above. Then,

- If $T_0 < +\infty$ and $T > T_0$, the system (S) is null-controllable at time T.
- If $T_0 > 0$ and $T < T_0$, the system (S) is not null-controllable at time T.

- Explicit formula (depending on the eigenelements) for the minimal null control time:

$$\psi[\lambda_{k,1},\dots,\lambda_{k,l}] = \sum_{j=1}^{l} \frac{\frac{\phi_{k,j}}{\mathcal{B}^*\phi_{k,j}}}{\prod\limits_{1\le i\ne j\le l}(\lambda_{k,j}-\lambda_{k,i})}.$$

- We recover known expressions of the minimal null control time when there is no condensation of eigenvalues or when $(\phi_\lambda)_\lambda$ is a Riesz basis: F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014)

Comments on the assumptions

$\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions for the wellposedness.

- Scalar control $U = \mathbb{R}$.
- Eigenvalues of $\mathcal{A}^*$.
  - Λ: positive simple eigenvalues of $\mathcal{A}^*$ satisfying (Spectrum) i.e. $\displaystyle\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty$.

    $\longrightarrow$ necessary controllability condition as $U = \mathbb{R}$
  - Weak gap condition: there exists $\rho > 0$ and $p \in \mathbb{N}^*$ such that

    $$\operatorname{Card}\left(\Lambda \cap [\mu, \mu + \rho]\right) \leq p.$$

    $\longrightarrow$ no assumption on "how close" are the eigenvalues
- $(\phi_\lambda)_{\lambda\in\Lambda}$ associated (normalized) eigenvectors.
  - complete family of eigenvectors (CFE) in X.

    $\longrightarrow$ no Riesz basis assumption
  - $\mathcal{B}^*\phi_\lambda \neq 0$, for all $\lambda \in \Lambda$.

    $\longrightarrow$ necessary controllability condition

$$\mathcal{A}y = \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + \exp(\partial_{xx}) \end{pmatrix} y, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \text{a nice scalar control operator} \end{pmatrix}.$$

$$\Lambda = \left\{ \lambda_{k,1} := k^2,\ \lambda_{k,2} := k^2 + e^{-k^2}\ ;\ k \in \mathbb{N}^* \right\} \quad \Longrightarrow \quad \#G_k = 2$$

$$T_0 = \limsup_{k\to\infty} \frac{1}{\lambda_{k,1}} \ln \max \left\{ \frac{1}{|\mathcal{B}^*\phi_{k,1}|}, \frac{1}{|\mathcal{B}^*\phi_{k,2}|}, \frac{\left\| \frac{\phi_{k,2}}{\mathcal{B}^*\phi_{k,2}} - \frac{\phi_{k,1}}{\mathcal{B}^*\phi_{k,1}} \right\|}{\lambda_{k,2} - \lambda_{k,1}} \right\} = 0.$$

Indeed,

$$\phi_{k,1} = \begin{pmatrix} e^{-k^2} \\ 1 \end{pmatrix} \varphi_k, \qquad \phi_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k,$$

imply

$$\mathcal{B}^*\phi_{k,1} = \mathcal{B}^*\phi_{k,2} = \text{ nice} \qquad \text{and} \qquad \|\phi_{k,2} - \phi_{k,1}\| = e^{-k^2} = \lambda_{k,2} - \lambda_{k,1}.$$

New phenomenon: the condensation of eigenvectors can compensate the condensation of eigenvalues.

$$y(T) = 0 \quad \Longleftrightarrow \quad \int_0^T e^{-\lambda(T-t)} u(t) \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, \ \forall \lambda \in \Lambda$$

$$\Longleftrightarrow \quad \int_0^T e^{-\lambda(T-t)} u(t) \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \psi_\lambda \right\rangle, \ \forall \lambda \in \Lambda$$

Look for u in the form

$$u : t \in (0,T) \mapsto -\sum_{k \geq 1} q_k(T-t)$$

where

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda t} q_k(t) \mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k, \\ \displaystyle\int_0^T e^{-\lambda_{k,j} t} q_k(t) \mathrm{d}t = e^{-\lambda_{k,j} T} \left\langle y_0, \psi_{k,j} \right\rangle, & \forall k \geq 1, \ \forall 1 \leq j \leq g_k. \end{cases}$$

The function q_k solves the moment problem inside the group G_k.

$\longrightarrow$ block resolution gives sharper results than using a biorthogonal family.

Resolution of the block moment problem

A. Benabdallah, F. Boyer & M. M. (2018)

Let $T \in (0,+\infty]$. For any $\varepsilon > 0$, there exists a constant $C > 0$ such that for any $k \geq 1$, for any $\omega_{k,1}, \dots, \omega_{k,g_k} \in \mathbb{C}$, there exists $q_k \in L^2(0,T;\mathbb{C})$ satisfying

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda t} q_k(t) \mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k, \\ \displaystyle\int_0^T e^{-\lambda_{k,j} t} q_k(t) \mathrm{d}t = \omega_{k,j}, & \forall 1 \leq j \leq g_k, \end{cases}$$

and

$$\| q_k \|_{L^2(0,T;\mathbb{C})} \leq C e^{\varepsilon \lambda_{k,1}} \max_{1 \leq l \leq g_k} \Big| \omega[\lambda_{k,1}, \dots, \lambda_{k,l}] \Big|.$$

Moreover, up to the factor $e^{\varepsilon \lambda_{k,1}}$, this last estimate is sharp.

- Local (depending only on the group G_k) estimate of q_k.
- Choosing $\omega_{k,j'} = \delta_{j',j}$ gives biorthogonal families with optimal estimates. Similar new result in presence of algebraically multiple eigenvalues which do not satisfy the classical gap condition.

Adaptation of H.O. Fattorini & D.L. Russell (1974).

First step: Resolution of the block moment problem in infinite time horizon

$$\begin{cases} \displaystyle\int_0^{+\infty} e^{-\lambda t} q_k(t)\mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k, \\ \displaystyle\int_0^{+\infty} e^{-\lambda_{k,j} t} q_k(t)\mathrm{d}t = \omega_{k,j}, & \forall 1 \leq j \leq g_k. \end{cases}$$

Use the isomorphism induced by the Laplace transform

$$\mathsf{L} : f \in L^2(0,+\infty;\mathbb{C}) \mapsto \left(F : \lambda \in \mathbb{C}^+ \mapsto \int_0^{+\infty} e^{-\lambda t} f(t)\mathrm{d}t \right) \in H^2(\mathbb{C}^+)$$

and for every $k \geq 1$, design $J_k \in H^2(\mathbb{C}^+)$ such that

$$\begin{gathered} J_k(\lambda) = 0, \qquad \forall \lambda \in \Lambda \backslash G_k, \\ J_k(\lambda_{k,j}) = \omega_{k,j}, \qquad \forall j \in \{1, \dots, g_k\}, \\ \int_{\mathbb{R}} |J_k(i\tau)|^2 \mathrm{d}\tau \leq C e^{\varepsilon \lambda_{k,1}} \max_{1 \leq l \leq g_k} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,l}] \right|. \end{gathered}$$

$$J_k(\lambda) = 0, \quad \forall \lambda \in \Lambda \backslash G_k, \qquad \text{and} \qquad J_k(\lambda_{k,j}) = \omega_{k,j}, \quad \forall j \in \{1, \dots, g_k\}.$$

Define

$$J_k : z \in \mathbb{C}^+ \mapsto \frac{P_k(z)}{(1+z)^p} W_k(z)$$

where

- W_k ensures orthogonality with respect to the other groups,

$$W_k(z) \approx \prod_{\lambda \in \Lambda \backslash G_k} \frac{\lambda - z}{\lambda + z}, \qquad \Longrightarrow \qquad J_k(\lambda) = 0, \quad \forall \lambda \in \Lambda \backslash G_k;$$

  $\longrightarrow$ term $e^{\varepsilon \lambda_{k,1}}$ in the estimate (gap between groups of eigenvalues)

- $(1+z)^p$ ensures sufficient regularity properties;
- P_k is the Lagrange interpolating polynomial ensuring $J_k(\lambda_{k,j}) = \omega_{k,j}$ for every $1 \leq j \leq g_k$.

  $\longrightarrow$ term $\max_{1 \leq l \leq g_k} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,l}] \right|$ in the estimate.

Overview of the resolution of the block moment problem

Second step: Bound (by contradiction) the norm of the inverse of the restriction mapping

$$q \in \overline{\operatorname{Span}\left\{t \mapsto e^{-\lambda t}\ ;\ \lambda \in \Lambda\right\}}^{L^2(0,+\infty;\mathbb{C})} \mapsto q_{|(0,T)}.$$

Back to controllability in time $T > T_0$: we want to set

$$u : t \in (0,T) \mapsto -\sum_{k\geq 1} q_k(T-t).$$

We have

$$\|q_k\|_{L^2(0,T)} \leq C e^{\varepsilon\lambda_{k,1}} \max_{1\leq l\leq g_k} \left|\omega[\lambda_{k,1},\dots,\lambda_{k,l}]\right|$$

where $\omega_{k,j} := e^{-\lambda_{k,j}T}\left\langle y_0, \psi_{k,j}\right\rangle$. Classical estimates for divided differences yield

$$\left|\omega[\lambda_{k,1},\dots,\lambda_{k,l}]\right| \leq C\|y_0\| e^{-\lambda_{k,1}T} \max_{1\leq j\leq l} \left\|\psi[\lambda_{k,1},\dots,\lambda_{k,j}]\right\|.$$

We get $u \in L^2(0,T)$, as $T > T_0$ with

$$T_0 = \limsup_{k\to\infty} \frac{\ln\left(\displaystyle\max_{1\leq l\leq g_k} \left\|\psi[\lambda_{k,1},\dots,\lambda_{k,l}]\right\|\right)}{\lambda_{k,1}}.$$

Lack of null controllability when $T < T_0$

Let $T > 0$ and assume null controllability at time T. Then,

$$y(T) = 0 \quad \iff \quad \int_0^T e^{-\lambda(T-t)} u(t) \mathrm{d}t = -\left\langle y_0, e^{-\lambda T} \psi_\lambda \right\rangle, \; \forall \lambda \in \Lambda$$

As the estimate given in the resolution of the block moment problem is sharp

$$\left| \omega[\lambda_{k,1}, \ldots, \lambda_{k,l}] \right| \leq C \|u\|_{L^2} \leq C \left\| y_0 \right\|.$$

As $\omega_{k,j} := e^{-\lambda_{k,j} T} \langle y_0, \psi_{k,j} \rangle$, classical estimates for divided differences yield

$$\|\psi[\lambda_{k,1}, \ldots, \lambda_{k,l}]\| \leq C e^{\lambda_{k,1} T} \quad \Longrightarrow \quad T \geq T_0.$$

Some extensions

- Similar expression of the minimal null control time for a given subspace Y_0 of initial conditions

$$T_0(Y_0) = \limsup_{k\to\infty} \frac{\ln\left(\max\limits_{1\le l\le g_k}\left\|\sum\limits_{j=1}^{l}\frac{\frac{P_{Y_0}^*\phi_{k,j}}{\mathcal{B}^*\phi_{k,j}}}{\prod\limits_{1\le i\ne j\le l}(\lambda_{k,j}-\lambda_{k,i})}\right\|\right)}{\lambda_{k,1}},$$

  with P_{Y_0} the orthogonal projection onto Y_0.

- Same result with complex-valued eigenvalues satisfying
  - there exists $\delta > 0$ such that $\operatorname{Re}\lambda \ge \delta|\lambda|$ and $\displaystyle\sum_{\lambda\in\Lambda}\frac{1}{|\lambda|} < +\infty$,
  - weak-gap condition: there exists $\rho > 0$ and $p \in \mathbb{N}^*$ such that

$$\operatorname{Card}\Big(\Lambda \cap \left([\mu, \mu+\rho] + i\mathbb{R}\right)\Big) \le p.$$

Uniform estimates

In the resolution of the block moment problem, in the estimate

$$\|q_k\|_{L^2(0,T;\mathbb{C})} \leq C e^{\varepsilon \lambda_{k,1}} \max_{1 \leq l \leq g_k} \Big| \omega[\lambda_{k,1}, \dots, \lambda_{k,l}] \Big|,$$

the constant C is uniform with respect to Λ in the class of sequences satisfying

- the weak-gap condition with parameters $p \in \mathbb{N}^*$ and $\rho > 0$
- the asymptotic behaviour : $\mathcal{N} : \mathbb{R} \to \mathbb{N}$ such that

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \geq \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} < \varepsilon, \qquad \forall \varepsilon > 0.$$

$\longrightarrow$ This could be used for applications to uniform null controllability of parameter dependant problems (space discretization, oscillating coefficients, degeneracy parameter etc...)

See for instance K. Bhandari & F. Boyer (2019): boundary control, from Robin to Dirichlet boundary conditions.

Algebraic multiplicity of eigenvalues

Similar characterization of T_0 (with generalized divided differences) if there exists $\eta \geq 1$ such that for any $\lambda \in \Lambda$, the algebraic multiplicity α_λ satisfies $\alpha_\lambda \leq \eta$.

- Assume that $\alpha_\mu = 2$ and $\alpha_\lambda = 1$ for all $\lambda \in \Lambda \backslash \{\mu\}$.
- Moment problem

$$\int_0^T e^{-\lambda t} q(t) \mathrm{d}t = \omega_\lambda^0, \quad \forall \lambda \in \Lambda, \qquad \text{and} \qquad \int_0^T (-t) e^{-\mu t} q(t) \mathrm{d}t = \omega_\mu^1.$$

Solve the moment problem associated with $\Lambda \cup \{\mu + h\}$ (simple eigenvalues)

$$\int_0^T e^{-\lambda t} q_h(t) \mathrm{d}t = \omega_\lambda^0, \quad \forall \lambda \in \Lambda, \qquad \text{and} \qquad \int_0^T e^{-(\mu+h)t} q_h(t) \mathrm{d}t = \omega_\mu^0 + h\omega_\mu^1.$$

Then,

$$\int_0^T \left(\frac{e^{-(\mu+h)t} - e^{-\mu t}}{h} \right) q_h(t) \mathrm{d}t = \frac{\omega_\mu^0 + h\omega_\mu^1 - \omega_\mu^0}{h} = \omega_\mu^1,$$

and pass to the limit using uniform bounds on q_h.

Back to the academic example

$$\mathcal{A}y = \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + f(-\partial_{xx}) \end{pmatrix} y, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \text{a nice scalar control operator} \end{pmatrix},$$

with $0 \leq f(k^2) < 2k+1$ and $\lim_{s \to +\infty} f(s) = 0$.

- Eigenvectors of $-\partial_{xx}$: $-\partial_{xx}\varphi_k = k^2 \varphi_k$. Thus,

$$\Lambda = \left\{ \lambda_{k,1} := k^2,\ \lambda_{k,2} := k^2 + f(k^2)\ ;\ k \in \mathbb{N}^* \right\}$$

$\longrightarrow$ possible very strong condensation of eigenvalues ! But

$$T_0 = \limsup_{k \to \infty} \frac{1}{\lambda_{k,1}} \ln \frac{\left\| \frac{\phi_{k,2}}{\mathcal{B}^* \phi_{k,2}} - \frac{\phi_{k,1}}{\mathcal{B}^* \phi_{k,1}} \right\|}{\lambda_{k,2} - \lambda_{k,1}} = 0.$$

- One can still have $T_0 > 0$ due to the action of $\mathcal{B}$ (if not nice).

Same phenomena on a PDE example

$$\begin{cases} \partial_t y(t,x) + \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + c(x) \end{pmatrix} y(t,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in (0,1), \end{cases}$$

For any $c \in L^2(0,1;\mathbb{R})$

- possible presence of algebraically double eigenvalues;
- possible strong condensation of eigenvalues;
- possible (finite number of) non observable modes.

There exists $Y_0 \subset (H^{-1}(0,1;\mathbb{R}))^2$ with finite codimension such that

- if $y_0 \notin Y_0$: not approximately controllable;
- if $y_0 \in Y_0$: null controllability in any time $T > 0$.

$\longrightarrow$ Inverse spectral theory: we can choose the algebraically double eigenvalues and the condensation of eigenvalues by choosing c.

The motivating example

E.H. Samb (2018)

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & q(x) \\ 0 & -d\partial_{xx} \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in (0,T), \\ y(0,x) = y_0(x), \quad x \in (0,1), \end{cases}$$

with $\sqrt{d} \notin \mathbb{Q}$.

- Weak-gap condition with $p = 2$.
- Explicit expression of the minimal time even if there is condensation of eigenvalues and the eigenvectors do not form a Riesz basis.

Ongoing work with F. Boyer (20??).

- Exact same assumptions as in the scalar case except '$\dim U = 1$' replaced by 'U a separable Hilbert space'.

F. Boyer & M. M. (20??)

For any $k \geq 1$, there exists a matrix M_k of size $g_k \times g_k$ such that

$$T_0 = \limsup_{k \to +\infty} \frac{\ln \rho(M_k)}{2\lambda_{k,1}}.$$

- The matrix M_k explicitely depends on the eigenelements of $\mathcal{A}^*$ and is of "reasonnable size".

Main ideas (for simple eigenvalues)

- Lift the block moment problem

$$\begin{cases} \displaystyle\int_0^T \left\langle q_k(t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda \right\rangle_U \mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k, \\ \displaystyle\int_0^T \left\langle q_k(t), e^{-\lambda_{k,j} t}\mathcal{B}^*\phi_{k,j} \right\rangle_U \mathrm{d}t = \omega_{k,j}, & \forall 1 \leq j \leq g_k, \end{cases}$$

to a vectorial block moment problem

$$\begin{cases} \displaystyle\int_0^T q_k(t)e^{-\lambda t}\mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k, \\ \displaystyle\int_0^T q_k(t)e^{-\lambda_{k,j} t}\mathrm{d}t = \Omega_{k,j} \in U, & \forall 1 \leq j \leq g_k. \end{cases}$$

- For any $(\Omega_{k,j})$ in the finite dimensional space $\mathrm{Span}(\mathcal{B}^*\phi_{k,1}, \dots, \mathcal{B}^*\phi_{k,g_k}) \subset U$, solve the vectorial moment problem (component by component using the scalar case) and compute its cost as a function of $(\Omega_{k,j})$.

- Minimize this cost under the constraints

$$\langle \Omega_{k,j}, \mathcal{B}^*\phi_{k,j} \rangle_U = \omega_{k,j}.$$

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -d\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y = 0, \quad t \in (0,T), x \in (0,1), \\ y(t,0) = \begin{pmatrix} 0 \\ u_1(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} u_2(t) \\ u_2(t) \end{pmatrix}. \end{cases}$$

- F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014).
  Assume that $u_2 = 0$ and $\sqrt{d} \notin \mathbb{Q}$. Then,

$$T_0 = \limsup_{k \to +\infty} \frac{-\ln |\lambda_{k+1} - \lambda_k|}{\lambda_k},$$

  and for any $\tau \in [0, +\infty]$, there exists $d \in (0, +\infty)$ such that $T_0 = \tau$.

- F. Boyer & M. M. (20??).
  Using both controls u_1 and u_2, for any $d > 0$,

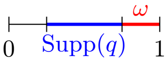
$$T_0 = 0.$$

Example with a distributed control: influence of the geometry

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & q(x) \\ 0 & -\partial_{xx} \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega(x) u(t,x) \end{pmatrix}, & t \in (0,T), x \in (0,1), \\ y(t,0) = y(t,1) = 0, \end{cases}$$

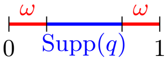
with $q(x) = \left(x - \frac{1}{2}\right) \mathbf{1}_{\left(\frac{1}{4}, \frac{3}{4}\right)}(x)$.

- F. Boyer & G. Olive (2014).



Not approximately controllable (for any time $T > 0$).

- F. Boyer & M. M. (20??).



Null controllable in any time $T > 0$.

Conclusion:

- Characterization of the minimal null control time under general assumptions.
- Block resolution of the moment problem gives sharper results than the use of biorthogonal families.
  New phenomenon: condensation of eigenvectors can compensate the condensation of eigenvalues.

Perspectives:

- Higher dimension i.e.

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} = +\infty \quad ?$$

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$$\sum_{\lambda\in\Lambda}\frac{1}{\lambda}=+\infty \quad ?$$

Thank you for your attention