On the robustness of the controllability of linear systems with respect to the introduction of age structuring

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General framework (I)

Many systems describing evolution of particles or living organisms (cells, bacteria, animals, ...) are described (within a linear approximation), by equations of the form

\[ \dot{p}(t) = Ap(t) + Bu(t), \quad p(0) = p_0, \]

where \( X \) the state space) is Hilbert, \( A : \mathcal{D}(A) \to X \) generates a \( C^0 \) semigroup, denoted \( \mathbb{T} \), in \( X \). \( U \) (the control space) is also Hilbert space and \( B \in \mathcal{L}(U, X_{-1}) \) is the control operator.

In many cases (namely for biological systems) the *age* of the considered particles (individuals) plays an important role. In particular, they can disappear (die) or get born.
General framework (II)

Grafting an age structure on a generic control system means assuming that $p$ depends on the age $a \in [0, a_\dagger]$ and that individuals can die (with a certain probability $\mu$) at any age or be born at a certain fertility rate $\beta$. The original system becomes

$$\dot{p}(t, a) + \frac{\partial p}{\partial a}(t, a) = Ap(t, a) - \mu(a)p(t, a) + \chi(a)Bu(t, a),$$

$$p(t, 0) = \int_0^{a_\dagger} \beta(a)p(t, a) \, da,$$

where $\chi$ is the characteristic function of an interval contained in $[0; a_\dagger)$.

- Does wellposedness of the original problem imply the wellposedness of the age structured one?

- Does controllability (in some sense) of the original problem imply the controllability of the age structured one?
Typical mortality and fertility rates
Example 1: the classical Lotka-McKendrick system

Take \( X = \mathbb{R}^n \) and \( U = \mathbb{R}^m \) with \( m \leq n \), with \( A \) a real \( n \times n \) matrix, \( B \) a real \( n \times m \) matrix, such that

\[
\text{rank} [B, AB, \ldots A^{n-1}B] = n. \tag{1}
\]

In particular, for \( X = U = \mathbb{C} \), \( A = 0 \) and \( B = 1 \) we obtain the age structured Lotka-McKendrick system:

\[
\begin{align*}
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} &= -\mu p + \chi_{(0,a_0)} u \quad (a \in (0,a_\dagger), \ t > 0), \\
p(0,a) &= p_0(a) \quad (a \in (0,a_\dagger)), \\
p(t,0) &= \int_0^{a_\dagger} \beta(a) p(a,t) da \quad (t > 0).
\end{align*}
\]

- \( p(t,a) \): distribution density of the population of age \( a \) at time \( t \);
- \( a^* \): maximal life expectancy;
- \( \mu(a), \beta(a) \): mortality and fertility rates.
Example 2 : transport of an age structured population

Let $\Omega = (0, L)$ and $\nu > 0$. We consider the following control problem

\[
\begin{aligned}
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \nu \frac{\partial p}{\partial x} + \mu(a)p &= 0, \\
(t, a, x) &\in (0, \tau) \times (0, a^+) \times \Omega, \\
p(t, a, 0) &= \chi_{(a_1, a_2)} u(t, a), \\
(t, a) &\in (0, \tau) \times (0, a^+), \\
p(t, 0, x) &= \int_0^{a^+} \beta(a) p(t, a, x) \, da, \\
(t, x) &\in (0, \tau) \times \Omega, \\
p(0, a, x) &= p_0(a, x) \\
(a, x) &\in \times (0, a^+) \times \Omega.
\end{aligned}
\]

We take $X = L^2(\Omega)$ and $U = \mathbb{R}$. The operator $A$ is defined by

\[\mathcal{D}(A) = \{ \varphi \in H^1(0, L) \mid \varphi(0) = 0 \}, \quad A\varphi = -\nu \frac{\partial \varphi}{\partial x}.\]

The control operator $B$ is defined by $Bu = \delta_0$, where $\delta_0$ is the Dirac mass at 0.
Example 3: Lotka-McKendrick system with diffusion

\[
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} + \mu(a, x)p - k \Delta p = u \chi(a_1, a_2) \times \omega, \quad a \in (0, a^*), \ x \in \Omega, \ t > 0,
\]

\[
p = 0 \ \text{or} \ \frac{\partial p}{\partial \nu} = 0, \quad a \in (0, a^*), \ x \in \partial \Omega, \ t > 0,
\]

\[
p(a, x, 0) = p_0(a, x), \quad a \in (0, a^*), \ x \in \Omega,
\]

\[
p(0, x, t) = \int_0^{a^*} \beta(a, x)p(a, t, x)da, \quad x \in \Omega, \ t > 0.
\]

- \(p(t, a, x)\): distribution density of the population of age \(a\) at spatial position \(x\) at time \(t\);
- \(a^*\): maximal life expectancy;
- \(k\): diffusion coefficient;
- \(\mu(a, x), \beta(a, x)\): mortality and fertility rates.
Other examples

- Population dynamics models with degenerate or fractional diffusion

- Schrödinger equation with age structure

\[ A = -i\Delta, \quad D(A) = H^2(\Omega) \cap H^1_0(\Omega), \quad B = \chi_0 \]
Some References on Analysis and Control for Population Dynamics (with Age Structuring)

• Semigroup properties: Chan and Guo, Di Blasio, Li et al., Langlais, Marinoschi, Walker, Webb, ...

• Optimal Control: Yong et al, Anita, ...

• Controllability problems: Ainseba, Anita, Barbu-Iannelli, Langlais, Traoré, Kavian

• Inverse problems: Traoré, Rundell, Filin, Perasso, Picart,…

• Numerical aspects: Lopez, Trigiante, Douglas, Milner, Huyer, Guo, Gerardo-Giorda, …
Outline

- Semigroup formulation
- $L^2$ null controllability
- Controllability with positivity constraints
- Conclusions and remarks
Semigroup Formulation
Spaces and operators

\[ \mathcal{X} = L^2(0, a_\dagger; X) \] is the extended state space,

\[ \mathcal{U} = L^2(0, a_\dagger; U) \] is the extended input space,

\[ \mathcal{D}(A) = \left\{ \varphi \in C([0, a_\dagger]; X) \mid \varphi(0) = \int_0^{a_\dagger} \beta(a) \varphi(a) \, da, -\frac{\partial \varphi}{\partial a} + A \varphi - \mu \varphi \in \mathcal{X} \right\}, \]

\[ A \varphi = -\frac{\partial \varphi}{\partial a} + A \varphi - \mu \varphi. \]

The control operator \( \mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{X}_{-1}) \) is defined by

\[ \mathcal{B}u = \chi_{(a_1, a_2)} Bu \quad (u \in \mathcal{U}), \]

where

\[ \mathcal{X}_{-1} = L^2(0, a_\dagger; X_{-1}) \]
Wellposedness

**Theorem.** (Maity, M.T, Zuazua, 2019)

Assume that $A$ generates a $C^0$ semigroup on $X$. Then $A$ generates a $C^0$ semigroup on $X$. Moreover, assume that $B$ is an admissible control operator for the semigroup $S$ generated by $A$. Then $B$ is an admissible control operator for the semigroup $T$ generated by $A$.

**Main steps of the proof:**

- “Guess” that $T_t \varphi = \left\{ \begin{array}{ll} \frac{\pi(a)}{\pi(a-t)} S_t \varphi(a-t), & t < a, \\ \pi(a) S_a b_\varphi(t-a) & t \geq a, \end{array} \right.$ where $\pi(a) = e^{-\int_0^a \mu(s) ds}$ and

  \[ b_\varphi(t) = \int_0^t \beta(a) \pi(a) S_a b_\varphi(t-a) + S_t \int_0^{a+t} \beta(a+t) \frac{\pi(a+t)}{\pi(a)} \varphi(a) \, da. \]

- Prove that $A$ is the generator of $T$. 
The dual system

**Proposition.** The operator $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \to \mathcal{X}$ defined by

$$\mathcal{D}(\mathcal{A}_0) = \left\{ \psi \in \mathcal{X} \mid q(t, a^+) = 0, \quad \frac{\partial \psi}{\partial a} - \mu \psi + A^* \psi \in \mathcal{X} \right\}, \quad \mathcal{A}_0 \psi = \frac{\partial \psi}{\partial a} - \mu \psi + A^* \psi,$$

generates the $C^0$-semigroup $\mathbb{T}^0$ on $\mathcal{X}$ given by

$$\mathbb{T}^0_t \varphi = \begin{cases} \frac{\pi(a)}{\pi(a+t)} S^*_t \varphi(a + t), & t < a^+ - a, \\ 0 & t \geq a^+ - a. \end{cases}$$

**Proposition.**

$$\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A}_0), \quad \mathcal{A}^* \psi = \frac{\partial \psi}{\partial a} - \mu \psi + A^* \psi + \beta(a) \psi(0).$$

Moreover, $\mathcal{B}^* \in \mathcal{L}(L^2(0, a^+; \mathcal{D}(\mathcal{A}^*)); \mathcal{U})$ is given by

$$\mathcal{B}^* \psi = \chi(a_1, a_2) B^* \psi.$$
$L^2$ null controllability
Main result

**Theorem.** (Maity, M.T. and Zuazua, 2019)
Assume that $\beta(a) = 0$ for $a \in (0, a_b)$, for some $a_b \in (0, a_{\dagger})$ and that $a_1 < a_b$. Let Assume that the pair $(A^*, B^*)$ is final state observable in time $\tau > \tau_0$, with

$$0 \leq \tau_0 < \bar{\tau}, \quad \bar{\tau} = \min\{a_2 - a_1, a_b - a_1\}.$$

Then the pair $(A^*, B^*)$ is final-state observable for every $\tau > a_1 + a_{\dagger} - a_2 + \tau_0$. In other words, for every $\tau > a_1 + a_{\dagger} - a_2 + \tau_0$ there exists $k_{\tau} > 0$ such that

$$\|T^*_\tau q_0\|_{\mathcal{X}}^2 \leq k_{\tau}^2 \int_0^\tau \|B^* T^*_\tau q_0\|_{\mathcal{U}}^2 \, dt, \quad (q_0 \in \mathcal{D}(A^*)). \quad (1)$$

**Corollary.** Under the above assumptions, the pair $(A, B)$ is null controllable in any time $\tau > a_1 + a_{\dagger} - a_2 + \tau_0$. 
Main steps of the proof

1) Note that, integrating along the characteristic lines,

\[ T_t^* q_0 = \begin{cases} \frac{\pi(a)}{\pi(a+t)} T_0^0 q_0(a+t) + \int_0^t \frac{\pi(a)}{\pi(a+t-s)} T_0^0 \beta(a+t-s)V(s) \, ds & t \leq a_\uparrow - a, \\ \int_t^{t+a-a_\uparrow} \frac{\pi(a)}{\pi(a+t-s)} T_0^0 \beta(a+t-s)V(s) \, ds & t > a_\uparrow - a, \end{cases} \]

where \( \pi(a) = e^{-\int_0^a \mu(r) \, dr} \) et \( V(t, a) := q(t, 0) \).

2) Prove that \( \| T_\tau^* q_0 \|^2 \ll \int_\eta^\tau \| T_t^* q_0(t, 0) \|^2 \, dt \) for some \( \eta > a_1 \). Obvious if \( \tau > a_\uparrow \), but quite tricky otherwise.

3) Prove that \( \int_0^\tau \int_\eta^\tau \| T_t^* q_0(t, 0) \|^2 \, dt \ll \int_0^\tau \int_{a_1}^{a_2} \| B T_t^* q_0 \|^2(t, a, ) \, da \, dt \).
Some Hints on Step 3 (I)

Use the fact that along the backward characteristic from \((t,0)\); i.e.

\[
\gamma(s) = (t - s, s) \quad (s \leq t \leq \tau),
\]

\(q\) is essentially the solution of the original equation. Then use the stanrd final state observability estimate for the heat equation (Lebeau-Robbiano, Fursikov-Imanuvilov, 1995).
Some Hints on Step 3 (II)

For $t \in (a, \tau)$, we set $\tilde{q}(t, a) = q(t, a) e^{-\int_0^a \mu(r) \, dr}$. Since $\beta(a) = 0$ for all $a \in (0, a_b)$, $\tilde{q}$ satisfies

$$\frac{\partial \tilde{q}}{\partial t} - \frac{\partial \tilde{q}}{\partial a} - A \tilde{q} = 0 \quad \text{for } t \geq 0, \quad a \in (0, a_b).$$

Set $w(s) = \tilde{q}(s, t - s)$ for $s \in (t - a_b, t)$. Then

$$\begin{cases}
\frac{\partial w}{\partial s} - Aw = 0 & (s \in (t - a_b, t)), \\
w(t - a_b) = q(t - a_b, a_b).
\end{cases}$$

From final state observability $\|w(t)\|^2 \leq c_1 e^{-\frac{c_2}{a_b - a_1}} \int_{t - a_b}^{t - a_1} \|B^* w(s)\|(s) \, ds,$

thus $\|\tilde{q}(t, 0)\|^2 \leq c_1 e^{-\frac{c_2}{a_b - a_1}} \int_{a_1}^{a_b} \|B^* \tilde{q}(t - s, s)\|^2 \, ds$. 
Controllability with positivity constraints
Steady states

Let $v_s \in L^\infty((0, a^\dagger) \times \Omega)$, $v_s \geq 0$ A non-negative function $p_s \in L^\infty((0, a^\dagger) \times \Omega)$ satisfying

\[
\begin{aligned}
\frac{\partial p_s}{\partial a} - L p_s + \mu(a) p_s &= m v_s & (a, x) \in (0, a^\dagger) \times \Omega, \\
\frac{\partial p_s}{\partial v_L} &= 0 & (a, x) \in (0, a^\dagger) \times \partial \Omega, \\
p_s(0, x) &= \int_0^{a^\dagger} \beta(a) p_s(a, x) \, da, & x \in \Omega,
\end{aligned}
\]

is said to be a non-negative steady state. Positive steady states are those non negative steady states such that there exist $a_* \in (0, a^\dagger)$ and $\delta > 0$ with

\[
p_{s, I}(a, x), p_{s, F}(a, x) \geq \delta \text{ a.e. on } [0, a_*] \times \overline{\Omega}.
\]

Such steady states exist at least if

\[
1 = R := \int_0^{a^\dagger} \beta(a) e^{-\int_0^{a} \mu(r) \, dr} \, da.
\]
Main result states

Assume that $p_{s,I}$ and $p_{s,F}$ are two positive steady states, thus there exist $a_\ast \in (0, a_\dagger)$ and $\delta > 0$ such that

$$p_{s,I}(a, x), p_{s,F}(a, x) \geq \delta \text{ a.e. on } [0, a_\ast] \times \overline{\Omega}.$$ 

Then there exist $\tau > 0$ and $\nu \in L^\infty((0, \tau) \times (0, a_\dagger) \times \Omega)$ such that the controlled system with

$$p_0(a, x) = p_{s,I}(a, x)$$

admits a unique solution $p$ satisfying

$$p(\tau, a, x) = p_{s,F}(a, x) \text{ for all } (a, x) \in (0, a_\dagger) \times \Omega.$$ 

Moreover, $p(\tau, a, x) \geq 0 \text{ for a.e. } (t, a, x) \in (0, \tau) \times (0, a_\dagger) \times \Omega.$
Proposition.

For every $\tau > 0$ there exists a constant $C_\tau > 0$ such that

$$
\|p\|_{L^\infty(\{0,\tau\} \times (0,a_\uparrow) \times \Omega)} \leq C \left( \|p_0\|_{L^\infty(\{0,a_\uparrow\} \times \Omega)} + \|u\|_{L^\infty(\{0,\tau\} \times (0,a_\uparrow) \times \Omega)} \right),
$$

Proof. Combine existing estimates on the heat kernel with characteristics method.

Proposition.

For every $\tau > a_1 + \max\{a_1, a_\uparrow - a_2\}$, there exists $k_\tau > 0$ such that the solution $q$ of the adjoint problem satisfies

$$
\int_0^{a_\uparrow} \left( \int_0^{a_2} \int_\Omega q^2(\tau, a, x) \, dx \right)^{\frac{1}{2}} \, da \leq k_\tau \int_0^\tau \int_{a_1}^{a_2} \int_\omega |q(t, a, x)| \, dx \, da \, dt \quad (q_0 \in X).
$$
Proof of Proposition 2

Let $p_{s,I}$ and $p_{s,F}$ be two non negative steady states and let $v_{s,I}$ and $v_{s,F}$ be the corresponding steady controls. We set

$$p_{s,r} = \left(1 - \frac{r}{N}\right)p_{s,I} + \frac{r}{N}p_{s,F}, \quad v_{r,k} = \left(1 - \frac{r}{N}\right)v_{s,I} + \frac{r}{N}v_{s,F} \quad (r = 0, 1, \ldots, N)$$

where $N \in \mathbb{N}$ is large enough. Using the $L^\infty$ estimates, we can steer $p_{s,r-1}$ to $p_{s,r}$, while preserving the positivity. Note that, like in the case of parabolic problems, the controllability time depends on the distance from $p_{s,I}$ to $p_{s,F}$. 
Conclusions and remarks

• We obtained controllability results for the linear Lotka-McKendrick system with diffusion and age structuring, in sharp time and without excluding low ages.

• We obtained controllability with positivity constraints, provided that the time is large enough.

• What about nonlinear models (mortality and/or fertility depending on the total population)?

• What about models involving competing populations?

• Feedback control?