Sufficient Conditions for the Boundary Controllability of Wave Equations with Transmission Conditions

Ludovick Gagnon

Université Côte d'Azur

October 20th, 2017

Outline

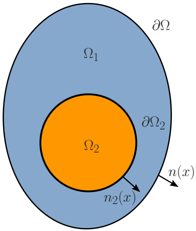
1. Introduction
   - Presentation of the problematic
   - Brief review of the controllability of the wave equation
2. Microlocal analysis of the wave equation with transmission condition
   - Definition of the generalized bicharacteristics
   - Propagation of the measures
3. Main results
   - Geometrical argument
   - Statement of the main results
   - Remarks on the $\Gamma(x_0)$ condition
4. Conclusion
5. Appendix : proof of the propagation lemma

Wave equations with transmission condition

Let $T > 0$. Let $\Omega \subset \mathbb{R}^2$ an open, **strictly convex**, simply connected domain of smooth boundary $\partial\Omega$. Let $\Omega_2 \subsetneq \Omega$ an open, **strictly convex**, simply connected domain of smooth boundary $\partial\Omega_2$. Define $\Omega_1 := \Omega \setminus \Omega_2$.

(WT)

$$\begin{cases} (\partial_t^2 - c_i^2\Delta)u^i(x,t) = 0, & (x,t) \in \Omega_i \times (0,T), \\ u^1(x,t) = u^2(x,t), & (x,t) \in \partial\Omega_2 \times (0,T), \\ c_1^2\partial_{n_2}u^1(x,t) = c_2^2\partial_{n_2}u^2(x,t), & (x,t) \in \partial\Omega_2 \times (0,T), \\ u^1(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u^i(0,x) = u_0^i(x), u_t^i(0,x) = u_1^i(x), & x \in \Omega_i \end{cases}$$



where $c_2 > c_1 > 0$ and $(u_0^1\mathbb{1}_{\Omega_1} + u_0^2\mathbb{1}_{\Omega_2}, u_1^1\mathbb{1}_{\Omega_1} + u_1^2\mathbb{1}_{\Omega_2}) \in H_0^1(\Omega) \times L^2(\Omega)$.

$$E(u^1,u^2)(t) := \sum_{i=1}^{2}\int_0^T\int_{\Omega_i}\left|u_t^i\right|^2 + c_i^2\left|\nabla u^i\right|^2 \,\mathrm{dxdt} \equiv E(u^1,u^2)(0)$$

For $\Gamma \subset \partial\Omega$ and $T > 0$, we consider the observability problem : $\exists c_T > 0$ such that

(obs) $$E(u^1,u^2)(t) \le c_T\int_0^T\int_\Gamma\left|\partial_n u^1(x,t)\right|^2 \,\mathrm{dxdt}.$$

Motivation : the wave equation

Let $T > 0$ and let Ω be defined as in the previous slide.

$$(W) \qquad \begin{cases} (\partial_t^2 - c^2\Delta)u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in \Omega. \end{cases}$$

Conserved energy :

$$E(u)(t) := \int_0^T \int_\Omega |u_t|^2 + c^2 |\nabla u|^2 \,\mathrm{dxdt} \equiv E(u)(0).$$

Observability problem : for given $T > 0$ and $\Gamma \subset \partial\Omega$, does $\exists c_T > 0$ such that

$$E(u)(t) \leq c_T \int_0^T \int_\Gamma |\partial_n u(x,t)|^2 \,\mathrm{dxdt}.$$

Motivation : the wave equation

Let $T > 0$ and let Ω be defined as in the previous slide.

$$(W) \qquad \begin{cases} (\partial_t^2 - c^2\Delta)u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in \Omega. \end{cases}$$

Conserved energy :

$$E(u)(t) := \int_0^T \int_\Omega |u_t|^2 + c^2 |\nabla u|^2 \,\mathrm{dxdt} \equiv E(u)(0).$$

Observability problem : for given $T > 0$ and $\Gamma \subset \partial\Omega$, does $\exists c_T > 0$ such that

$$E(u)(t) \leq c_T \int_0^T \int_\Gamma |\partial_n u(x,t)|^2 \,\mathrm{dxdt}.$$

Proof by contradiction : let $\{u_k\}_{k\in\mathbb{N}}$ be a renormalized sequence ($E(u_k) = 1$) such that

$$\int_0^T \int_\Gamma |\partial_n u_k(x,t)|^2 \,\mathrm{dxdt} \to 0.$$

We associate a defect measure μ acting on $(x,t,\xi,\tau) \in T_b^*(\Omega\times]0,T[)$.

Motivation : the wave equation

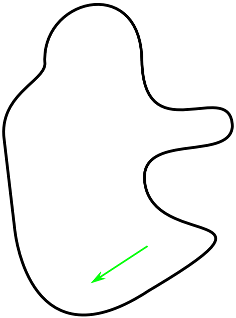
Let $T^*(\Omega \times \mathbb{R}_t) = \{(t, x, \tau, \xi) \in \mathbb{R}^6 \,|\, (t, x) \in \mathbb{R}_t \times \Omega\}$. The principal symbol of $P = \partial_t^2 - c^2 \Delta$ is given by

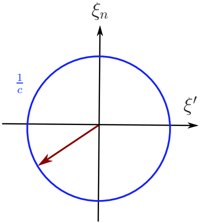
$$p(t, x; \tau, \xi) = \tau^2 - c^2 |\xi|^2.$$

The bicharacteristics γ propagates in

$$\mathrm{Char}\ P := \{(t, x, \tau, \xi) \in T^*(\Omega \times \mathbb{R}_t) \,|\, p(t, x; \tau, \xi) = 0\}$$

in straight line and at constant speed





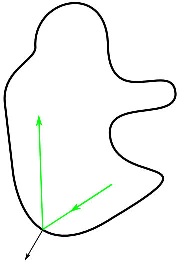
Near $\partial\Omega$

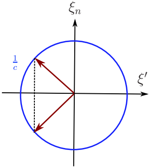
Let $(x_n, x') \in \partial\Omega$, then locally $\Delta = \partial^2_{x_n} + R(x_n, x', D_{x'})$ where $R(x_n, x', D_{x'})$ is a second order tangential elliptic operator of real principal symbol $r(x_n, x', \xi')$. We have

$$p(t, x, \tau, \xi', \xi^n)(\sigma_0) = \tau^2 - c^2(|\xi^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi')^2) = 0$$

We decompose $T^*(\partial\Omega \times \mathbb{R}_t) = \mathcal{H} \cup \mathcal{G}^+ \cup G^- \cup \mathcal{E}$ where

$$\begin{aligned}
\mathcal{H} &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) > |\tau|/c\}\,,\\
\mathcal{G}^+ &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) = |\tau|/c\}\,,\\
\mathcal{G}^- &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) = |\tau|/c\}\,,\\
\mathcal{E} &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) < |\tau|/c\}\,.
\end{aligned}$$





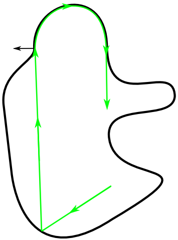
Near $\partial\Omega$

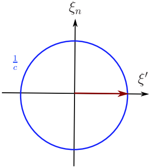
Let $(x_n, x') \in \partial\Omega$, then locally $\Delta = \partial^2_{x_n} + R(x_n, x', D_{x'})$ where $R(x_n, x', D_{x'})$ is a second order tangential elliptic operator of real principal symbol $r(x_n, x', \xi')$. We have

$$p(t, x, \tau, \xi', \xi^n)(\sigma_0) = \tau^2 - c^2(|\xi^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi')^2) = 0$$

We decompose $T^*(\partial\Omega \times \mathbb{R}_t) = \mathcal{H} \cup \mathcal{G}^+ \cup G^- \cup \mathcal{E}$ where

$$\begin{aligned}
\mathcal{H} &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) > |\tau|/c\}\,, \\
\mathcal{G}^+ &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) = |\tau|/c\}\,, \\
\mathcal{G}^- &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) = |\tau|/c\}\,, \\
\mathcal{E} &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) < |\tau|/c\}\,.
\end{aligned}$$





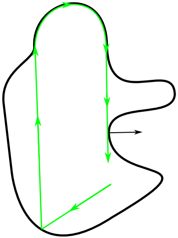
Near $\partial\Omega$

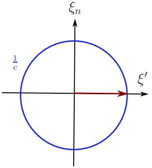
Let $(x_n, x') \in \partial\Omega$, then locally $\Delta = \partial^2_{x_n} + R(x_n, x', D_{x'})$ where $R(x_n, x', D_{x'})$ is a second order tangential elliptic operator of real principal symbol $r(x_n, x', \xi')$. We have

$$p(t, x, \tau, \xi', \xi^n)(\sigma_0) = \tau^2 - c^2(|\xi^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi')^2) = 0$$

We decompose $T^*(\partial\Omega \times \mathbb{R}_t) = \mathcal{H} \cup \mathcal{G}^+ \cup G^- \cup \mathcal{E}$ where

$$\begin{aligned}
\mathcal{H} &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) > |\tau|/c\}\,,\\
\mathcal{G}^+ &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) = |\tau|/c\}\,,\\
\mathcal{G}^- &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) = |\tau|/c\}\,,\\
\mathcal{E} &:= \{(t, x, \tau, \xi) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi) < |\tau|/c\}\,.
\end{aligned}$$

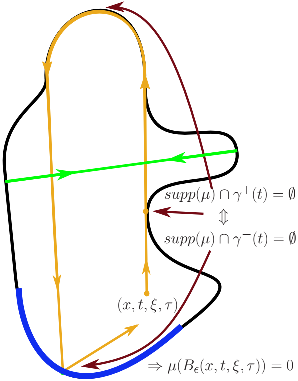




Motivation : the wave equation

Bardos, Lebeau, Rauch, '92 :

1. $\mu(B_\epsilon(x,t,\xi,\tau)) = \mu(\gamma(B_\epsilon(x,t,\xi,\tau)))$
2. $\mathrm{supp}(\mu) \cap \gamma^-(t) = \emptyset \Leftrightarrow \mathrm{supp}(\mu) \cap \gamma^+(t) = \emptyset$
3. $\mu = 0$ on $\Gamma \times]0, T[$
4. if $\mu = 0$ on $\Omega \times]0, T[$, then $u_k \to 0$



Definition (Geometric Control Condition (GCC))

We say that Γ and $T > 0$ satisfies the geometric control condition if and only if every ray of the optic geometry of Ω intersects Γ at a non-diffractive point in time T.

Propagation of the bicharacteristics in Ω_i

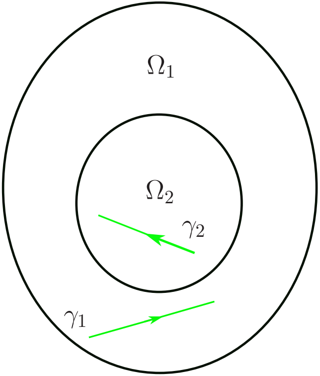
Let $T^*(\Omega_i \times \mathbb{R}_t) = \{(t, x, \tau_i, \xi_i) \in \mathbb{R}^6 \,|\, (t, x) \in \mathbb{R}_t \times \Omega_i\}$. The principal symbol of $P_i = \partial_t^2 - c_i^2 \Delta$ is given by

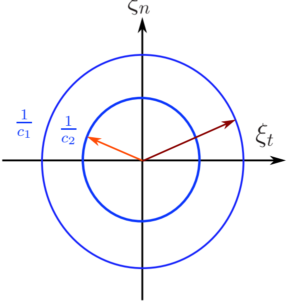
$$p_i(t, x; \tau_i, \xi_i) = \tau_i^2 - c_i^2 |\xi_i|^2.$$

The bicharacteristics γ_i propagates in

$$\mathrm{Char}\ P_i := \{(t, x, \tau_i, \xi_i) \in T^*(\Omega_i \times \mathbb{R}_t) \,|\, p_i(t, x; \tau_i, \xi_i) = 0\}$$

in straight line and at constant speed





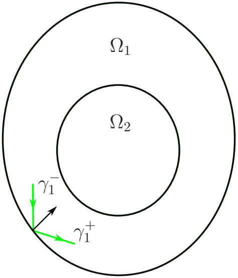
Near $\partial\Omega$

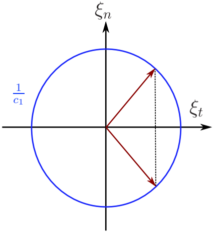
Let $(x_n, x') \in \partial\Omega$, then locally $\Delta = \partial^2_{x_n} + R(x_n, x', D_{x'})$ where $R(x_n, x', D_{x'})$ is a second order tangential elliptic operator of real principal symbol $r(x_n, x', \xi_1')$. We have

$$p_1(t, x, \tau_1, \xi_1', \xi_1^n)(\sigma_0) = \tau_1^2 - c_1^2(|\xi_1^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi_1')^2) = 0$$

We decompose $T^*(\partial\Omega \times \mathbb{R}_t) = \mathcal{H} \cup \mathcal{G}^+ \cup \mathcal{E}$ where

$$\mathcal{H} := \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi_1) > |\tau_1|/c_1\},$$
$$\mathcal{G}^+ := \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi_1) = |\tau_1|/c_1\},$$
$$\mathcal{E} := \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \,|\, r(x_n, x', \xi_1) < |\tau_1|/c_1\}.$$





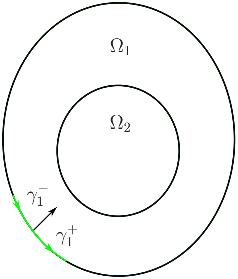
Near $\partial\Omega$

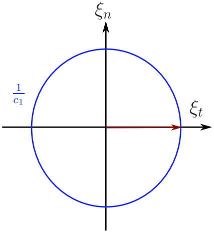
Let $(x_n, x') \in \partial\Omega$, then locally $\Delta = \partial_{x_n}^2 + R(x_n, x', D_{x'})$ where $R(x_n, x', D_{x'})$ is a second order tangential elliptic operator of real principal symbol $r(x_n, x', \xi_1')$. We have

$$p_1(t, x, \tau_1, \xi_1', \xi_1^n)(\sigma_0) = \tau_1^2 - c_1^2(|\xi_1^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi_1')^2) = 0$$

We decompose $T^*(\partial\Omega \times \mathbb{R}_t) = \mathcal{H} \cup \mathcal{G}^+ \cup \mathcal{E}$ where

$$\begin{aligned}
\mathcal{H} &:= \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \mid r(x_n, x', \xi_1) > |\tau_1|/c_1\}\,, \\
\mathcal{G}^+ &:= \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \mid r(x_n, x', \xi_1) = |\tau_1|/c_1\}\,, \\
\mathcal{E} &:= \{(t, x, \tau_1, \xi_1) \in T^*(\partial\Omega) \mid r(x_n, x', \xi_1) < |\tau_1|/c_1\}\,.
\end{aligned}$$





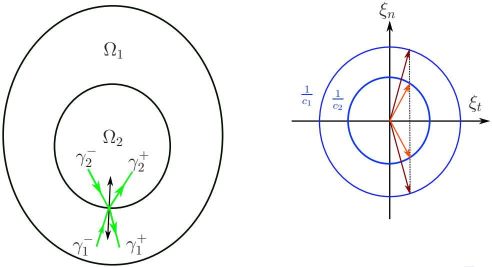
Near $\partial\Omega_2$

$$p_i(t,x,\tau_i,\xi_i',\xi_i^n)(\sigma_0) = \tau_i^2 - c_i^2(|\xi_i^n|^2 + r(x_n(\sigma_0),x'(\sigma_0),\xi_i')^2) = 0,$$

$$T^*(\partial\Omega_2\times\mathbb{R}_t) = (\mathcal{H}_1\times\mathcal{H}^2)\cup(\mathcal{H}^1\times\mathcal{G}^{2,+})\cup(\mathcal{H}^1\times\mathcal{E}^2)\cup(\mathcal{G}^{1,-}\times\mathcal{E}^2)\cup(\mathcal{E}^1\times\mathcal{E}^2),$$

where

$$\begin{aligned}
\mathcal{H}^1\times\mathcal{H}^2 &= \{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T)) \mid |\xi_i'|<|\tau_i|/c_i,\ c_1^2\xi_1^n = -c_2^2\xi_2^n\},\\
\mathcal{H}^1\times\mathcal{G}^{2,+} &= \{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T)) \mid |\xi_1'|<|\tau_1|/c_1, |\xi_2'| = |\tau_2|/c_2\},\\
\mathcal{H}^1\times\mathcal{E}^2 &= \{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T)) \mid |\xi_1'|<|\tau_1|/c_1, |\xi_2'| > |\tau_2|/c_2\},\\
\mathcal{G}^{1,-}\times\mathcal{E}^2 &= \{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T)) \mid |\xi_1'| = |\tau_1|/c_1, |\xi_2'| > |\tau_2|/c_2\},\\
\mathcal{E}^1\times\mathcal{E}^2 &= \{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T)) \mid |\xi_i'| > |\tau_i|/c_i\}.
\end{aligned}$$



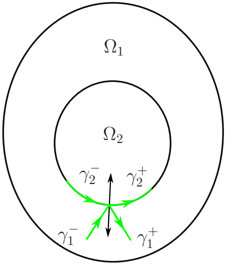
Near $\partial\Omega_2$

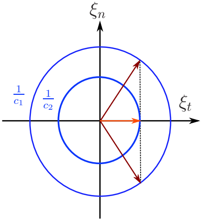
$$p_i(t,x,\tau_i,\xi_i',\xi_i^n)(\sigma_0)=\tau_i^2-c_i^2(|\xi_i^n|^2+r(x_n(\sigma_0),x'(\sigma_0),\xi_i')^2)=0,$$

$$T^*(\partial\Omega_2\times\mathbb{R}_t)=(\mathcal{H}_1\times\mathcal{H}^2)\cup(\mathcal{H}^1\times\mathcal{G}^{2,+})\cup(\mathcal{H}^1\times\mathcal{E}^2)\cup(\mathcal{G}^{1,-}\times\mathcal{E}^2)\cup(\mathcal{E}^1\times\mathcal{E}^2),$$

where

$$\begin{aligned}
\mathcal{H}^1\times\mathcal{H}^2 &=\{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T))\mid |\xi_i'|<|\tau_i|/c_i,\, c_1^2\xi_1^n=-c_2^2\xi_2^n\},\\
\mathcal{H}^1\times\mathcal{G}^{2,+} &=\{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T))\mid |\xi_1'|<|\tau_1|/c_1, |\xi_2'|=|\tau_2|/c_2\},\\
\mathcal{H}^1\times\mathcal{E}^2 &=\{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T))\mid |\xi_1'|<|\tau_1|/c_1, |\xi_2'|>|\tau_2|/c_2\},\\
\mathcal{G}^{1,-}\times\mathcal{E}^2 &=\{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T))\mid |\xi_1'|=|\tau_1|/c_1, |\xi_2'|>|\tau_2|/c_2\},\\
\mathcal{E}^1\times\mathcal{E}^2 &=\{(\rho_1,\rho_2)\in T^*(\partial\Omega_2\times(0,T))\mid |\xi_i'|>|\tau_i|/c_i\}.
\end{aligned}$$





Near $\partial\Omega_2$

$$p_i(t,x,\tau_i,\xi_i',\xi_i^n)(\sigma_0) = \tau_i^2 - c_i^2(|\xi_i^n|^2 + r(x_n(\sigma_0),x'(\sigma_0),\xi_i')^2) = 0,$$

$$T^*(\partial\Omega_2 \times \mathbb{R}_t) = (\mathcal{H}_1 \times \mathcal{H}^2) \cup (\mathcal{H}^1 \times \mathcal{G}^{2,+}) \cup (\mathcal{H}^1 \times \mathcal{E}^2) \cup (\mathcal{G}^{1,-} \times \mathcal{E}^2) \cup (\mathcal{E}^1 \times \mathcal{E}^2),$$

where

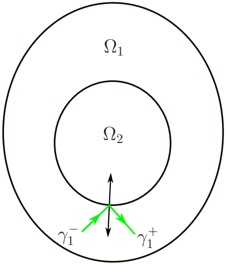
$$\mathcal{H}^1 \times \mathcal{H}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_i'| < |\tau_i|/c_i, c_1^2\xi_1^n = -c_2^2\xi_2^n\},$$

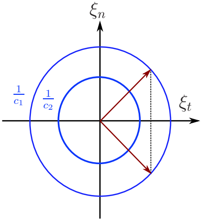
$$\mathcal{H}^1 \times \mathcal{G}^{2,+} = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_1'| < |\tau_1|/c_1, |\xi_2'| = |\tau_2|/c_2\},$$

$$\mathcal{H}^1 \times \mathcal{E}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_1'| < |\tau_1|/c_1, |\xi_2'| > |\tau_2|/c_2\},$$

$$\mathcal{G}^{1,-} \times \mathcal{E}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_1'| = |\tau_1|/c_1, |\xi_2'| > |\tau_2|/c_2\},$$

$$\mathcal{E}^1 \times \mathcal{E}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_i'| > |\tau_i|/c_i\}.$$





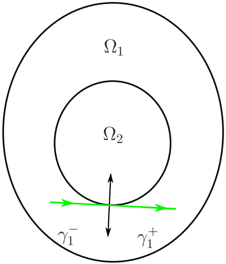
Near $\partial\Omega_2$

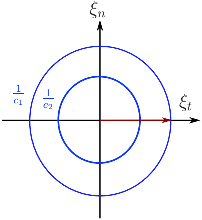
$$p_i(t,x,\tau_i,\xi_i',\xi_i^n)(\sigma_0) = \tau_i^2 - c_i^2(|\xi_i^n|^2 + r(x_n(\sigma_0), x'(\sigma_0), \xi_i')^2) = 0,$$

$$T^*(\partial\Omega_2 \times \mathbb{R}_t) = (\mathcal{H}_1 \times \mathcal{H}^2) \cup (\mathcal{H}^1 \times \mathcal{G}^{2,+}) \cup (\mathcal{H}^1 \times \mathcal{E}^2) \cup (\mathcal{G}^{1,-} \times \mathcal{E}^2) \cup (\mathcal{E}^1 \times \mathcal{E}^2),$$

where

$$\mathcal{H}^1 \times \mathcal{H}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_i'| < |\tau_i|/c_i, c_1^2\xi_1^n = -c_2^2\xi_2^n\},$$
$$\mathcal{H}^1 \times \mathcal{G}^{2,+} = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_1'| < |\tau_1|/c_1, |\xi_2'| = |\tau_2|/c_2\},$$
$$\mathcal{H}^1 \times \mathcal{E}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_1'| < |\tau_1|/c_1, |\xi_2'| > |\tau_2|/c_2\},$$
$$\mathcal{G}^{1,-} \times \mathcal{E}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_1'| = |\tau_1|/c_1, |\xi_2'| > |\tau_2|/c_2\},$$
$$\mathcal{E}^1 \times \mathcal{E}^2 = \{(\rho_1,\rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid |\xi_i'| > |\tau_i|/c_i\}.$$





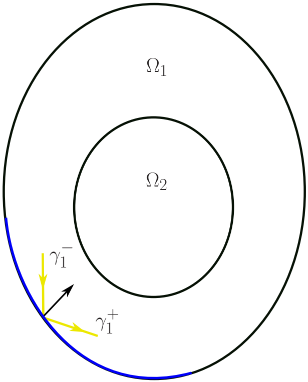
Propagation of the measures on $\partial\Omega$

Let μ_1 and μ_2 be the defect measures attached to u_1 and u_2. Then

Theorem (Bardos, Lebeau, Rauch, '92)

Let $\rho \in \mathcal{H} \cup \mathcal{G}^{2,+}$. Then,

$$\left((\gamma_1^-) \cap supp(\mu_1)\right) = \emptyset \Leftrightarrow \left((\gamma_1^+) \cap supp(\mu_1)\right) = \emptyset.$$



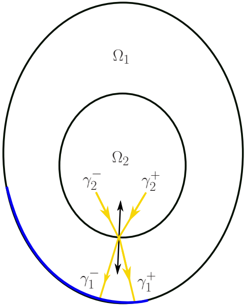
Propagation of the measures on $\partial\Omega_2$

An adaptation of the work of Burq and Lebeau '01 on Lamé system gives

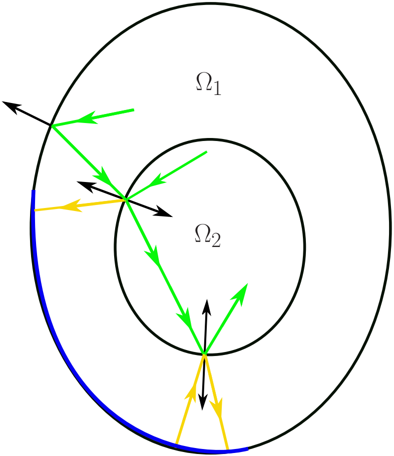
Corollary (G., '17)

For $\rho_2 \in \mathcal{H}^2 \times \mathcal{G}^{2,+}$, *we have the following equivalence*

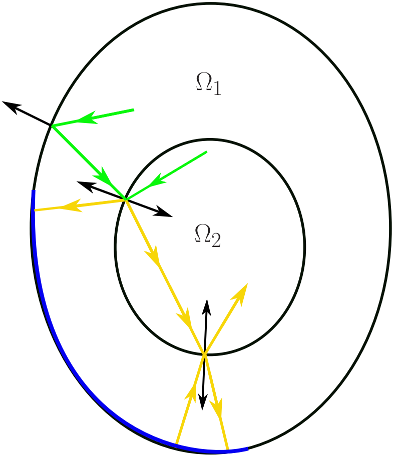
$$\left((\gamma_1^-) \cap supp(\mu_1)\right) \cup \left((\gamma_2^-) \cap supp(\mu_2)\right) = \emptyset$$
$$\Updownarrow$$
$$\left((\gamma_1^+) \cap supp(\mu_1)\right) \cup \left((\gamma_2^+) \cap supp(\mu_2)\right) = \emptyset$$



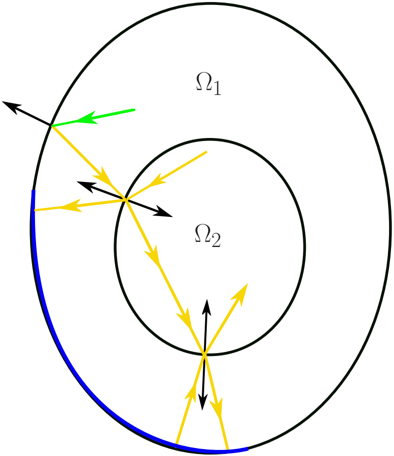
Recursive use of propagation of the measures on $\partial\Omega_2$



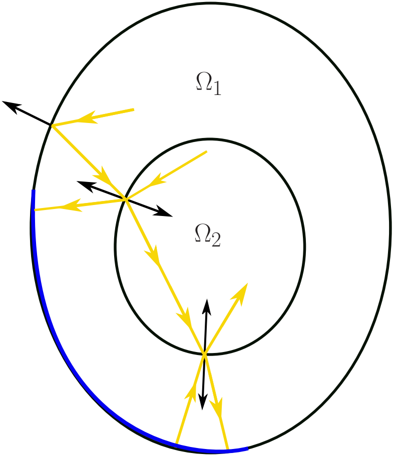
Recursive use of propagation of the measures on $\partial\Omega_2$



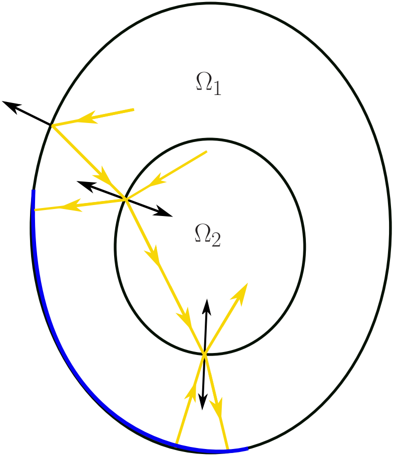
Recursive use of propagation of the measures on $\partial\Omega_2$



Recursive use of propagation of the measures on $\partial\Omega_2$



Recursive use of propagation of the measures on $\partial\Omega_2$



Remarks :

1. GCC alone is not sufficient
2. The propagation results are sufficient for the observability in the case $\Gamma = \partial\Omega$ (noticed by Lebeau, Le Rousseau, Terpolilli and Trélat, '14).

Geometrical construction

For $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$, consider

$$\Gamma(x_0) := \{x \in \partial\Omega \,|\, \langle (x - x_0), n(x)\rangle > 0\}.$$

Step 1 : Propagation of $\Gamma(x_0)$

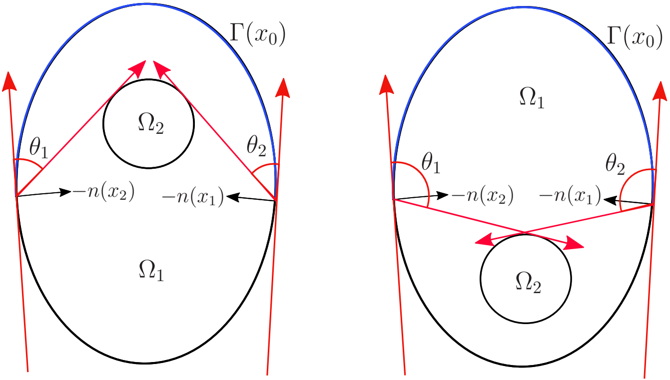


Figure: Left : case where Ω_2 is included in $\Gamma(x_0)$ - right : case where Ω_2 is not included in $\Gamma(x_0)$

Geometrical construction

For $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$, consider

$$\Gamma(x_0) := \{x \in \partial\Omega \,|\, \langle (x - x_0), n(x) \rangle > 0\}.$$

Step 1 : Propagation of $\Gamma(x_0)$

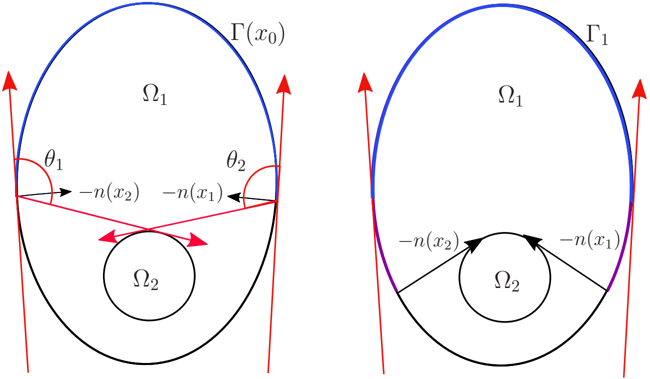


Figure: Left : case where Ω_2 is not included in $\Gamma(x_0)$ - right : Propagation of Γ

Geometrical construction

For $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$, consider

$$\Gamma(x_0) := \{x \in \partial\Omega \,|\, \langle (x - x_0), n(x) \rangle > 0\}.$$

Step 1 : Propagation of $\Gamma(x_0)$

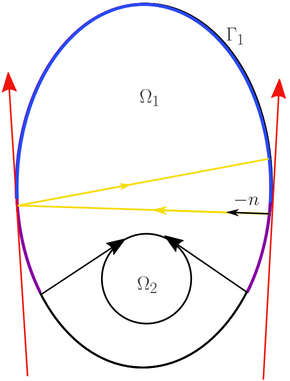


Figure: Propagation of the bicharacteristics in the $-n$ direction

Geometrical construction

For $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$, consider

$$\Gamma(x_0) := \{x \in \partial\Omega \,|\, \langle (x - x_0), n(x) \rangle > 0\}.$$

Step 1 : Propagation of $\Gamma(x_0)$

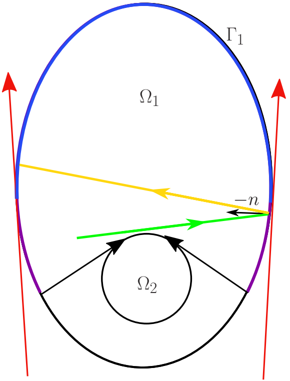
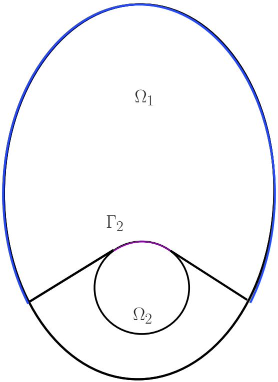


Figure: Confined rays argument

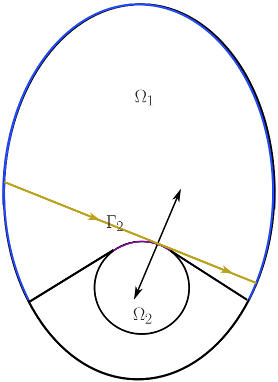
Geometrical construction

Step 2 : Creation of Γ_2



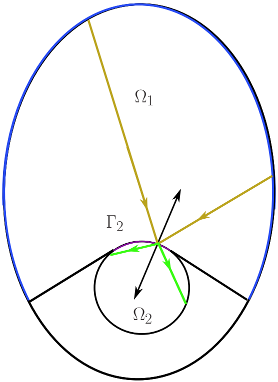
Geometrical construction

Step 2 : Creation of Γ_2



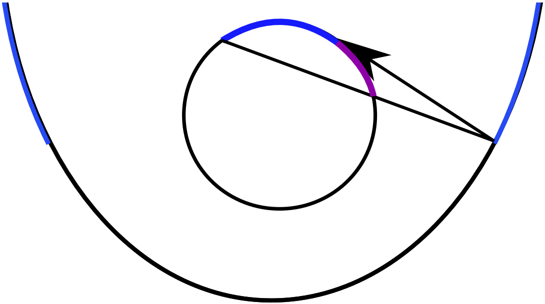
Geometrical construction

Step 2 : Creation of Γ_2



Geometrical construction

Step 3 : Extension of Γ_2



Geometrical construction

Step 3 : Extension of Γ_2

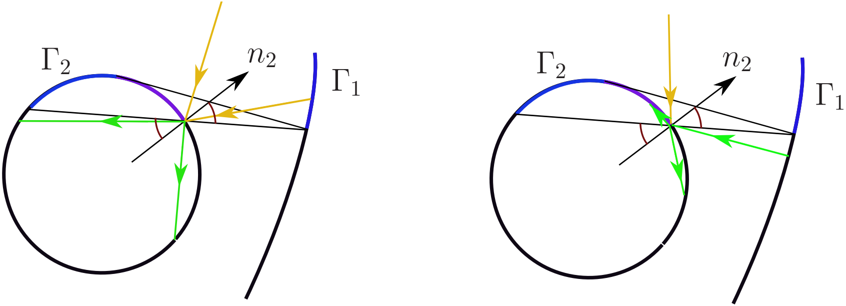


Figure: Left : γ_1^- and γ_1^+ intersects Γ_1. Right : γ_1^- does not intersect Γ_1 but γ_2^+ hits the extension of Γ_2.

Geometrical construction

Step 3 : Extension of Γ_2

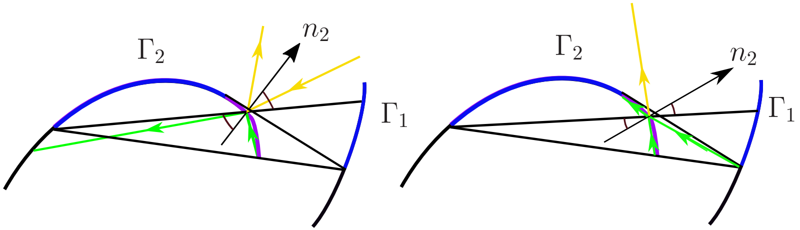
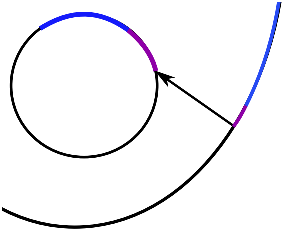


Figure: Left : γ_1^- and γ_1^+ intersects Γ_1. Right : γ_1^- does not intersect Γ_1 but γ_2^+ hits the extension of Γ_2.

In the second case, it may happen that γ_2^+ intersects the purple region once again. One iterates the same argument. We conclude in a finite number of iterations since there are no contact of infinite order of $\partial\Omega_2$ with its tangents.

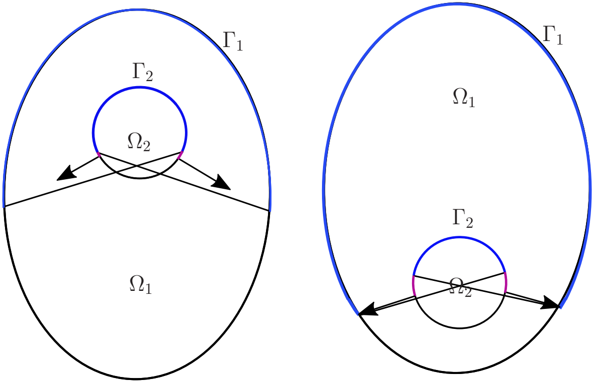
Geometrical construction

Step 4 : extension of Γ_1



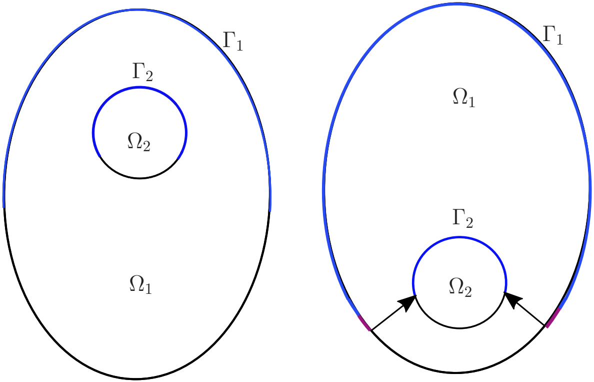
Geometrical construction

Step 5 : Iteration of step 3 and 4



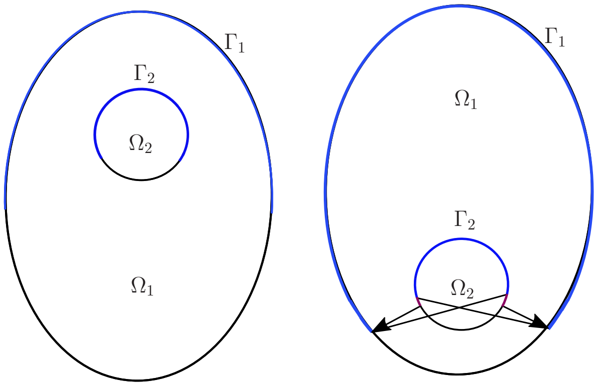
Geometrical construction

Step 5 : Iteration of step 3 and 4



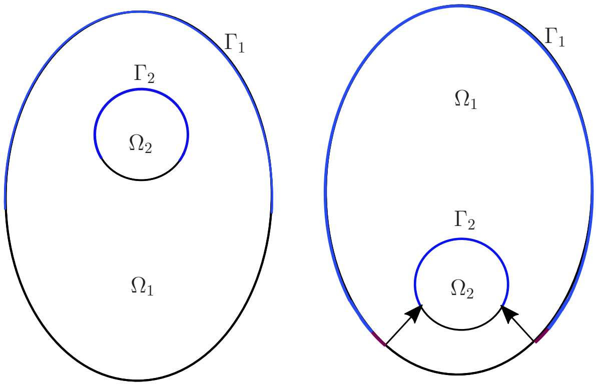
Geometrical construction

Step 5 : Iteration of step 3 and 4



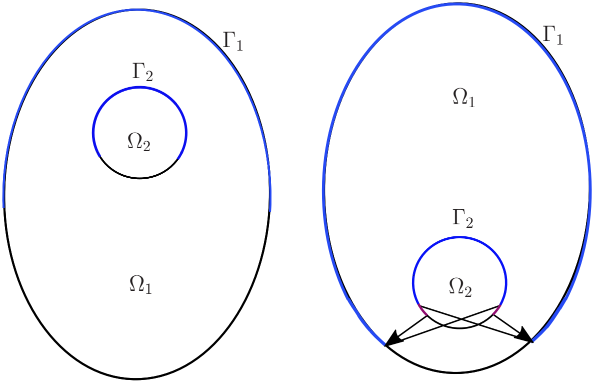
Geometrical construction

Step 5 : Iteration of step 3 and 4



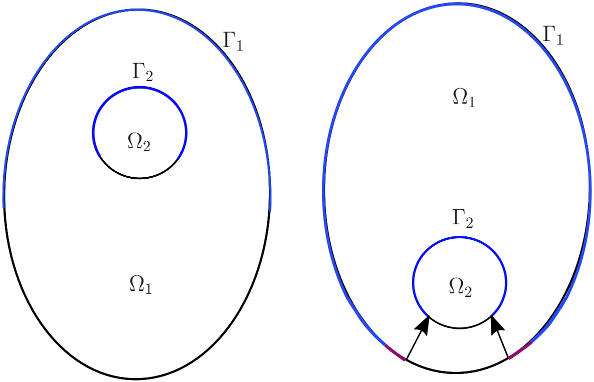
Geometrical construction

Step 5 : Iteration of step 3 and 4



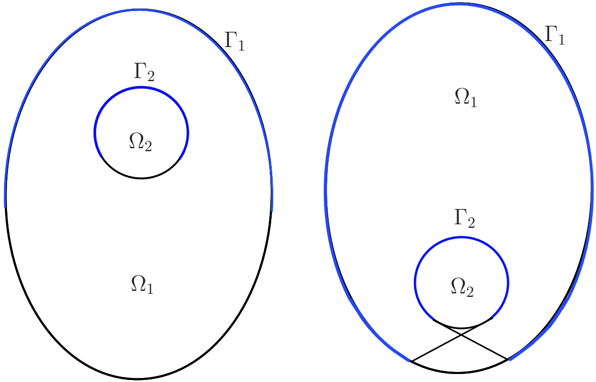
Geometrical construction

Step 5 : Iteration of step 3 and 4



Geometrical construction

Step 5 : Iteration of step 3 and 4



Geometrical construction

Final step : definition of Ω_1^f

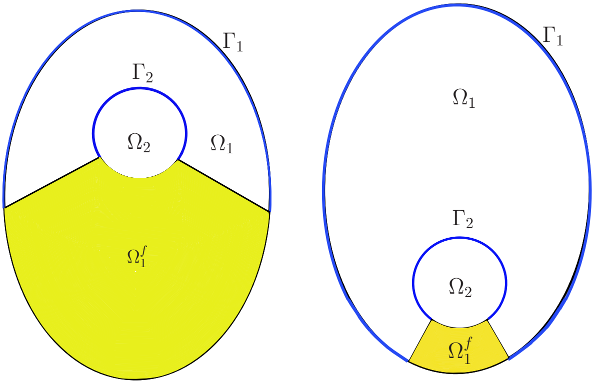


Figure: Definition of Ω_1^f

Uniformly escaping geometry

Collision map :

$$\mathcal{F} : (\partial\Omega \cup \partial\Omega_2) \times \mathbb{R}^2 \longrightarrow (\partial\Omega \cup \partial\Omega_2) \times \mathbb{R}^2, \quad (x,\xi) \longmapsto (x^1,\xi^1).$$

Projections :

$$\Pi_x(x,\xi) = x, \qquad \Pi_\xi(x,\xi) = \xi.$$

Parametrizations :

$$\partial\Omega : \delta(s), s \in [0,1), \qquad \partial\Omega_2 : \delta_2(s), s \in [0,1),$$

Definition (Uniformly escaping geometry)

We say that Ω_1^f is a uniformly escaping geometry if the application

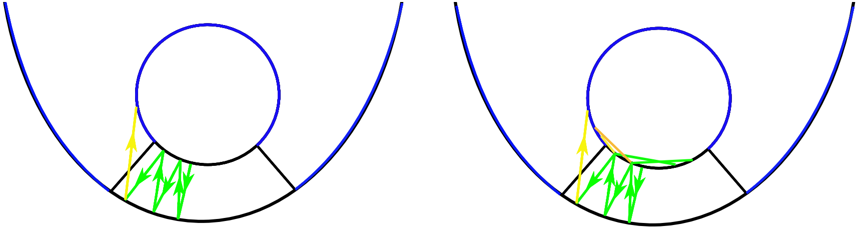
$$\mathcal{M} : (\partial\Omega_2 \setminus \Gamma_2) \times \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x,\xi) \longmapsto \langle \xi, n\left(\Pi_x\left(\mathcal{F}\left(x,\xi\right)\right)\right)^\perp \rangle$$

is nondecreasing for $s \mapsto \mathcal{M}(\delta_2(s), n_2(\delta_2(s))), \delta_2(s) \in \partial\Omega_2 \setminus \Gamma_2$.

Statement of the main results : UEG case

Theorem (G. ,'17)

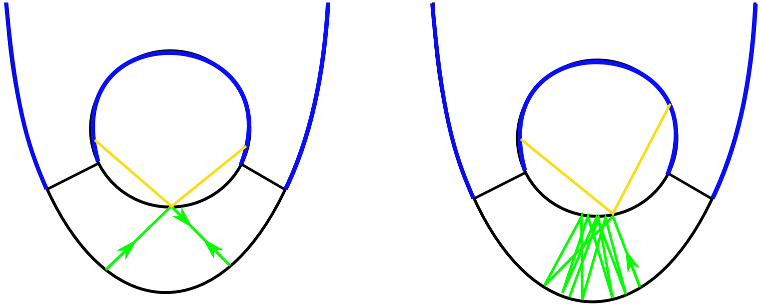
Let $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $\Gamma(x_0)$. Suppose Ω_1^f is a uniformly escaping geometry, then for every $c_2 > c_1$, there exists a time $T > 0$ such that (WT) *is observable.*



Statement of the main results : non-UEG case

Theorem (G. ,'17)

Let $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $\Gamma(x_0)$. Suppose Ω_1^f is not a uniformly escaping geometry but if Ω_1^f and $c_2 > c_1$ are such that for every $x \in \partial\Omega \setminus \Gamma_1$ and $\xi \in \mathbb{R}^2$ such that $\Pi_x(\mathcal{F}(x,\xi)) \in \partial\Omega_2 \setminus \Gamma_2$ and $\Pi_x(\mathcal{F}^2(x,\xi)) \in \partial\Omega \setminus \Gamma_1$, the transmitted rays in Ω_2 are such that $\Pi_x(\mathcal{F}_2^2(x,\xi^{\pm})) \in \Gamma_2$, then (WT) *is observable in some time $T > 0$.*



Remarks on the $\Gamma(x_0)$ condition

Could we just used Γ connected and satisfying GCC instead of $\Gamma(x_0)$?

Remarks on the $\Gamma(x_0)$ condition

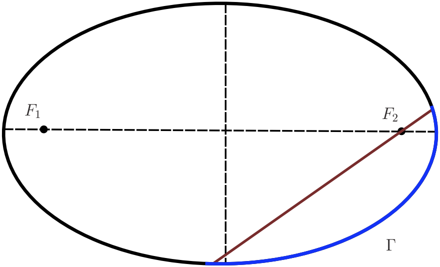
Could we just used Γ connected and satisfying GCC instead of $\Gamma(x_0)$? No.

Remarks on the $\Gamma(x_0)$ condition

Could we just used Γ connected and satisfying GCC instead of $\Gamma(x_0)$? No.

Lemma (G., '17)

There exists strictly convex domains $\Omega \subset \mathbb{R}^2$ and $\Gamma \subset \partial\Omega$ connected such that GCC holds but $\Gamma \subsetneq \Gamma(x_0)$ for any $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$.

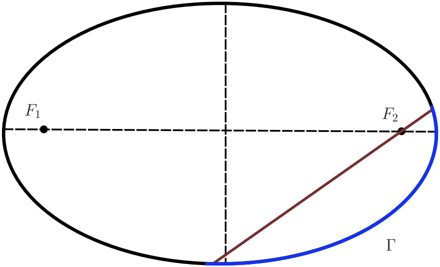


Remarks on the $\Gamma(x_0)$ condition

Could we just used Γ connected and satisfying GCC instead of $\Gamma(x_0)$? No.

Lemma (G., '17)

There exists strictly convex domains $\Omega \subset \mathbb{R}^2$ and $\Gamma \subset \partial\Omega$ connected such that GCC holds but $\Gamma \subsetneq \Gamma(x_0)$ for any $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$.

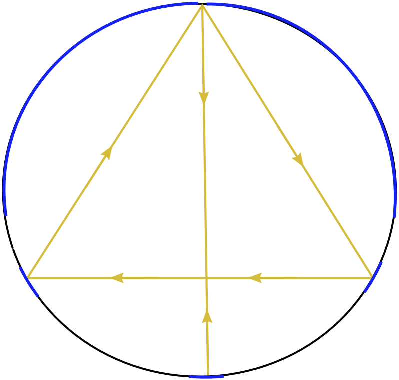


Corollary (G., '17)

There exists strictly convex domains $\Omega \subset \mathbb{R}^2$ and $\Gamma \subset \partial\Omega$ such that Γ and $\partial\Omega \setminus \Gamma$ are observable regions.

Remarks on the $\Gamma(x_0)$ condition

If Ω is convex, then $\Gamma(x_0)$ is connected. An example where GCC holds but is not given by $\Gamma(x_0)$ was given by BLR, '92.



Later on, it was proved

Theorem (Miller, '02)

The $\Gamma(x_0)$ condition implies GCC.

Proof

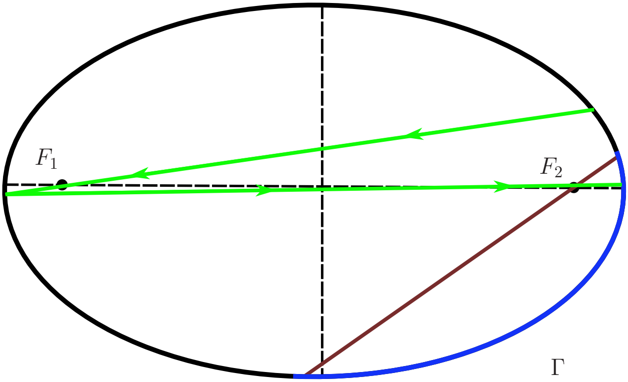


Figure: Convergence to the main axis

Proof

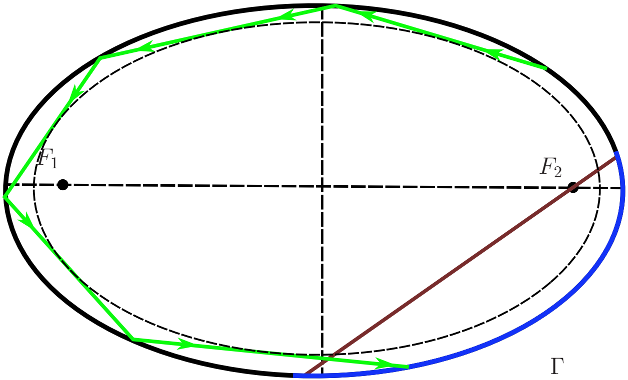


Figure: Elliptic caustic

Proof

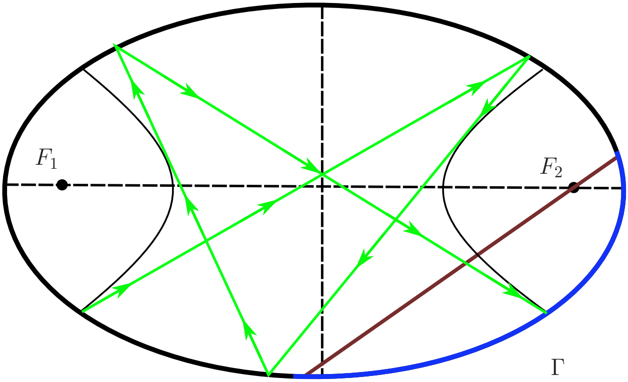


Figure: Hyperbolic caustic

Conclusion

Open problems

1. Higher dimensions
2. Case of the ellipse
3. Case where Γ is not connected

Perspectives

1. Exponential decay with memory term
2. Scattering at infinity

Proof of the propagation of the support of the measures

Let $(\rho_1, \rho_2) \in \mathcal{H}^1 \times (\mathcal{H}^2 \cup \mathcal{G}^{2,+})$. Since $\rho_1 \in \mathcal{H}^1$, we write

$$P_i = -c_i^2(D_x - \Lambda^+(x,y,D_y))(D_x - \Lambda^-(x,y,D_y)) + T(x,y,D_y)$$
$$P_i = -c_i^2(D_x - \widetilde{\Lambda}^-(x,y,D_y))(D_x - \widetilde{\Lambda}^+(x,y,D_y)) + \widetilde{T}(x,y,D_y)$$

$\Lambda^\pm$ and $\widetilde{\Lambda}^\pm$ are tangential pseudodifferential operators of order 1 and such that $\sigma_1(\Lambda^\pm) = \pm\sqrt{r}$ and $\sigma_1(\widetilde{\Lambda}^\pm) = \pm\sqrt{r}$ and where T and $\widetilde{T}$ are tangential pseudodifferential operators of order $-\infty$. Suppose $\gamma_1^+ \cup \mathrm{supp}(\mu_1) = \emptyset$,

$$c_1^2 Q_0(D_x u_1^k|_{x=0} - \Lambda^- u_1^k|_{x=0}) \to 0 \text{ in } L^2(Y).$$

Using the transmission conditions,

$$c_1^2 Q_0 \left(\frac{D_x u_2^k}{c_2^2}\bigg|_{x=0} - \Lambda^- u_2^k|_{x=0} \right) \to 0 \text{ in } L^2(Y).$$

Suppose $\gamma_2^- \cup \mathrm{supp}(\mu_2) = \emptyset$.

1. If $\rho_2 \in H^2$, we get the additional relation

$$Q_0 \left(D_x u_2^k|_{x=0} - c_2^2 \Lambda^- u_2^k|_{x=0} \right) \to 0 \text{ in } L^2(Y).$$

   which allow us to conclude.

2. If $\rho_2 \in \mathcal{G}^{2,+}$, then we use

Lemma

$\rho_2 \in supp(\mu_2) \Leftrightarrow \rho_2 \in supp(meas(u_{2\,|x=0})) \cup supp(meas(D_{n_2} u_2|_{x=0}))$.