Sharp control time of exact boundary controllability of 1-D coupled hyperbolic systems

Long Hu

Joint with Guillaume Olive

1School of Mathematics, Shandong University
2Institute of Mathematics, Jagiellonian University

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Outline

1. Introduction

2. Exact boundary controllability of linear hyperbolic conservation laws $M = 0$

3. Exact boundary controllability of linear hyperbolic balance laws $M \neq 0$

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Outline

1 Introduction

2 Exact boundary controllability of linear hyperbolic conservation laws $M = 0$

3 Exact boundary controllability of linear hyperbolic balance laws $M \neq 0$

4 Concluding remarks
Let $T > 0$ fixed. Our system considered here is

$$ \frac{\partial y}{\partial t}(t, x) = \Lambda(x) \frac{\partial y}{\partial x}(t, x) + M(x)y(t, x), \quad t \in (0, T), \ x \in (0, 1), \quad (1.1) $$

where,

- $y = (y_1, \cdots, y_n)^T$ is a vector function of $(t, x)$;
- $M(\cdot) \in L^\infty((0, 1); \mathbb{R}^{n \times n})$ with entries $m_{ij}(x) \ (1 \leq i, j \leq n)$;
- $\Lambda(\cdot) \in C^{0,1}([0, 1]; \mathbb{R}^{n \times n})$ is diagonal with $p \geq 1$ distinct negative eigenvalues and $m := n - p$ distinct positive eigenvalues. Therefore, we assume that

$$ \Lambda(x) = \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix}, \quad (1.2) $$

where $\Lambda_+(x) = \text{diag}(\lambda_1(x), \cdots, \lambda_p(x))$ and $\Lambda_-(x) = \text{diag}(\lambda_{p+1}(x), \cdots, \lambda_n(x))$ are diagonal submatrices satisfying

$$ \lambda_1(x) < \cdots < \lambda_p(x) < 0 < \lambda_{p+1}(x) < \cdots < \lambda_{p+m}(x), \quad \forall x \in [0, 1]. \quad (1.3) $$
The boundary conditions on $x = 0$ and $x = 1$ are given by:

\[
y_+(t, 0) = Qy_-(t, 0), \quad y_-(t, 1) = u(t), \quad \forall 0 < t < T,
\]

in which $Q \in \mathbb{R}^{p \times m}$ is a constant matrix, $u \in L^2((0, T); \mathbb{R}^m)$ is chosen as boundary control function.

**Remark**

Such linear coupled system involves many physical models of balance laws (without vanishing characteristic speeds):

- Telegrapher equations (Heaviside, O. (1892)); $2 \times 2$ system
- (Linearized) Saint-Venant equations (Barré de Saint-Venant (1871)); $2 \times 2$ system
- (Linearized) Saint-Venant-Exner equations (Exner(1920,1925); Hudson-Sweby (2003)); $3 \times 3$ system
- (Linearized) Heat-exchangers (Allievi(1903); G.Bastin & J.-M.Coron (2016)); $6 \times 6$ system;
- ...
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An excellent book:
Problem Description

Exact boundary controllability problem in time $T$

Let $T > 0$. For any given $\varphi, \psi \in L^2((0, 1); \mathbb{R}^n)$. Does there exist boundary control $u \in L^2((0, T); \mathbb{R}^m)$ such that the solution of the control system (1.1) satisfies

$$y(0, x) = \varphi(x), \quad (1.5)$$
$$y(T, x) = \psi(x)? \quad (1.6)$$

Remark

- $T$ is what we call the control time. ((1.1) is not exact controllable if $T = +\infty$);
- Null boundary controllability: (1.6) is replaced by
  $$y(T, x) = 0, \ 0 < x < 1;$$
- Approximate boundary controllability: (1.6) is replaced by
  $$\|y(T, x) - \psi(x)\|_{L^2(0,1)} \leq \varepsilon, \ \forall \varepsilon > 0.$$
State of the art

- D. L. Russell (1978): First order 1-D linear hyperbolic systems:
  \[ y_t = \Lambda(x)y_x + M(x)y. \]

- J.L. Lions (1988): Linear n-D \((n \geq 1)\) wave equation:
  \[ y_{tt} - \Delta y = 0. \]

- E. Zuazua (1990, 1993): Semilinear wave equation:
  \[ y_{tt} - \Delta y = f(y). \]

  \[ y_t = A(y)y_x + M(y). \]
State of the art

Complements


For the quasilinear hyperbolic system

\[ y_t = A(y)y_x + M(y), \]

with \( A(0) = \text{diag}(\lambda_1(0), \ldots, \lambda_{p+m}(0)) \) satisfying the order relation

\[ \lambda_1(0) < \cdots < \lambda_p(0) < 0 < \lambda_{p+1}(0) < \cdots < \lambda_{p+m}(0) \]

and with the nonlinear boundary conditions

\[ x = 0 : y_s = G_s(t, y_{p+1}, \cdots, y_{p+m}) \quad (s = 1, \cdots, p), \tag{1.7} \]
\[ x = 1 : y_r = G_r(t, y_1, \cdots, y_p) + u_r(t) \quad (r = p + 1, \cdots, p + m), \]

in which \( G_i(t, 0, \cdots, 0) \equiv 0 \quad (i = 1, \cdots, n). \)
One can realize the (local) exact boundary controllability by means of $u$ under the framework of $C^1$ norm if

- $p \leq m$;

$$T > T_1 := \frac{1}{|\lambda_p(0)|} + \frac{1}{\lambda_{p+1}(0)};$$

- On the non-control side $x = 0$ in a neighborhood of $y = 0$ satisfies

$$y_s = G_s(t, y_{p+1}, \ldots, y_{p+m}) \quad (s = 1, \ldots, p)$$
$$\Leftrightarrow y_{m+\bar{r}} = \overline{G}_{m+\bar{r}}(t, y_1, \ldots, y_p, y_{p+1}, \ldots, y_m) \quad (\bar{r} = 1, \ldots, p).$$

### Remark

- D.L. Russell (1978) has already given the same result under the case $m = p$ for the linear case on the exact controllability.
Remark continued

Coupling conditions (1.9) on the non-control side $x = 0$ in our linear case considered here means

the $p \times p$ matrix formed from the last $p$ rows

and the last $p$ columns of $Q$ is invertible, (1.10)

which can be replaced by a little weaker assumption (in particular for the case $m > p$):

$$\text{rank } Q = p.$$ (1.11)
L. Hu (2015)

For the case $F(y) = 0$ i.e.

$$\frac{\partial y}{\partial t} = A(y) \frac{\partial y}{\partial x} \quad (t, x) \in [0, T] \times [0, 1],$$

the exact control time of the system (1.12) can be improved as

$$T > T_2 := \max \left( \frac{1}{|\lambda_p(0)|} + \frac{1}{\lambda_{m+1}(0)}, \frac{1}{|\lambda_{p+1}(0)|} \right). \quad (1.13)$$

Moreover, for the linear case, there exists $Q$ with $\text{rank}(Q) = p$, such that the system (1.12) is not exactly controllable in time $T < T_2$.

Remark

$T_1 \geq T_2$ if $m \geq p$ and $T_1 > T_2$ if $m > p$
An extra recent known result

With an extra conditions on $Q$, i.e. $\forall i \in \{1, \ldots, p\}$:

the $i \times i$ matrix formed from last $i$ rows

and the last $i$ columns of $Q$ is invertible. \hfill (1.14)


- If $m = 1$, (1.1) is exactly controllable in time $T \geq T_3$. where

$$T_3 = \max_{i \in \{1, \ldots, p\}} (T_i(\Lambda) + T_{m+i}(\Lambda), T_{p+1}(\Lambda)).$$ \hfill (1.15)

with $T_i(\Lambda) := \int_0^1 \frac{1}{|\lambda_i(\xi)|} d\xi$, $i = 1, \ldots, n$;

- If $m \geq 2$, $m \geq p$, (1.1) is “almost” (except a countable number of $M$) exactly controllable in time $T \geq T_3$;

- If $p = 1$ or 2, $m = 2$, $\Lambda$ is constant, $M$ is analytic in a neighborhood of a closed subinterval of $[0, 1]$, $Q_{p1} \neq 0$, then (1.1) is exactly controllable in time $T > T_3$;

- If $M = 0$, (1.1) is not exactly controllable in time $T$ for every $T < T_3$ and every $Q$ with (1.14).
From now on, let us keep in mind the order of the relation

\[
\begin{cases}
0 < T_1(\Lambda) < \ldots < T_p(\Lambda), \\
0 < T_{p+m}(\Lambda) < \ldots < T_{p+1}(\Lambda),
\end{cases}
\]

then one easily sees that

\[ T_3 \leq T_2 \leq T_1 \]  \quad (1.17)

Moreover,

\[ T_2 < T_1, \text{ if } m > p, \]  \quad (1.18)

and

\[ T_3 < T_2, \text{ if } m \geq p > 1. \]  \quad (1.19)

In fact, J.-M. Coron & H.-M. Nguyen (2018) was focused on null-controllability, the exact controllability results are the direct consequence of it.
In general, better structure conditions on $Q$ imply a better time for the exact controllability of linear hyperbolic systems. However, we do not know

- what happens if such structure conditions on $Q$ does not satisfy (especially only assume $\text{rank}(Q) = p$);
- what happens just before the control time (except the particular cases of systems in Hu 2015 and J.-M. Coron & H.-M. Nguyen 2018 on $M = 0$);
- what happens on the general hyperbolic balance laws.

**Question**

Can we get the critical control time for linear hyperbolic system $T_c$ such that the exact controllability holds if $T \geq T_c$ and does not hold if $T < T_c$?

- D. L. Russell (1978) already raised this question in his famous survey paper (for the null controllability setting and for the homogeneous case).
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Definition of canonical form matrix (LH & G. Olive (2019), see also Dopico & Johnson & Molera (2006))

We say that a matrix $Q^0 \in \mathbb{R}^{p \times m}$ is in canonical form if there exist distinct column indices $c_1(Q^0), \ldots, c_p(Q^0) \in \{1, \ldots, m\}$ such that:

$$
\forall i \in \{1, \ldots, p\}, \quad \begin{cases} 
q^0_{i, c_i(Q^0)} \neq 0, \\
q^0_{i,j} = 0, & \forall j > c_i(Q^0), \quad j \not\in \{c_{i+1}(Q^0), \ldots, c_p(Q^0)\}, \\
q^0_{i,j} = 0, & \forall j < c_i(Q^0).
\end{cases}
$$

(2.1)
Some examples:

\[ Q_0^1 = \begin{pmatrix} 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_0^2 = \begin{pmatrix} 0 & 0 & 4 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q_0^3 = \begin{pmatrix} 1 & 4 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \]

The matrices \( Q_0^1 \) and \( Q_0^2 \) are both in canonical form, with \( c_3(Q_0^1) = 4, \ c_2(Q_0^1) = 3, \ c_1(Q_0^1) = 2 \) and \( c_3(Q_0^2) = 2, \ c_2(Q_0^2) = 1, \ c_1(Q_0^2) = 3 \). However, \( Q_0^3 \) is not in canonical form because there is no \( c_3(Q_0^3) \) that simultaneously satisfies the second and third conditions of (2.1).
If $Q^0 \in \mathbb{R}^{p \times m}$ is in canonical form, then necessarily:

- The indices $c_1(Q^0), \ldots, c_p(Q^0)$ are unique.
- $q^0_{i,j} = 0$ for every $i \in \{1, \ldots, p\}$ and $j \notin \{c_1(Q^0), \ldots, c_p(Q^0)\}$.
- $\text{rank} \ Q^0 = p$.
- $q^0_{k,c_i(Q^0)} = 0$, $\forall k > i$, $\forall i \in \{1, \ldots, p\}$. 
Proposition: canonical $UL$-decomposition (LH & G.Olive 2019)

Let $Q \in \mathbb{R}^{p \times m}$ with rank $Q = p$. Then, there exists a unique $Q^0 \in \mathbb{R}^{p \times m}$ such that the following two properties hold:

- There exists $L \in \mathbb{R}^{m \times m}$ such that $QL = Q^0$ with $L$ lower triangular ($\ell_{ij} = 0$ if $i < j$) and with only ones on its diagonal ($\ell_{ii} = 1$ for every $i$).
- $Q^0$ is in canonical form.

We call $Q^0$ the canonical form of $Q$.

Definition-canonical form of full low rank matrix

Let $Q \in \mathbb{R}^{p \times m}$ with rank $Q = p$. We define $c_1(Q), \ldots, c_p(Q) \in \{1, \ldots, m\}$ by

$$c_i(Q) = c_i(Q^0),$$

where $Q^0$ is the canonical form of $Q$ defined by above Proposition.
Preliminary: A toy example

Let \( Q_1 = \begin{pmatrix} 4 & 6 & 3 & -1 \\ 8 & -1 & 5 & 3 \\ 2 & -1 & 1 & 1 \end{pmatrix} \). We look at the last row, take the last nonzero entry as pivot and do the column substitutions \( C_3 \leftarrow C_3 - C_4 \), \( C_2 \leftarrow C_2 + C_4 \) and \( C_1 \leftarrow C_1 - 2C_4 \), so that

\[
Q_1 \leftarrow \begin{pmatrix} 6 & 5 & 4 & -1 \\ 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
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$Q_1 \leftarrow \begin{pmatrix} 6 & 5 & 4 & -1 \\ 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} 2 & 1 & 4 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
Preliminary: A toy example

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\[
Q_1 \leftarrow \begin{pmatrix} 6 & 5 & 4 & -1 \\ 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} 2 & 1 & 4 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = Q_1^0
\]
Main results in decoupled case

After such a long but necessary preparation, we now state our main result on the linear hyperbolic conservation laws

**Theorem on $M = 0$ (LH & G. Olive (2019))**

Let $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ satisfy (1.2) and (1.3), and $Q \in \mathbb{R}^{p \times m}$ be fixed. For every $T > 0$, (1.1) with $M = 0$ is exactly controllable in time $T$ if, and only if, the following two properties hold:

(i) $\text{rank } Q = p$.

(ii) $T \geq \max_{i \in \{1, \ldots, p\}} (T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda))$.

The proof of this theorem is based on the characteristic methods + canonical $UL$-decomposition for the full low rank matrix $Q$ + careful use of the order relation between $T_i(\Lambda)$. 
Brief idea on proof on toy example

\[
\begin{cases}
  y_{1t} + y_{1x} = 0, \\
  y_{2t} - \frac{1}{2} y_{2x} = 0, \\
  y_{3t} - y_{3x} = 0,
\end{cases}
\]  

(2.2)

and the boundary conditions

\[
x = 0 : y_1 = ay_2 + by_3, \\
x = 1 : y_2 = u(t); y_3 = v(t).
\]

where \(u\) and \(v\) are considered as the control functions

Let \(Q = (a, b)\). Then

- If \(\text{rank}(Q) = 0\) (i.e. \(a = b = 0\)), (2.2) is not exact controllable.
- If \(\text{rank}(Q) = 1\), (2.2) is exact controllable if and only if

\[
\begin{cases}
  T \geq 3, \text{if } b = 0, \text{ with } Q^0 = (a, 0), L = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \forall c \in \mathbb{R}, \\
  T \geq 2, \text{if } b \neq 0, \text{ with } Q^0 = (0, b), L = \begin{pmatrix} 1 & 0 \\ -a/b & 1 \end{pmatrix}.
\end{cases}
\]
Main results in decoupled case

Remarks

- The previous Theorem implies that $T_c$ for the linear conservation laws is

\[
T_c = \begin{cases} 
  \max_{i \in \{1, \ldots, p\}} \left( T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda) \right), & \text{if } \text{rank}(Q) = p; \\
  +\infty, & \text{otherwise.}
\end{cases}
\]

we denote it by $T_c(\Lambda, Q)$ hereafter;

- If $\text{rank}(Q) = p$, then

\[
\max_{(c_1, \ldots, c_p) \in \{1, \ldots, m\}} \text{subject to } c_j \neq c_k, j \neq k \}
T_c(\Lambda, Q) = T_1,
\]

the maximum is reached with $c_p(Q) = 1$.

- If $\text{rank}(Q) = p$, then

\[
\min_{(c_1, \ldots, c_p) \in \{1, \ldots, m\}} \text{subject to } c_j \neq c_k, j \neq k \}
T_c(\Lambda, Q) = T_3,
\]

the minimum is reached if $Q$ satisfy (1.14) i.e. $c_i(Q) = m - p + i \forall 1 \leq i \leq p$. 

L. Hu (Shandong Univ.)
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Counterexample

We want to extend our $T_c(\Lambda, Q)$ for the conservation laws to the general hyperbolic balance laws ($M \neq 0$) we denote as $T_c(\Lambda, M, Q)$, obviously, we have

$$T_c(\Lambda, 0, Q) = T_c(\Lambda, Q).$$

However, some counterexamples show us these two quantities may not agree in principle.


There exists a nontrivial hyperbolic balance laws

$$y_t = \Lambda(x)y_x + \Sigma(x)y$$

(\Sigma \neq 0), such that $T = T_3$ (which is $T_c(\Lambda, Q)$ if we assume (1.14)) is not enough to guarantee the null-controllability (and exact-controollability) in the framework of $L^2$ norm, which, however, is not the case if $\Sigma \equiv 0$. 
Thanks to these counterexamples, we loose our requirement a little bit. We denote by $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$ the sharp (minimal, infimal) time for the exact controllability of balance laws (1.1), that is

$$T_{\text{inf}}(\Lambda, M, Q) = \inf \{T > 0, \ (1.1) \text{ is exactly controllable in time } T\} . \quad (3.2)$$

Obviously, $T_{\text{inf}}(\Lambda, M, Q)$ is unique and satisfy

- If $T > T_{\text{inf}}(\Lambda, M, Q)$, then (1.1) is exactly controllable in time $T$.
- If $T < T_{\text{inf}}(\Lambda, M, Q)$, then (1.1) is not exactly controllable in time $T$.
- $T_{\text{inf}}(\Lambda, 0, Q) = T_c(\Lambda, Q)$.
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Obviously, $T_{\text{inf}}(\Lambda, M, Q)$ is unique and satisfy

- If $T > T_{\text{inf}}(\Lambda, M, Q)$, then (1.1) is exactly controllable in time $T$.
- If $T < T_{\text{inf}}(\Lambda, M, Q)$, then (1.1) is not exactly controllable in time $T$.
- $T_{\text{inf}}(\Lambda, 0, Q) = T_c(\Lambda, Q)$.

**Question:** What is $T_{\text{inf}}(\Lambda, M, Q)$?
Main results on $M \neq 0$

**Theorem on $M \neq 0$ (LH & G. Olive (2019))**

Let $\Lambda \in C^{0,1}([0, 1])^{n \times n}$ satisfy (1.2) and (1.3), $M \in L^\infty(0, 1)^{n \times n}$ and $Q \in \mathbb{R}^{p \times m}$ be fixed. We have:

$$T_{\text{inf}}(\Lambda, M, Q) = T_{\text{inf}}(\Lambda, 0, Q) = T_c(\Lambda, Q).$$

(3.3)
Remark

Observe that the expression (3.3) of $T_{\text{inf}} (\Lambda, M, Q)$ does not depend on $M$. This means that the internal coupling terms $M(x)y(t, x)$ in (1.1) have almost no impact on the controllability properties of this system. All our attention should then be on the coupling on the boundary $Q$ and the decoupled system.
Sketch of the Proof

1) We rewrite (1.1) as an equivalent abstract evolution system:

\[
\begin{aligned}
\frac{d}{dt} y(t) &= A_M y(t) + B u(t), \quad t \in (0, T), \\
y(0) &= y^0,
\end{aligned}
\]  

(also denoted by \((A_M, B)\)), in which

- the unbounded linear operator \(A_M : D(A_M) \subset L^2(0, 1)^n \rightarrow L^2(0, 1)^n\) is defined, for every \(y \in D(A_M)\) by

\[
A_M y(x) = \Lambda(x) \frac{\partial y}{\partial x}(x) + M(x)y(x), \quad x \in (0, 1),
\]

with domain

\[
D(A_M) = \{ y \in H^1(0, 1)^n, \quad y_+(0) = Q y_-(0), \quad y_-(1) = 0 \}.
\]

- the control operator \(B \in \mathcal{L}(\mathbb{R}^m, D(A_M^*)')\) is given for every \(u \in \mathbb{R}^m\) and \(z \in D(A_M^*)\) by

\[
\langle Bu, z \rangle_{D(A_M^*)', D(A_M^*)} = u \cdot \Lambda_-(1) z_-(1).
\]
with

\[ D(A_M^*) = \{ z \in H^1(0, 1)^n, \quad z_+(1) = 0, \quad z_-(0) = R^* z_+(0) \} , \]

where \( R := -\Lambda_+(0)Q\Lambda_-(0)^{-1} \in \mathbb{R}^{p \times m} \), and for every \( z \in D(A_M^*) \),

\[ A_M^* z(x) = -\Lambda(x) \frac{\partial z}{\partial x}(x) + \left( -\frac{\partial \Lambda}{\partial x}(x) + M(x)^* \right) z(x), \quad x \in (0, 1). \quad (3.6) \]

One can see that the system \((A_M, B)\) is well defined in \( C^0([0, T]; L^2(0, 1)^n) \) and

\[ y(t) = S_{A_M}(t)y^0 + \Phi_M(t)u, \quad \forall t \geq 0, \quad (3.7) \]

where \( S_{A_M}(t)(t \geq 0) \) is the \( C_0 \) semigroup with the generator \( A_M \), \( \Phi_M(t) \in \mathcal{L}(L^2(0, +\infty; \mathbb{R}^m)) \), \( D(A_M^*)' \) is the so-called input map of \((A_M, B)\) (see M. Tucsnak & G. Weiss (2009)) i.e.

\[ \Phi_M(t)u = \int_0^t S_{A_M}(t - s)Bu(s) \, ds, \quad \forall u \in L^2(0, +\infty; \mathbb{R}^m). \]
2) Compactness-uniqueness arguments: A flexible method to analysis the exact controllability of partial differential equations:

- J. Rauch & M. Taylor (1974): Exponential decay of some hyperbolic equations;
- J.L. Lions (1988): Plate equation;
- .....
Sketch of the Proof

Here we use an abstract version on it.

**Lemma 1** (M. Duprez & G. Olive (2018) and LH & G. Olive (2019))

Let $H$ and $U$ be two Hilbert spaces. Let $A_1 : D(A_1) \subset H \rightarrow H$ be the generator of a $C_0$-semigroup on $H$ and let $B \in \mathcal{L}(U, D(A_1^*))'$ be admissible for $A_1$. Let $P \in \mathcal{L}(H)$ be a bounded operator and $A_2 = A_1 + P$ with $D(A_2) = D(A_1)$. For $i = 1, 2$, let $\Phi_i(T) \in \mathcal{L}(L^2(0, +\infty; U), H)$ be the input map of $(A_i, B)$ at time $T \geq 0$, and let

$$T_{\inf}(A_i, B) = \inf \{ T > 0, \ (A_i, B) \text{ is exactly controllable in time } T \} \in [0, +\infty].$$

We assume that:

(i) For $i = 1, 2$, $(A_i, B)$ satisfies the Fattorini-Hautus test, i.e.

$$\ker(\lambda - A_i^*) \cap \ker B^* = \{0\}, \ \forall \lambda \in \mathbb{C}. \quad (3.8)$$

(ii) $\Phi_1(T)^* - \Phi_2(T)^*$ is compact for every $T > 0$.

Then, we have $T_{\inf}(A_2, B) = T_{\inf}(A_1, B)$. 
sketch of proof of the Lemma 1

Let $T_1 > 0$ be such that $(A_1, B)$ is exactly controllable in time $T_1$ and we only show $T_{\text{inf}} (A_2, B) \leq T_1$. By assumption and duality there exists $C > 0$ such that, for every $z^1 \in H$,

$$\|z^1\|_H^2 \leq C \int_0^{T_1} \|\Phi_1(T_1)^* z^1(t)\|^2_U \, dt,$$

so that, we have the following inequality with compact remainder

$$\|z^1\|_H^2 \leq 2C \left( \int_0^{T_1} \|\Phi_2(T_1)^* z^1(t)\|^2_U \, dt + \int_0^{T_1} \| (\Phi_1(T_1)^* - \Phi_2(T_1)^*) z^1(t)\|^2_U \, dt \right)$$

Since $\Phi_1(T_1)^* - \Phi_2(T_1)^*$ is assumed to be compact and that $(A_2, B)$ satisfies the Fattorini-Hautus test. Therefore, we can apply theorem in M. Duprez & G. Olive (2018) and obtain that $(A_2, B)$ is exactly controllable in time $T_1 + \varepsilon (\forall \varepsilon > 0)$. 
3) Let

\[ A_1 = A_{\partial \Lambda \partial x}, \quad A_2 = A_M, \quad P = M - \frac{\partial \Lambda}{\partial x}. \]

Once the assumptions of this Lemma has been checked, we obtain

\[ T_{\inf} (\Lambda, M, Q) = T_{\inf} \left( \Lambda, \frac{\partial \Lambda}{\partial x}, Q \right) = T_{\inf} (\Lambda, 0, Q). \]

We find that (i) of the lemma is easy to be checked. Let \( \lambda \in \mathbb{C} \) and \( z \in D(A_M^*) \) be such that \( A_M^* z = \lambda z \) and \( B^* z = 0 \). Thus, \( z \in H^1(0,1)^n \) solves the system of O.D.E.

\[
\begin{cases}
\frac{\partial z}{\partial x}(x) = -\Lambda(x)^{-1} \left( \lambda \text{Id}_{\mathbb{R}^{n \times n}} + \frac{\partial \Lambda}{\partial x}(x) - M(x)^* \right) z(x), \quad x \in (0,1), \\
z(1) = 0,
\end{cases}
\]

so that \( z = 0 \) by uniqueness.
Sketch of the Proof

4) HOWEVER, to check (ii) is never straightforward (for us) (see remark below for the straightforward answer). We develop the following lemma to check (ii)

**Lemma 2 (LH & G.Olive (2019))**

For every $T > 0$, there exist a Hilbert space $\tilde{H}$, a compact operator $F \in \mathcal{L}(H, \tilde{H})$ and $C > 0$ such that

$$
\int_0^T \left\| B^* V \tilde{z}(t) \right\|^2_U dt + \int_0^T \left\| V \tilde{z}(t) \right\|^2_H dt \leq C \left\| Fz^0 \right\|^2_{\tilde{H}}, \quad \forall z^0 \in D(A_1^*),
$$

where $V \tilde{z}(t) = \int_0^t S_{A_1}(t-s)^* P S_{A_1}(s)^* z^0 ds$. Then $\Phi_1(T)^* - \Phi_2(T)^*$ is compact for every $T > 0$.

**Remark**

D.L. Russell (1978, p. 657) said (ii) is true. “A somewhat involved, but not conceptually difficult, argument allows one to see that the operator differences $S^* - S_d^*$, $C^* - C_d^*$ are both compact”, where $C^* - C_d^*$ corresponds to $\Phi_M(T)^* - \Phi_{M_d}(T)^*$ in our notation, where $M_d$ denotes the diagonal part of $M$. 
Sketch of the Proof

**sketch of proof of the Lemma 2**

∀\(z^0 \in D(A_1^*) = D(A_2^*)\), we have

\[(\Phi_1(T)^* - \Phi_2(T)^*)z^0(t) = B^* S_{A_1}(T-t)^* z^0 - B^* S_{A_2}(T-t)^* z^0 \text{ for a.e. } t \in (0, T)\]

Therefore, we can apply \(B^*\) to obtain the following identity:

\[
B^* S_{A_1}(t)^* z^0 - B^* S_{A_2}(t)^* z^0 = -B^* \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds \\
+ B^* \int_0^t S_{A_1}(t-s)^* P^* (S_{A_1}(s)^* z^0 - S_{A_2}(s)^* z^0) \, ds.
\]
Sketch of the Proof

Sketch of proof of the Lemma 2

then we obtain

\[
\int_0^T \| B^* S_{A_1}(t)^* z^0 - B^* S_{A_2}(t)^* z^0 \|_U^2 \, dt \leq \\
C \left( \int_0^T \left\| B^* \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds \right\|_U^2 \, dt \right. \\
+ \int_0^T \left\| \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds \right\|_H^2 \, dt \right). 
\]

The second term on the right hand side of above estimate is based on the estimate of Volterra integral equation (see A.F. Neves & H.S. Ribeiro & O. Lopes (1986) and LH & G. Olive (2019)).
5) We observe that the assumption in Lemma 2 only concerns the semigroup of the unperturbed system \((A_1, B)\). This makes this lemma usable in practice. By using the basic functional equation of semigroups, we can have easier sufficient conditions to check (ii):

**Lemma 3 (LH & G. Olive (2019))**

There exist \(\varepsilon > 0\), a Hilbert space \(\widehat{H}\), a function \(G \in L^2(0, \varepsilon; \mathcal{L}(H, \widehat{H}))\) with \(G(t)\) compact for a.e. \(t \in (0, \varepsilon)\) and \(C > 0\) such that, for a.e. \(t \in (0, \varepsilon)\),

\[
\|B^*V\tilde{z}(t)\|_U + \|V\tilde{z}(t)\|_H \leq C \|G(t)z^0\|_{\widehat{H}}, \quad \forall z^0 \in D(A_1^*),
\]

where \(V\tilde{z}(t) = \int_0^t S_{A_1}(t-s)^*P^*S_{A_1}(s)^*z^0 \, ds\). Then \(\Phi_1(T)^* - \Phi_2(T)^*\) is compact for every \(T > 0\).

**Remark**

The assumption in Lemma 3 has to be checked only for small times, which makes the computation easier in our case (Notice: \(P\) here is only assumed to be bounded).
6) We now check (ii) by using the assumption in Lemma 3. we do it for

$$\varepsilon = T_1(\Lambda),$$

since

$$(V \tilde{z}(t))_i(x) = \int_0^t \left( S_A \frac{\partial \Lambda}{\partial x} (t-s) P^* \tilde{z}(s) \right)_i(x) ds, 0 < t < \varepsilon.$$  

with $\tilde{z}(t) = S_A \frac{\partial \Lambda}{\partial x} (t)^* z^0$ satisfying the decoupled transport equations

$$\begin{cases}
\frac{\partial \tilde{z}}{\partial t}(t, x) = -\Lambda(x) \frac{\partial \tilde{z}}{\partial x}(t, x), \\
\tilde{z}_+(t, 1) = 0, \quad \tilde{z}_-(t, 0) = R^* \tilde{z}_+(t, 0), \\
\tilde{z}(0, x) = z^0(x),
\end{cases}$$

$t \in (0, \varepsilon), x \in (0, 1)$.  \hfill (3.9)
By using the characteristic method, we can directly calculate that 
\((V \tilde{z}(t))_i(x)\) is a sum of terms of the form

\[
K f(x) = \int_{J(x)} k(\beta(s, x)) f(\alpha(s, x)) \, ds,
\]

in which

- \(k \in L^\infty(0, 1)\) and \(f \in L^2(0, 1)\);
- \(\alpha\) and \(\beta\) are \(C^1\) functions and

\[
\frac{\partial \alpha}{\partial s}(s, x) \neq 0, \quad \frac{\partial \beta}{\partial s}(s, x) \neq 0.
\]

By change of variable, one can see

\[
K f(x) = \int_0^1 h(\xi, x) f(\xi) \, d\xi, \quad \text{with} \quad h \in L^\infty((0, 1) \times (0, 1)),
\]

and \(\forall x_0 \in [0, 1],\ K f(x_0) \in \mathbb{R}\), then such kind of operator is indeed compact.
7) A natural candidate for the function $G$ of Lemma 3 will then be the function defined for every $t \in (0, \varepsilon)$ and $z^0 \in L^2(0, 1)^n$ by

$$G(t)z^0 = \left( (V \tilde{z}(t))_{p+1} (1), \ldots, (V \tilde{z}(t))_n (1), (V \tilde{z}(t))_1, \ldots, (V \tilde{z}(t))_n \right). \tag{3.12}$$

$G(t)$ satisfies the assumptions of Lemma 3 (compact $\forall 0 < t < \varepsilon$ and $G \in L^2(0, \varepsilon; \mathcal{L}(L^2(0, 1)^n, \mathbb{R}^m \times L^2(0, 1)^n))$. 


Outline

1. Introduction

2. Exact boundary controllability of linear hyperbolic conservation laws $M = 0$

3. Exact boundary controllability of linear hyperbolic balance laws $M \neq 0$

4. Concluding remarks
Let us go back to the question we asked in the previous slide focusing on $T_c(\Lambda, M, Q)$. It is obviously now that if $T_c(\Lambda, M, Q)$ exists, then

$$T_c(\Lambda, M, Q) = T_{inf}(\Lambda, M, Q) = T_{inf}(\Lambda, 0, Q) = T_c(\Lambda, Q).$$

Thanks to the counterexamples found by D.L. Russell (1978) and J.-M. Coron & H.-M. Nguyen (2018), we now know that there exists $M \neq 0$, such that

$$T_c(\Lambda, M, Q) \text{ does NOT exist (not because of } T_c(\Lambda, M, Q) = +\infty)!!!$$

An explicit characterization on which $M$ (not small perturbation) can guarantee the existence of $T_c(\Lambda, M, Q)$ would be of great interest (By using the methods of J.-M. Coron & H.-M. Nguyen (2018), one might see this is “almost” true except a countable number of $M$).
Pas un jour sans contrôle

Thank you for your attention!