Sharp control time of exact boundary controllability of 1-D coupled hyperbolic systems

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Outline

1. Introduction
2. Exact boundary controllability of linear hyperbolic conservation laws $M = 0$
3. Exact boundary controllability of linear hyperbolic balance laws $M \neq 0$
4. Concluding remarks

Outline

Introduction

Let $T > 0$ fixed. Our system considered here is

$$\frac{\partial y}{\partial t}(t,x) = \Lambda(x)\frac{\partial y}{\partial x}(t,x) + M(x)y(t,x), \quad t \in (0,T), x \in (0,1), \tag{1.1}$$

where,

- $y = (y_1, \cdots, y_n)^T$ is a vector function of (t,x);
- $M(\cdot) \in L^\infty((0,1);\mathbb{R}^{n\times n})$ with entries $m_{ij}(x)$ $(1 \leq i,j \leq n)$;
- $\Lambda(\cdot) \in C^{0,1}([0,1];\mathbb{R}^{n\times n})$ is diagonal with $p \geq 1$ distinct negative eigenvalues and $m := n - p$ distinct positive eigenvalues. Therefore, we assume that

$$\Lambda(x) = \begin{pmatrix} \Lambda_+(x) & 0 \\ 0 & \Lambda_-(x) \end{pmatrix}, \tag{1.2}$$

where $\Lambda_+(x) = \mathrm{diag}(\lambda_1(x), \cdots, \lambda_p(x))$ and $\Lambda_-(x) = \mathrm{diag}(\lambda_{p+1}(x), \cdots, \lambda_n(x))$ are diagonal submatrices satisfying

$$\lambda_1(x) < \cdots < \lambda_p(x) < 0 < \lambda_{p+1}(x) < \cdots < \lambda_{p+m}(x), \quad \forall x \in [0,1]. \tag{1.3}$$

The boundary conditions on $x = 0$ and $x = 1$ are given by:

$$y_+(t, 0) = Qy_-(t, 0), \quad y_-(t, 1) = u(t), \ \forall 0 < t < T, \tag{1.4}$$

in which $Q \in \mathbb{R}^{p\times m}$ is a constant matrix, $u \in L^2((0, T); \mathbb{R}^m)$ is chosen as boundary control function.

Remark

The well-posedness under L^2 framework: see G. Bastin $\&$ J.-M. Coron (2016) on using Lumer-Philipps theorem and LH $\&$G.Olive (2019) on the perturbation arguments.

Introduction

Such linear coupled system involves many physical models of balance laws (without vanishing characteristic speeds):

- Telegrapher equations (Heaviside, O. (1892)); 2×2 system
- (Linearized) Saint-Venant equations (Barré de Saint-Venant (1871)); 2×2 system
- (Linearized) Saint-Venant-Exner equations (Exner(1920,1925); Hudson-Sweby (2003)); 3×3 system
- (Linearized) Heat-exchangers (Allievi(1903); G.Bastin $\&$ J.-M.Coron (2016)); 6×6 system;
- $\cdots$

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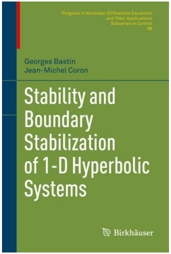
An excellent book:



Problem Description

Exact boundary controllability problem in time T

Let $T > 0$. For any given $\varphi, \psi \in L^2((0,1);\mathbb{R}^n)$. Does there exist boundary control $u \in L^2((0,T);\mathbb{R}^m)$ such that the solution of the control system (1.1) satisfies

$$y(0,x) = \varphi(x), \tag{1.5}$$
$$y(T,x) = \psi(x)? \tag{1.6}$$

Remark

- T is what we call the control time. ((1.1) is not exact controllable if $T = +\infty$);
- Null boundary controllability: (1.6) is replaced by

$$y(T,x) = 0, 0 < x < 1;$$

- Approximate boundary controllability: (1.6) is replaced by

$$\|y(T,x) - \psi(x)\|_{L^2(0,1)} \leq \varepsilon,\ \forall \varepsilon > 0.$$

State of the art

- D. L. Russell (1978): First order 1-D linear hyperbolic systems:

$$y_t = \Lambda(x) y_x + M(x) y.$$

- J.L. Lions (1988): Linear n-D $(n \geq 1)$ wave equation:

$$y_{tt} - \Delta y = 0.$$

- E. Zuazua (1990, 1993): Semilinear wave equation:

$$y_{tt} - \Delta y = f(y).$$

- T.T. Li, B.P. Rao, B.-Y. Zhang (1998, 2002, 2003, 2010): 1-D quasilinear hyperbolic systems:

$$y_t = A(y) y_x + M(y).$$

State of the art

Complements

- Vanishing speeds: JMC, O.Glass and Z.Q. Wang (2010), T.T. Li and B.P. Rao (2009), LH and Z. Q. Wang (2016) etc.
- Global steady states controllability for the Saint-Venant equations: M. Gugat (2003), M. Gugat and G. Leugering (2003, 2009). Friction and slopes are allowed in the last paper.
- Generalization: A(t, x, y): Z.Q. Wang (2007).
- BV solutions: F.Ancona and A.Marson (1998), Th. Horsin (1998), A. Bressan and G.Coclite (2002), O.Glass (2007,2014), V.Perrollaz (2012) etc.

Exact boundary controllability (T.T. Li, B.P. Rao and B.Y. Zhang (1998, 2002,2003, 2010))

For the quasilinear hyperbolic system

$$y_t = A(y)y_x + M(y),$$

with $A(0) = diag(\lambda_1(0), \cdots, \lambda_{p+m}(0))$ satisfying the order relation

$$\lambda_1(0) < \cdots < \lambda_p(0) < 0 < \lambda_{p+1}(0) < \cdots < \lambda_{p+m}(0)$$

and with the nonlinear boundary conditions

$$\begin{aligned} x = 0 : y_s &= G_s(t, y_{p+1}, \cdots, y_{p+m}) \ (s = 1, \cdots, p), \\ x = 1 : y_r &= G_r(t, y_1, \cdots, y_p) + u_r(t) \ (r = p+1, \cdots, p+m), \end{aligned} \quad (1.7)$$

in which $G_i(t, 0, \cdots, 0) \equiv 0 \ \ (i = 1, \cdots, n)$.

State of the art

One can realize the (local) exact boundary controllability by means of u under the framework of C^1 norm if

- $p \leq m$;
- $$T > T_1 := \frac{1}{|\lambda_p(0)|} + \frac{1}{\lambda_{p+1}(0)}; \tag{1.8}$$
- On the non-control side $x = 0$ in a neighborhood of $y = 0$ satisfies

$$\begin{aligned} & y_s = G_s(t, y_{p+1}, \cdots, y_{p+m}) \quad (s = 1, \cdots, p) \\ \Leftrightarrow & y_{m+\bar{r}} = \overline{G}_{m+\bar{r}}(t, y_1, \cdots, y_p, y_{p+1}, \cdots, y_m) \quad (\bar{r} = 1, \cdots, p). \end{aligned} \tag{1.9}$$

Remark

- D.L. Russell (1978) has already given the same result under the case $m = p$ for the linear case on the exact controllability.

Remark continued

Coupling conditions (1.9) on the non-control side $x = 0$ in our linear case considered here means

$$\text{the } p \times p \text{ matrix formed from the last } p \text{ rows and the last } p \text{ columns of } Q \text{ is invertible,} \quad (1.10)$$

which can be replaced by a little weaker assumption (in particular for the case $m > p$):

$$\operatorname{rank} Q = p. \quad (1.11)$$

State of the art

L. Hu (2015)

For the case $F(y) = 0$ i.e.

$$\frac{\partial y}{\partial t} = A(y)\frac{\partial y}{\partial x} \quad (t, x) \in [0, T] \times [0, 1], \tag{1.12}$$

the exact control time of the system (1.12) can be improved as

$$T > T_2 := \max\left(\frac{1}{|\lambda_p(0)|} + \frac{1}{\lambda_{m+1}(0)}, \frac{1}{|\lambda_{p+1}(0)|}\right). \tag{1.13}$$

Moreover, for the linear case, there exists Q with $rank(Q) = p$, such that the system (1.12) is not exactly controllable in time $T < T_2$.

Remark

$T_1 \geq T_2$ if $m \geq p$ and $T_1 > T_2$ if $m > p$

An extra recent known result

With an extra conditions on Q, i.e. $\forall i \in \{1, \dots, p\}$:

the $i \times i$ matrix formed from last i rows

and the last i columns of Q is invertible. (1.14)

J.-M. Coron & H.-M. Nguyen(2018)

- If $m = 1$, (1.1) is exactly controllable in time $T \geq T_3$. where

$$T_3 = \max_{i \in \{1, \dots, p\}} (T_i(\Lambda) + T_{m+i}(\Lambda), T_{p+1}(\Lambda)). \tag{1.15}$$

  with $T_i(\Lambda) := \int_0^1 \frac{1}{|\lambda_i(\xi)|} d\xi, \; i = 1, \cdots, n$;
- If $m \geq 2$, $m \geq p$, (1.1) is "almost" (except a countable number of M) exactly controllable in time $T \geq T_3$;
- If $p = 1$ or 2, $m = 2$, Λ is constant, M is analytic in a neighborhood of a closed subinterval of $[0, 1]$, $Q_{p1} \neq 0$, then (1.1) is exactly controllable in time $T > T_3$;
- If $M = 0$, (1.1) is not exactly controllable in time T for every $T < T_3$ and every Q with (1.14).

Remarks continued

- From now on, let us keep in mind the order of the relation

$$\begin{cases} 0 < T_1(\Lambda) < \ldots < T_p(\Lambda), \\ 0 < T_{p+m}(\Lambda) < \ldots < T_{p+1}(\Lambda), \end{cases} \tag{1.16}$$

then one easily sees that

$$T_3 \leq T_2 \leq T_1 \tag{1.17}$$

Moreover,

$$T_2 < T_1, \text{ if } m > p, \tag{1.18}$$

and

$$T_3 < T_2, \text{ if } m \geq p > 1. \tag{1.19}$$

- In fact, J.-M. Coron $\&$ H.-M. Nguyen (2018) was focused on null-controllability, the exact controllability results are the direct consequence of it.

State of the art

In general, better structure conditions on Q imply a better time for the exact controllability of linear hyperbolic systems. However, we do not know

- what happens if such structure conditions on Q does not satisfy (especially only assume $rank(Q) = p$);
- what happens just before the control time (except the particular cases of systems in Hu 2015 and J.-M. Coron & H.-M. Nguyen 2018 on $M = 0$);
- what happens on the general hyperbolic balance laws.

Question

Can we get the critical control time for linear hyperbolic system T_c such that the exact controllability holds if $T \geq T_c$ and does not hold if $T < T_c$?

- D. L. Russell (1978) already raised this question in his famous survey paper (for the null controllability setting and for the homogeneous case).

Outline

Preliminary

Definition of canonical form matrix (LH & G. Olive (2019), see also Dopico & Johnson & Molera (2006))

We say that a matrix $Q^0 \in \mathbb{R}^{p\times m}$ is in canonical form if there exist distinct column indices $c_1(Q^0), \dots, c_p(Q^0) \in \{1, \dots, m\}$ such that:

$$\forall i \in \{1, \dots, p\}, \quad \begin{cases} q^0_{i,c_i(Q^0)} \neq 0, \\ q^0_{i,j} = 0, \quad \forall j > c_i(Q^0), \quad j \notin \{c_{i+1}(Q^0), \dots, c_p(Q^0)\}, \\ q^0_{i,j} = 0, \quad \forall j < c_i(Q^0). \end{cases} \tag{2.1}$$

Preliminary

Some examples:

$$Q_1^0 = \begin{pmatrix} 0 & \boxed{1} & 4 & -1 \\ 0 & 0 & \boxed{2} & 3 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}, \ Q_2^0 = \begin{pmatrix} 0 & 0 & \boxed{4} \\ \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 0 \end{pmatrix}, \ Q_3^0 = \begin{pmatrix} 1 & 4 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The matrices Q_1^0 and Q_2^0 are both in canonical form, with $c_3(Q_1^0) = 4$, $c_2(Q_1^0) = 3$, $c_1(Q_1^0) = 2$ and $c_3(Q_2^0) = 2$, $c_2(Q_2^0) = 1$, $c_1(Q_2^0) = 3$. However, Q_3^0 is not in canonical form because there is no $c_3(Q_3^0)$ that simultaneously satisfies the second and third conditions of (2.1).

Preliminary

Remarks

If $Q^0 \in \mathbb{R}^{p\times m}$ is in canonical form, then necessarily:

- The indices $c_1(Q^0), \dots, c_p(Q^0)$ are unique.
- $q^0_{i,j} = 0$ for every $i \in \{1, \dots, p\}$ and $j \notin \{c_1(Q^0), \dots, c_p(Q^0)\}$.
- $\operatorname{rank} Q^0 = p$.
- $q^0_{k,c_i(Q^0)} = 0, \quad \forall k > i, \quad \forall i \in \{1, \dots, p\}$.

Preliminary

Proposition: canonical UL-decomposition (LH & G.Olive 2019)

Let $Q \in \mathbb{R}^{p\times m}$ with $\operatorname{rank} Q = p$. Then, there exists a unique $Q^0 \in \mathbb{R}^{p\times m}$ such that the following two properties hold:

- There exists $L \in \mathbb{R}^{m\times m}$ such that

  $$QL = Q^0$$

  with L lower triangular ($\ell_{ij} = 0$ if $i < j$) and with only ones on its diagonal ($\ell_{ii} = 1$ for every i).
- Q^0 is in canonical form.

We call Q^0 the canonical form of Q.

Definition-canonical form of full low rank matrix

Let $Q \in \mathbb{R}^{p\times m}$ with $\operatorname{rank} Q = p$. We define $c_1(Q), \dots, c_p(Q) \in \{1, \dots, m\}$ by

$$c_i(Q) = c_i(Q^0),$$

where Q^0 is the canonical form of Q defined by above Proposition.

Preliminary: A toy example

Let $Q_1 = \begin{pmatrix} 4 & 6 & 3 & -1 \\ 8 & -1 & 5 & 3 \\ 2 & -1 & 1 & 1 \end{pmatrix}$. We look at the last row, take the last nonzero entry as pivot and do the column substitutions $C_3 \leftarrow C_3 - C_4$, $C_2 \leftarrow C_2 + C_4$ and $C_1 \leftarrow C_1 - 2C_4$, so that

$$Q_1 \leftarrow \begin{pmatrix} 6 & 5 & 4 & -1 \\ 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Main results in decoupled case

After such a long but necessary preparation, we now state our main result on the linear hyperbolic conservation laws

Theorem on $M = 0$ (LH & G. Olive (2019))

Let $\Lambda \in C^{0,1}([0,1])^{n\times n}$ satisfy (1.2) and (1.3), and $Q \in \mathbb{R}^{p\times m}$ be fixed. For every $T > 0$, (1.1) with $M = 0$ is exactly controllable in time T if, and only if, the following two properties hold:

(i) $\operatorname{rank} Q = p$.

(ii) $T \geq \max_{i\in\{1,\dots,p\}}(T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda))$.

The proof of this theorem is based on the characteristic methods + canonical UL-decomposition for the full low rank matrix $Q+$ careful use of the order relation between $T_i(\Lambda)$.

Brief idea on proof on toy example

$$\begin{cases} y_{1t} + y_{1x} = 0, \\ y_{2t} - \frac{1}{2} y_{2x} = 0, \\ y_{3t} - y_{3x} = 0, \end{cases} \tag{2.2}$$

and the boundary conditions

$$x = 0 : y_1 = a y_2 + b y_3,$$
$$x = 1 : y_2 = u(t); y_3 = v(t).$$

where u and v are considered as the control functions
Let $Q = (a, b)$. Then

- If $rank(Q) = 0$(i.e. $a = b = 0$), (2.2) is not exact controllable.
- If $rank(Q) = 1$, (2.2) is exact controllable if and only if

$$\begin{cases} T \geq 3, \text{if } \; b = 0, \; \text{with } Q^0 = (a, 0), L = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \forall c \in \mathbb{R}, \\ T \geq 2, \text{if } \; b \neq 0, \; \text{with } Q^0 = (0, b), L = \begin{pmatrix} 1 & 0 \\ -a/b & 1 \end{pmatrix}. \end{cases}$$

Main results in decoupled case

Remarks

- The previous Theorem implies that T_c for the linear conservation laws is

$$T_c = \begin{cases} \max_{i\in\{1,\dots,p\}}(T_{p+1}(\Lambda), T_i(\Lambda) + T_{p+c_i(Q)}(\Lambda)), \text{if } rank(Q) = p; \\ +\infty, \quad \text{otherwise.} \end{cases}$$

  we denote it by $T_c(\Lambda, Q)$ hereafter;
- If $rank(Q) = p$, then

$$\max_{\substack{(c_1,\dots,c_p)\in\{1,\dots,m\} \\ c_j\neq c_k,\, j\neq k}} T_c(\Lambda, Q) = T_1,$$

  the maximum is reached with $c_p(Q) = 1$.
- If $rank(Q) = p$, then

$$\min_{\substack{(c_1,\dots,c_p)\in\{1,\dots,m\} \\ c_j\neq c_k,\, j\neq k}} T_c(\Lambda, Q) = T_3,$$

  the minimum is reached if Q satisfy (1.14) i.e. $c_i(Q) = m - p + i \ \forall 1 \leq i \leq p$.

Outline

Counterexample

We want to extend our $T_c(\Lambda, Q)$ for the conservation laws to the general hyperbolic balance laws ($M \neq 0$) we denote as $T_c(\Lambda, M, Q)$, obviously, we have

$$T_c(\Lambda, 0, Q) = T_c(\Lambda, Q).$$

However, some counterexamples show us these two quantities may not agree in principle.

Counterexamples: D.L. Russell (1978) and J.-M. Coron & H.-M. Nguyen (2018)

There exists a nontrivial hyperbolic balance laws

$$y_t = \Lambda(x) y_x + \Sigma(x) y \tag{3.1}$$

($\Sigma \neq 0$), such that $T = T_3$ (which is $T_c(\Lambda, Q)$ if we assume (1.14)) is not enough to guarantee the null-controllability (and exact-controollability) in the framework of L^2 norm, which, however, is not the case if $\Sigma \equiv 0$.

Thanks to these counterexamples, we loose our requirement a little bit. We denote by $T_{\inf}(\Lambda, M, Q) \in [0, +\infty]$ the sharp (minimal, infimal) time for the exact controllability of balance laws (1.1), that is

$$T_{\inf}(\Lambda, M, Q) = \inf\{T > 0, \quad \text{(1.1) is exactly controllable in time } T\}. \tag{3.2}$$

Obviously, $T_{\inf}(\Lambda, M, Q)$ is unique and satisfy

- If $T > T_{\inf}(\Lambda, M, Q)$, then (1.1) is exactly controllable in time T.
- If $T < T_{\inf}(\Lambda, M, Q)$, then (1.1) is not exactly controllable in time T.
- $T_{\inf}(\Lambda, 0, Q) = T_c(\Lambda, Q)$.

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- If $T < T_{\inf}(\Lambda, M, Q)$, then (1.1) is not exactly controllable in time T.
- $T_{\inf}(\Lambda, 0, Q) = T_c(\Lambda, Q)$.

Question: What is $T_{inf}(\Lambda, M, Q)$?

Main results on $M \neq 0$

Theorem on $M \neq 0$ (LH & G. Olive (2019))

Let $\Lambda \in C^{0,1}([0,1])^{n\times n}$ satisfy (1.2) and (1.3), $M \in L^\infty(0,1)^{n\times n}$ and $Q \in \mathbb{R}^{p\times m}$ be fixed. We have:

$$T_{\inf}(\Lambda, M, Q) = T_{\inf}(\Lambda, 0, Q) = T_c(\Lambda, Q). \tag{3.3}$$

Main results on $M \neq 0$

Remark

Observe that the expression (3.3) of $T_{\text{inf}}(\Lambda, M, Q)$ does not depend on M. This means that the internal coupling terms $M(x)y(t,x)$ in (1.1) have almost no impact on the controllability properties of this system. All our attention should then be on the coupling on the boundary Q and the decoupled system.

Sketch of the Proof

1) We rewrite (1.1) as an equivalent abstract evolution system:

$$\begin{cases} \dfrac{d}{dt} y(t) = A_M y(t) + Bu(t), \quad t \in (0,T), \\ y(0) = y^0, \end{cases} \tag{3.4}$$

(also denoted by (A_M, B)), in which

- the unbounded linear operator $A_M : D(A_M) \subset L^2(0,1)^n \longrightarrow L^2(0,1)^n$ is defined, for every $y \in D(A_M)$ by

  $$A_M y(x) = \Lambda(x) \frac{\partial y}{\partial x}(x) + M(x) y(x), \quad x \in (0,1),$$

  with domain

  $$D(A_M) = \left\{ y \in H^1(0,1)^n, \quad y_+(0) = Q y_-(0), \quad y_-(1) = 0 \right\}.$$

- the control operator $B \in \mathcal{L}(\mathbb{R}^m, D(A_M^*)')$ is given for every $u \in \mathbb{R}^m$ and $z \in D(A_M^*)$ by

  $$\langle Bu, z \rangle_{D(A_M^*)', D(A_M^*)} = u \cdot \Lambda_-(1) z_-(1). \tag{3.5}$$

with

$$D(A_M^*) = \left\{ z \in H^1(0,1)^n, \quad z_+(1) = 0, \quad z_-(0) = R^* z_+(0) \right\},$$

where $R := -\Lambda_+(0) Q \Lambda_-(0)^{-1} \in \mathbb{R}^{p \times m}$, and for every $z \in D(A_M^*)$,

$$A_M^* z(x) = -\Lambda(x) \frac{\partial z}{\partial x}(x) + \left(-\frac{\partial \Lambda}{\partial x}(x) + M(x)^* \right) z(x), \quad x \in (0,1). \quad (3.6)$$

- One can see that the system (A_M, B) is well defined in $C^0([0,T]; L^2(0,1)^n)$ and

$$y(t) = S_{A_M}(t) y^0 + \Phi_M(t) u, \quad \forall t \geq 0, \quad (3.7)$$

where $S_{A_M}(t)(t \geq 0)$ is the C_0 semigroup with the generator A_M, $\Phi_M(t) \in \mathcal{L}(L^2(0,+\infty; \mathbb{R}^m), D(A_M^*)')$ is the so-called input map of (A_M, B) (see M. Tucsnak & G. Weiss (2009)) i.e.

$$\Phi_M(t) u = \int_0^t S_{A_M}(t-s) B u(s)\, ds, \forall u \in L^2(0,+\infty; \mathbb{R}^m).$$

2) Compactness-uniqueness arguments: A flexible method to analysis the exact controllability of partial differential equations:

- J. Rauch & M. Taylor (1974); Exponential decay of some hyperbolic equations;
- S. Dolecki & D.L. Russell (1977): Observability inequality on Banach space;
- J.L. Lions(1988): Plate equation;
- E. Zuazua (1987, 1991): Plate equation, Wave equation;
- C. Bardos, G. Lebeau & J. Rauch(1992): Wave equation;
- L. Rosier (1997): KdV equation;
- .....

Sketch of the Proof

Here we use an abstract version on it.

Lemma 1(M. Duprez & G. Olive (2018) and LH & G. Olive (2019))

Let H and U be two Hilbert spaces. Let $A_1 : D(A_1) \subset H \longrightarrow H$ be the generator of a C_0-semigroup on H and let $B \in \mathcal{L}(U, D(A_1^*)')$ be admissible for A_1. Let $P \in \mathcal{L}(H)$ be a bounded operator and $A_2 = A_1 + P$ with $D(A_2) = D(A_1)$. For $i = 1, 2$, let $\Phi_i(T) \in \mathcal{L}(L^2(0, +\infty; U), H)$ be the input map of (A_i, B) at time $T \geq 0$, and let

$$T_{\inf}(A_i, B) = \inf \{T > 0, \quad (A_i, B) \text{ is exactly controllable in time } T\} \in [0, +\infty].$$

We assume that:

(i) For $i = 1, 2$, (A_i, B) satisfies the Fattorini-Hautus test, i.e.

$$\ker(\lambda - A_i^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}. \tag{3.8}$$

(ii) $\Phi_1(T)^* - \Phi_2(T)^*$ is compact for every $T > 0$.

Then, we have $T_{\inf}(A_2, B) = T_{\inf}(A_1, B)$.

Sketch of the Proof

sketch of proof of the Lemma 1

Let $T_1 > 0$ be such that (A_1, B) is exactly controllable in time T_1 and we only show $T_{\inf}(A_2, B) \leq T_1$. By assumption and duality there exists $C > 0$ such that, for every $z^1 \in H$,

$$\left\|z^1\right\|_H^2 \leq C \int_0^{T_1} \left\|\Phi_1(T_1)^* z^1(t)\right\|_U^2 \, dt,$$

so that, we have the following inequality with compact remainder

$$\left\|z^1\right\|_H^2 \leq 2C \left(\int_0^{T_1} \left\|\Phi_2(T_1)^* z^1(t)\right\|_U^2 \, dt + \int_0^{T_1} \left\|\left(\Phi_1(T_1)^* - \Phi_2(T_1)^*\right) z^1(t)\right\|_U^2 \, dt \right)$$

Since $\Phi_1(T_1)^* - \Phi_2(T_1)^*$ is assumed to be compact and that (A_2, B) satisfies the Fattorini-Hautus test. Therefore, we can apply theorem in M. Duprez & G. Olive (2018) and obtain that (A_2, B) is exactly controllable in time $T_1 + \varepsilon (\forall \varepsilon > 0)$.

3) Let

$$A_1 = A_{\frac{\partial \Lambda}{\partial x}}, \quad A_2 = A_M, \quad P = M - \frac{\partial \Lambda}{\partial x}.$$

Once the assumptions of this Lemma has been checked, we obtain

$$T_{\inf}(\Lambda, M, Q) = T_{\inf}\left(\Lambda, \frac{\partial \Lambda}{\partial x}, Q\right) = T_{\inf}(\Lambda, 0, Q).$$

We find that (i) of the lemma is easy to be checked. Let $\lambda \in \mathbb{C}$ and $z \in D(A_M^*)$ be such that $A_M^* z = \lambda z$ and $B^* z = 0$. Thus, $z \in H^1(0,1)^n$ solves the system of O.D.E.

$$\begin{cases} \dfrac{\partial z}{\partial x}(x) = -\Lambda(x)^{-1}\left(\lambda \mathrm{Id}_{\mathbb{R}^{n\times n}} + \dfrac{\partial \Lambda}{\partial x}(x) - M(x)^*\right) z(x), \quad x \in (0,1), \\ z(1) = 0, \end{cases}$$

so that $z = 0$ by uniqueness.

Sketch of the Proof

4) HOWEVER, to check (ii) is never straightforward (for us) (see remark below for the straightforward answer). We develop the following lemma to check (ii)

Lemma 2 (LH & G.Olive (2019))

For every $T > 0$, there exist a Hilbert space $\widetilde{H}$, a compact operator $F \in \mathcal{L}(H, \widetilde{H})$ and $C > 0$ such that

$$\int_0^T \|B^* V \tilde{z}(t)\|_U^2 \, dt + \int_0^T \|V \tilde{z}(t)\|_H^2 \, dt \leq C \left\|F z^0\right\|_{\widetilde{H}}^2, \quad \forall z^0 \in D(A_1^*),$$

where $V\tilde{z}(t) = \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds$. Then $\Phi_1(T)^* - \Phi_2(T)^*$ is compact for every $T > 0$.

Remark

D.L. Russell (1978, p. 657) said (ii) is true. "A somewhat involved, but not conceptually difficult, argument allows one to see that the operator differences $S^* - S_d^*$, $C^* - C_d^*$ are both compact", where $C^* - C_d^*$ corresponds to $\Phi_M(T)^* - \Phi_{M_d}(T)^*$ in our notation, where M_d denotes the diagonal part of M.

Sketch of the Proof

sketch of proof of the Lemma 2

$\forall z^0 \in D(A_1^*) = D(A_2^*)$, we have

$(\Phi_1(T)^* - \Phi_2(T)^*)z^0(t) = B^* S_{A_1}(T-t)^* z^0 - B^* S_{A_2}(T-t)^* z^0$ for a.e. $t \in (0,T)$

Therefore, we can apply B^* to obtain the following identity:

$$B^* S_{A_1}(t)^* z^0 - B^* S_{A_2}(t)^* z^0 = -B^* \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds$$
$$+ B^* \int_0^t S_{A_1}(t-s)^* P^* \left(S_{A_1}(s)^* z^0 - S_{A_2}(s)^* z^0 \right) \, ds.$$

Sketch of the Proof

sketch of proof of the Lemma 2

then we obtain

$$\int_0^T \left\| B^* S_{A_1}(t)^* z^0 - B^* S_{A_2}(t)^* z^0 \right\|_U^2 \, dt \leq$$
$$C \left(\int_0^T \left\| B^* \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds \right\|_U^2 \, dt \right.$$
$$\left. + \int_0^T \left\| \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds \right\|_H^2 \, dt \right).$$

The second term on the right hand side of above estimate is based on the estimate of Volterra integral equation (see A.F. Neves & H.S. Ribeiro & O. Lopes (1986) and LH & G. Olive (2019)).

Sketch of the Proof

5) We observe that the assumption in Lemma 2 only concerns the semigroup of the unperturbed system (A_1, B). This makes this lemma usable in practice. By using the basic functional equation of semigroups, we can have easier sufficient conditions to check (ii):

Lemma 3 (LH & G.Olive (2019))

There exist $\varepsilon > 0$, a Hilbert space $\widehat{H}$, a function $G \in L^2(0, \varepsilon; \mathcal{L}(H, \widehat{H}))$ with $G(t)$ compact for a.e. $t \in (0, \varepsilon)$ and $C > 0$ such that, for a.e. $t \in (0, \varepsilon)$,

$$\|B^* V \tilde{z}(t)\|_U + \|V \tilde{z}(t)\|_H \leq C \left\|G(t) z^0\right\|_{\widehat{H}}, \quad \forall z^0 \in D(A_1^*),$$

where $V\tilde{z}(t) = \int_0^t S_{A_1}(t-s)^* P^* S_{A_1}(s)^* z^0 \, ds$. Then $\Phi_1(T)^* - \Phi_2(T)^*$ is compact for every $T > 0$.

Remark

The assumption in Lemma 3 has to be checked only for small times, which makes the computation easier in our case (Notice: P here is only assumed to be bounded).

6) We now check (ii) by using the assumption in Lemma 3. we do it for

$$\varepsilon = T_1(\Lambda),$$

since

$$(V\tilde{z}(t))_i\,(x) = \int_0^t \left(S_{A_{\frac{\partial\Lambda}{\partial x}}}(t-s)^* P^* \tilde{z}(s)\right)_i (x)\,ds, 0 < t < \varepsilon.$$

with $\tilde{z}(t) = S_{A_{\frac{\partial\Lambda}{\partial x}}}(t)^* z^0$ satisfying the decoupled transport equations

$$\begin{cases} \dfrac{\partial \tilde{z}}{\partial t}(t,x) = -\Lambda(x)\dfrac{\partial \tilde{z}}{\partial x}(t,x), \\ \tilde{z}_+(t,1) = 0, \quad \tilde{z}_-(t,0) = R^*\tilde{z}_+(t,0), \\ \tilde{z}(0,x) = z^0(x), \end{cases} \qquad t \in (0,\varepsilon), x \in (0,1). \tag{3.9}$$

Sketch of the Proof

By using the characteristic method, we can directly calculate that $(V\tilde{z}(t))_i\,(x)$ is a sum of terms of the form

$$Kf(x) = \int_{J(x)} k(\beta(s,x))f(\alpha(s,x))\,ds, \tag{3.10}$$

in which

- $k \in L^\infty(0,1)$ and $f \in L^2(0,1)$;
- α and β are C^1 functions and

$$\frac{\partial \alpha}{\partial s}(s,x) \neq 0, \frac{\partial \beta}{\partial s}(s,x) \neq 0.$$

- By change of variable, one can see

$$Kf(x) = \int_0^1 h(\xi,x)f(\xi)\,d\xi, \qquad \text{with} \quad h \in L^\infty((0,1)\times(0,1)), \tag{3.11}$$

and $\forall x_0 \in [0,1]$, $Kf(x_0) \in \mathbb{R}$, then such kind of operator is indeed compact.

Sketch of the Proof

7) A natural candidate for the function G of Lemma 3 will then be the function defined for every $t \in (0, \varepsilon)$ and $z^0 \in L^2(0,1)^n$ by

$$G(t)z^0 = \left((V\tilde{z}(t))_{p+1}(1), \ldots, (V\tilde{z}(t))_n(1), (V\tilde{z}(t))_1, \ldots, (V\tilde{z}(t))_n \right). \tag{3.12}$$

$G(t)$ satisfies the assumptions of Lemma 3 (compact $\forall 0 < t < \varepsilon$ and $G \in L^2(0, \varepsilon; \mathcal{L}(L^2(0,1)^n, \mathbb{R}^m \times L^2(0,1)^n))$.

Outline

Concluding remarks

Let us go back to the question we asked in the previous slide focusing on $T_c(\Lambda, M, Q)$. It is obviously now that if $T_c(\Lambda, M, Q)$ exists, then

$$T_c(\Lambda, M, Q) = T_{inf}(\Lambda, M, Q) = T_{inf}(\Lambda, 0, Q) = T_c(\Lambda, Q).$$

Thanks to the counterexamples found by D.L. Russell (1978) and J.-M. Coron &H.-M. Nguyen (2018), we now know that there exists $M \neq 0$, such that

$T_c(\Lambda, M, Q)$ does NOT exist (not because of $T_c(\Lambda, M, Q) = +\infty$)!!!

An explicit characterization on which M (not small perturbation) can guarantee the existence of $T_c(\Lambda, M, Q)$ would be of great interest (By using the methods of J.-M. Coron &H.-M. Nguyen (2018), one might see this is "almost" true except a countable number of M).

Pas un jour sans contrôle

Thank you for your attention !