Quantification of the unique continuation property for the heat equation with lateral Cauchy data

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Introduction

- Bounded domain Ω of $\mathbb{R}^d$
- Bounded interval of time $(0,T)$
- Subdomain Γ_0 of $\partial\Omega$ $(\Gamma_1 = \partial\Omega \setminus \overline{\Gamma_0})$

Heat equation with lateral Cauchy data:

For f in $\Omega \times (0,T)$ and (g_0, g_1) on $\Gamma_0 \times (0,T)$, find u in $\Omega \times (0,T)$ s.t.

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times (0,T) \\ u = g_0 & \text{on } \Gamma_0 \times (0,T) \\ \partial_n u = g_1 & \text{on } \Gamma_0 \times (0,T) \end{cases}$$

Unique continuation: if $f, g_0, g_1 = 0$ then $u = 0$ (Holmgren's Theorem)

$\longrightarrow$ but an **ill-posed** problem !

Quantification: if $f, g_0, g_1 = 0$ are small, how small is u ?

State of the art (extremely partial)

Laplace equation: logarithmic stability estimate in Ω

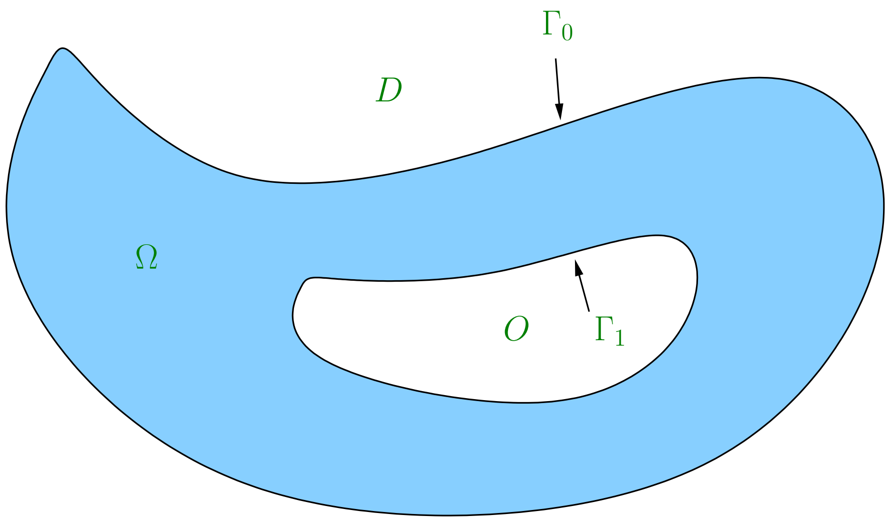
1. Smooth domain Ω: John (1960); $\cdots$; Phung (2003)
2. Lipschitz domain Ω: Alessandrini, Rondi, Rosset & Vessella (2009); Bourgeois & Dardé (2010)

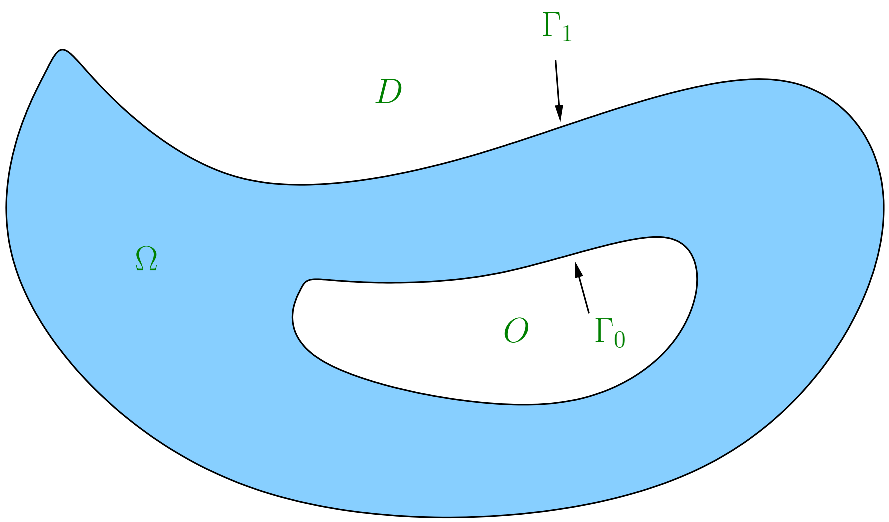
Heat equation: stability estimate in $\Omega \times (0, T)$?

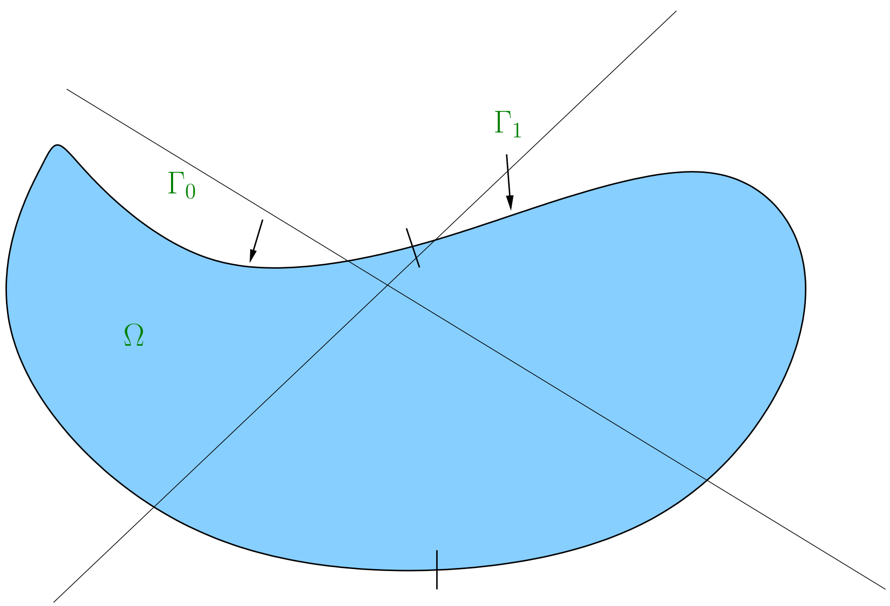
1. Review papers/Books: Lavrentiev, Romanov & Shishatskii; Isakov; Yamamoto; Vessella
2. Truncated domain $\Omega_\varepsilon \times (\varepsilon, T - \varepsilon)$: Hölder estimate
3. Whole domain $\Omega \times (0, T)$ but $\Gamma_0 = \partial\Omega$: logarithmic estimate (Klibanov, 2006)
4. Truncated domain $\Omega \times (\varepsilon, T - \varepsilon)$: logarithmic estimate (Boulakia, 2016: Stokes equation)

A geometric assumption

- $\Omega = D \setminus \overline{O}$, with $O \Subset D$
- O and D of class C^2
- Either $\Gamma_0 = \partial O$ or $\Gamma_0 = \partial D$







Main result

Notation: $Q = \Omega \times (0,T)$, $\Sigma_0 = \Gamma_0 \times (0,T)$, $P = \partial_t - \Delta$

Main Theorem:

For all $s \in (0,1)$, there exists $C > 0$ s.t. for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ with

$$\|u\|_{H^1(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \leq M$$

$$\|Pu\|_{L^2(Q)} + \|u\|_{H^1(\Sigma_0)} + \|\partial_n u\|_{L^2(\Sigma_0)} \leq \varepsilon$$

then

$$\|u\|_{L^2(0,T;H^1(\Omega))} \leq C \frac{M}{\log^s\left(2 + M/\varepsilon\right)}$$

- This theorem implies **unique continuation**
- **Conditional stability**: a bound M is needed for u

$$H^2(Q) \subset H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \subset H^{2,1}(Q)$$

Main result

Main Theorem (compact formulation):

For all $s \in (0,1)$, there exists $C > 0$ s.t.

for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$:

$$\|u\|_{L^2(0,T;H^1(\Omega))} \leq C \frac{\|u\|_{H^1(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))}}{\log^s \left(2 + \frac{\|u\|_{H^1(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))}}{\|Pu\|_{L^2(Q)} + \|u\|_{H^1(\Sigma_0)} + \|\partial_n u\|_{L^2(\Sigma_0)}} \right)}$$

Remarks:

- Extension of (Phung, 2003) for the Laplace equation ($\partial_t u = 0$)
- Quasi-optimal in a certain sense ($s = 1$ cannot be improved)
- Applications to inverse problems (convergence of quasi-reversibility)

and approximate controllability (cost of the control)

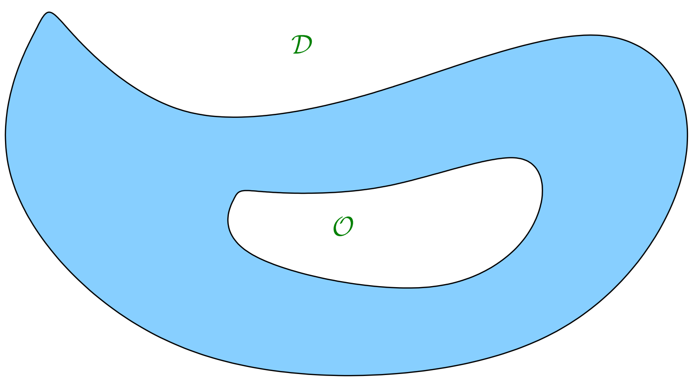
A Carleman weight

Theorem (see Tucsnak & Weiss, 2009)

$\mathcal{D} \subset \mathbb{R}^d$: bounded and connected domain of class C^2, with $\mathcal{O} \Subset \mathcal{D}$

There exists $\eta \in C^2(\overline{\mathcal{D}})$ s.t.

1. $\eta(x) > 0$ for all $x \in \mathcal{D}$
2. $\eta(x) = 0$ for all $x \in \partial\mathcal{D}$
3. $|\nabla \eta(x)| > 0$ for all $x \in \overline{\mathcal{D}} \setminus \mathcal{O}$



Slight improvement: item 2 is replaced by $\eta \equiv d(\cdot, \partial\mathcal{D})$ in $\mathcal{V} = \overline{\mathcal{N}} \cap \overline{\mathcal{D}}$

$\mathcal{N}$: open neighborhood of $\partial\mathcal{D}$

Case of the Laplacian

Carleman estimate : $\varphi(x) := e^{\lambda\eta(x)}$, there exists $C > 0$ such that for large λ and s, for all $v \in H^2(\Omega)$

$$s^3\lambda^4 \int_\Omega \varphi^3 v^2 \, e^{2s\phi} \, dx + s\lambda^2 \int_\Omega \varphi |\nabla v|^2 \, e^{2s\phi} \, dx \leq C \int_\Omega (\Delta v)^2 \, e^{2s\phi} \, dx$$
$$+ C \, s^3\lambda^3 \int_{\partial\Omega} \varphi^3 v^2 \, e^{2s\phi} \, ds + C \, s\lambda \int_{\partial\Omega} \varphi |\nabla v|^2 \, e^{2s\phi} \, ds$$

Truncation of the domain:

$$\Omega_\varepsilon = \{x \in \Omega, \ d(x, \Gamma_1) > \varepsilon\}$$

$$\begin{cases} \chi_\varepsilon(x) = 1 \quad \text{if} \quad d(x, \Gamma_1) \geq 2\varepsilon \\ \chi_\varepsilon(x) = 0 \quad \text{if} \quad d(x, \Gamma_1) \leq \varepsilon \end{cases}$$

$\longrightarrow$ Carleman estimate is applied to $v = \chi_\varepsilon u$ for $u \in H^2(\Omega)$

Case of the Laplacian

Estimate in Ω_ε: fix λ, define $g(z) = e^{\lambda z}$

By definition of η: $0 \leq \eta \leq L := \|\eta\|_\infty$ and $\eta > \varepsilon$ in Ω_ε

$$\text{(LHS)} \geq C s\, e^{2sg(3\varepsilon)} \int_{\Omega_{3\varepsilon}} (u^2 + |\nabla u|^2)\, dx$$

Since $\Delta(\chi_\varepsilon u) = \chi_\varepsilon \Delta u + 2\nabla\chi_\varepsilon \cdot \nabla u + u\Delta\chi_\varepsilon$

$$\text{(RHS)} \leq C\, e^{2sg(L)} \int_\Omega (\Delta u)^2\, dx + (C/\varepsilon^4)\, e^{2sg(2\varepsilon)} \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} (u^2 + |\nabla u|^2)\, dx$$
$$+ C s^3\, e^{2sg(L)} \int_{\Gamma_0} (u^2 + |\nabla u|^2)\, ds$$

From (LHS) $\leq$ (RHS), we obtain

$$\|u\|^2_{H^1(\Omega_{3\varepsilon})} \leq C s^2\, e^{2sp_1} (\|\Delta u\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Gamma_0)} + \|\partial_n u\|^2_{L^2(\Gamma_0)})$$
$$+ (C/s\varepsilon^4)\, e^{-2sp_2} \|u\|^2_{H^1(\Omega)}$$

with $p_1(\varepsilon) = g(L) - g(3\varepsilon) > 0$ and $p_2(\varepsilon) = g(3\varepsilon) - g(2\varepsilon) > 0$

Case of the Laplacian

Estimate in Ω_ε: there exist $C, C_1, C_2 > 0$ such that for large s and small ε, for all $u \in H^2(\Omega)$

$$\|u\|_{H^1(\Omega_\varepsilon)} \leq e^{C_1 s}(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^1(\Gamma_0)} + \|\partial_n u\|_{L^2(\Gamma_0)}) \\ +C\varepsilon^{-2} e^{-C_2 s\varepsilon} \|u\|_{H^1(\Omega)}$$

For fixed ε: optimization of $s \longrightarrow$ Hölder estimate (see Robbiano, 91): there exist $C(\varepsilon) > 0$ and $\nu(\varepsilon) \in (0,1)$ s.t. for all $u \in H^2(\Omega)$

$$\|u\|_{H^1(\Omega_\varepsilon)} \leq C(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^1(\Gamma_0)} + \|\partial_n u\|_{L^2(\Gamma_0)})^\nu \|u\|_{H^1(\Omega)}^{1-\nu}$$

Remark: Hölder estimate implies uniqueness

$$\Delta u = 0 \text{ in } \Omega,\ u = 0 \text{ and } \partial_n u = 0 \text{ in } \Gamma_0 \Longrightarrow u = 0 \text{ in } \Omega$$

How to complement the estimate in the whole domain Ω **?**

Case of the Laplacian

Estimate in $\Omega \setminus \Omega_\varepsilon$: Hardy's inequality (see Phung, 2003)

For all $r \in (0, 1/2)$, for all $v \in H^r(\Omega)$

$$\left\| \frac{v}{d^r(\cdot, \partial\Omega)} \right\|_{L^2(\Omega)} \leq C \, \|v\|_{H^r(\Omega)}.$$

Since $d(\cdot, \partial\Omega) \leq \varepsilon$ in $R_\varepsilon = \Omega \setminus \Omega_\varepsilon$,

$$\|u\|_{L^2(R_\varepsilon)} \leq C \, \varepsilon^{r/2} \|u\|_{H^{1/2}(\Omega)} \leq C \, \varepsilon^{r/2} \|u\|^{1/2}_{L^2(\Omega)} \|u\|^{1/2}_{H^1(\Omega)}$$

$$\leq C \, \eta \, \|u\|_{L^2(\Omega)} + \frac{\varepsilon^r}{\eta} \|u\|_{H^1(\Omega)} \quad (\eta > 0)$$

Apply last estimate to $\partial_i u$, $i = 1, \cdots, d$, then

$$\|u\|_{H^1(R_\varepsilon)} \leq C \, \eta \, \|u\|_{H^1(\Omega)} + \frac{\varepsilon^r}{\eta} \|u\|_{H^2(\Omega)}$$

Case of the Laplacian

Global estimate on Ω:

Elimination of s + adding estimates on Ω_ε and $\Omega \setminus \Omega_\varepsilon$ + absorbing $\|u\|_{H^1(\Omega)}$ term for appropriate $\eta \longrightarrow$

for all $r \in (0,1)$, there exists some $c > 0$ s.t. for small ε, for all $u \in H^2(\Omega)$,

$$\|u\|_{H^1(\Omega)} \leq e^{c/\varepsilon}(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^1(\Gamma_0)} + \|\partial_n u\|_{L^2(\Gamma_0)}) + \varepsilon^r \|u\|_{H^2(\Omega)}$$

Final result:

Optimisation in $\varepsilon \longrightarrow$ for all $s \in (0,1)$, there exists $C > 0$ s.t.

for all $u \in H^2(\Omega)$

$$\|u\|_{H^1(\Omega)} \leq C \frac{\|u\|_{H^2(\Omega)}}{\log^s \left(2 + \frac{\|u\|_{H^2(\Omega)}}{\|\Delta u\|_{L^2(\Omega)} + \|u\|_{H^1(\Gamma_0)} + \|\partial_n u\|_{L^2(\Gamma_0)}}\right)}$$

About the optimality

Theorem: for $d = 2$, choose $\Omega = B(O,2) \setminus \overline{B(O,1)}$ and $\Gamma_0 = \partial B(O,1)$

Assume there exists a non decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ s.t. for all $u \in H^2(\Omega)$

$$\|u\|_{H^1(\Omega)} \leq \frac{\|u\|_{H^2(\Omega)}}{f\left(\frac{\|u\|_{H^2(\Omega)}}{\|\Delta u\|_{L^2(Q)} + \|u\|_{H^1(\Gamma_0)} + \|\partial_n u\|_{L^2(\Gamma_0)}}\right)}$$

Then there exists some $C > 0$ such that for large A,

$$f(A) \leq C \log(A)$$

Proof: take harmonic functions $v_p(x) = r^p \sin(p\,\theta)$ for large p

Case of the heat equation

Uniform Carleman estimate (Fernandez-Cara & Guerrero, 2006) : for $L = \|\eta\|_\infty$ and $\theta(t) = t(T-t)$

$$\phi(x,t) = \frac{e^{\lambda(2L+\eta(x))} - e^{4\lambda L}}{\theta(t)} \quad \xi(x,t) = \frac{e^{\lambda(2L+\eta(x))}}{\theta(t)}$$

There exist some $C > 0$ such that for all T, for large λ and ρ, for all $u \in H^2(Q)$

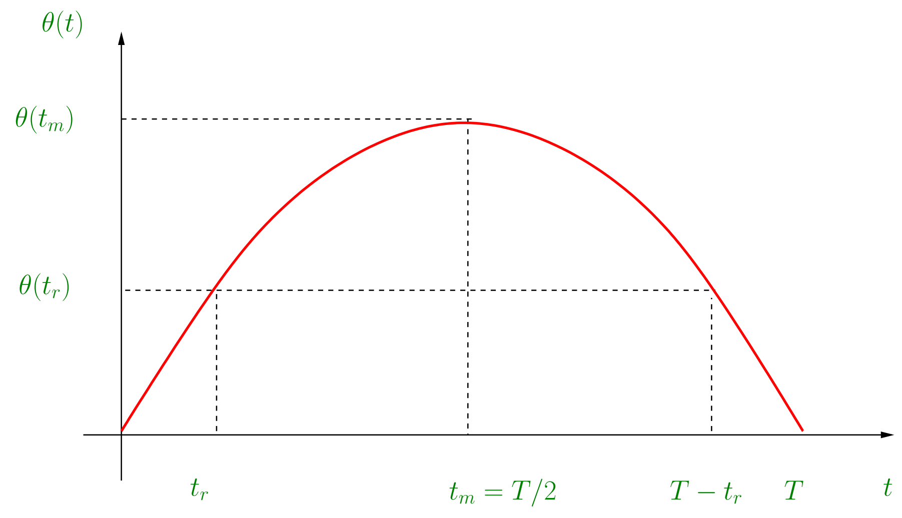
$$\begin{aligned} s^3\lambda^4 \int_Q \xi^3 u^2 \, e^{2s\phi} \, dxdt + s\lambda^2 \int_Q \xi |\nabla u|^2 \, e^{2s\phi} \, dxdt \\ \leq C \int_Q (Pu)^2 \, e^{2s\phi} \, dxdt + C \, s^3\lambda^3 \int_\Sigma \xi^3 u^2 \, e^{2s\phi} \, dsdt \\ + C \, s\lambda \int_\Sigma \xi |\nabla u|^2 \, e^{2s\phi} \, dsdt + C \, \frac{1}{s\lambda} \int_\Sigma \frac{1}{\xi} (\partial_t u)^2 \, e^{2s\phi} \, dsdt \end{aligned}$$

where $s = \rho(T + T^2)$ and $\Sigma = \partial\Omega \times (0,T)$

Case of the heat equation

$\longrightarrow$ Let us try to apply Carleman estimate to $v = \chi_\varepsilon u$ for $u \in H^2(Q)$, fix λ and define $g(z) = e^{4\lambda L} - e^{\lambda(2L+z)}$

Threshold coefficient: $\gamma \in (0,1)$ s.t. $\theta(t_r) = \gamma\,\theta(t_m)$



There exists $C > 0$ such that for large s and small ε, for all $u \in H^2(Q)$

$$\|u\|_{L^2(t_r,T-t_r;H^1(\Omega_{3\varepsilon}))} \leq C\,e^{s\,p_1}\Big(\|Pu\|_{L^2(Q)} + \|u\|_{H^1(\Sigma_0)} + \|\partial_n u\|_{L^2(\Sigma_0)}\Big)$$
$$+C\,\varepsilon^{-2}\,e^{-s\,p_2}\,\|u\|_{L^2(0,T;H^1(\Omega))}$$

with

$$p_1(\varepsilon,\gamma) = \frac{g(3\varepsilon)}{\theta(t_r)} - \frac{g(L)}{2\theta(t_m)}, \quad p_2(\varepsilon,\gamma) = \frac{g(2\varepsilon)}{\theta(t_m)} - \frac{g(3\varepsilon)}{\theta(t_r)}$$

Case of the heat equation

Remark: $p_1 > 0$ but we also need $p_2 > 0$ with

$$p_2 = \frac{g(2\varepsilon)}{\theta(t_r)} \left(\gamma - \frac{g(3\varepsilon)}{g(2\varepsilon)} \right)$$

$\longrightarrow$ we have to choose $\gamma = 1 - c^2\varepsilon$ for small ε

Then

$$t_r = \frac{T}{2} - c\sqrt{\varepsilon}\,\frac{T}{2}, \quad T - t_r = \frac{T}{2} + c\sqrt{\varepsilon}\,\frac{T}{2}$$

$\longrightarrow$ We obtain an estimate on a time interval centered at $T/2$ and of length $\sim \sqrt{\varepsilon}$!

Remedy: use the **uniform** Carleman estimate on a sequence of small intervals of length $\sim \varepsilon$ and cover the whole interval $(\varepsilon, T - \varepsilon)$

Case of the heat equation

Carleman estimate on (t_-^n, t_+^n) for $n \in \mathbb{N}$ with

$$\begin{cases} t_-^n = c\sqrt{\varepsilon}\, n\, \tau \\ t_+^n = \tau + c\sqrt{\varepsilon}\, n\, \tau \end{cases} \quad \text{where} \quad \tau = \frac{2\varepsilon}{1 - c\sqrt{\varepsilon}}$$

$\longrightarrow$ **Estimate** on $(\tilde{t}_-^n, \tilde{t}_+^n)$ with

$$\begin{cases} \tilde{t}_-^n = \dfrac{\tau}{2} + c\sqrt{\varepsilon} \left(n - \dfrac{1}{2}\right) \tau \\ \tilde{t}_+^n = \dfrac{\tau}{2} + c\sqrt{\varepsilon} \left(n + \dfrac{1}{2}\right) \tau \end{cases}$$

Remark:

$$\tilde{t}_-^0 = \varepsilon, \qquad \tilde{t}_-^{n+1} = \tilde{t}_+^n$$

$$t_+^n - t_-^n = \tau \sim \varepsilon \qquad \tilde{t}_+^n - \tilde{t}_-^n = c\sqrt{\varepsilon}\, \tau \sim \varepsilon^{3/2}$$

$\longrightarrow$ Add the estimates on each $(\tilde{t}_-^n, \tilde{t}_+^n)$

Case of the heat equation

Estimate in $Q_\varepsilon = \Omega_\varepsilon \times (\varepsilon, T-\varepsilon)$: there exist $\nu, C, C_1, C_2 > 0$ such that for large ρ and small ε, for all $u \in H^2(Q)$

$$\|u\|_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_\varepsilon))} \leq e^{C_1\rho/\varepsilon}\Big(\|Pu\|_{L^2(Q)} + \|u\|_{H^1(\Sigma_0)} + \|\partial_n u\|_{L^2(\Sigma_0)}\Big) \\ + C\,\varepsilon^{-\nu}\,e^{-C_2\rho}\,\|u\|_{L^2(0,T;H^1(\Omega))}$$

For fixed ε: optimization of $\rho \longrightarrow$ Hölder estimate: there exist $C(\varepsilon) > 0$ and $\nu(\varepsilon) \in (0,1)$ s.t. for all $u \in H^2(Q)$

$$\|u\|_{L^2(\varepsilon,T-\varepsilon;H^1(\Omega_\varepsilon))} \\ \leq C(\|Pu\|_{L^2(Q)} + \|u\|_{H^1(\Sigma_0)} + \|\partial_n u\|_{L^2(\Sigma_0)})^\nu \|u\|^{1-\nu}_{L^2(0,T;H^1(\Omega))}$$

Remark: Hölder estimate implies uniqueness

$$Pu = 0 \text{ in } Q,\ u = 0 \text{ and } \partial_n u = 0 \text{ in } \Sigma_0 \Longrightarrow u = 0 \text{ in } Q$$

Hardy's inequality in Q to complement the estimate in $Q \setminus Q_\varepsilon$

An application: estimate the initial condition

Notation: for u defined in Q, $u_0 := u|_{t=0}$

Corollary: for all $s \in (0,1)$, there exists $C > 0$ s.t. for all $u \in H^1(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ satisfying $Pu = 0$ in Q:

$$\|u_0\|_{L^2(\Omega)} \leq C \frac{\|u\|_{H^1(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))}}{\log^{\frac{3}{4}s}\left(2 + \frac{\|u\|_{H^1(0,T;H^1(\Omega))}+\|u\|_{L^2(0,T;H^2(\Omega))}}{\|u\|_{H^1(\Sigma_0)}+\|\partial_n u\|_{L^2(\Sigma_0)}}\right)}$$

Proof: we use the fact that $u_0 \in H^{3/2}(\Omega)$ + interpolation theory (Lions & Magenes)

Convergence rate of quasi-reversibility

Ill-posed problem: assume $u \in H^2(Q)$ satisfies, for $f, g_0, g_1 \in L^2(Q) \times H^{3/2}(\Sigma_0) \times H^{1/2}(\Sigma_0)$:

$$\begin{cases} Pu = \partial_t u - \Delta u = f & \text{in} \quad Q \\ u = g_0 & \text{on} \quad \Sigma_0 \\ \partial_n u = g_1 & \text{on} \quad \Sigma_0 \end{cases}$$

Notation: $V = \{v \in H^2(Q),\, v|_{\Sigma_0} = 0,\, \partial_n v|_{\Sigma_0} = 0\}$

Regularized problem (Lattès & Lions, Klibanov): for $\varepsilon > 0$ find $u_\varepsilon \in H^2(Q)$ s.t.

$$\begin{cases} (Pu_\varepsilon, Pv)_{L^2(Q)} + \varepsilon(u_\varepsilon, v)_{H^2(Q)} = (f, Pv)_{L^2(Q)} & \text{for all} \quad v \in V \\ u_\varepsilon = g_0 & \text{on} \quad \Sigma_0 \\ \partial_n u_\varepsilon = g_1 & \text{on} \quad \Sigma_0 \end{cases}$$

Convergence rate of quasi-reversibility

Convergence rate for $u_\varepsilon - u$?

Corollary: for all $s \in (0,1)$, there exists a constant $C > 0$ such that for all $u \in H^2(Q)$ solving the ill-posed heat equation for data (f, g_0, g_1) and small $\varepsilon > 0$,

$$\|u_\varepsilon - u\|_{L^2(0,T;H^1(\Omega))} \leq C \frac{\|u\|_{H^2(Q)}}{\log^s(1/\varepsilon)}$$

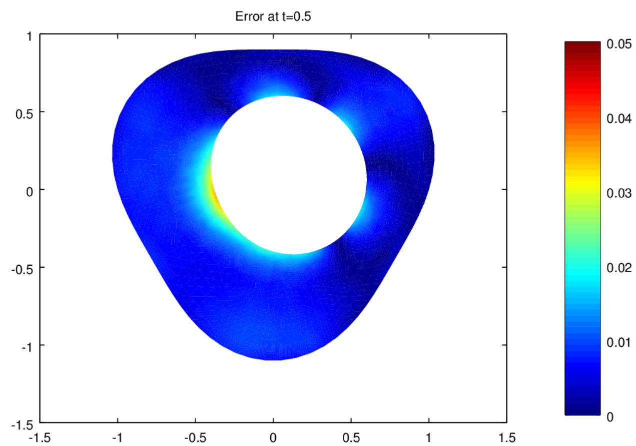
Proof: we have the estimates for all ε:

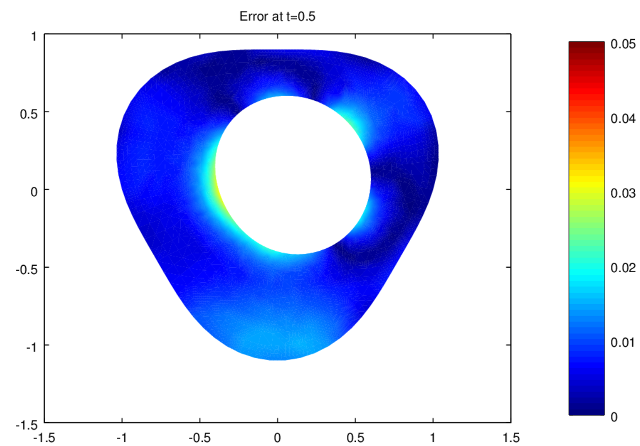
$$\|u_\varepsilon - u\|_{H^2(Q)} \leq \|u\|_{H^2(Q)} \qquad \|P(u_\varepsilon - u)\|_{L^2(Q)} \leq \sqrt{\varepsilon}\, \|u\|_{H^2(Q)}$$

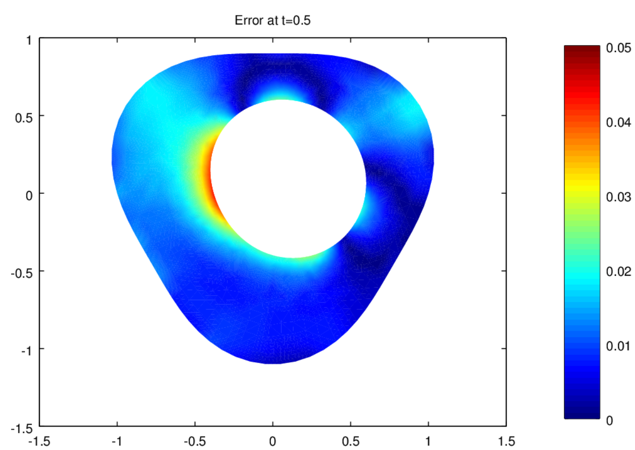
$+$ u_ε and u have the same boundary conditions on Σ_0

$\longrightarrow$ we plug these estimates in the main theorem

Reconstructions with quasi-reversibility







Noise on the data of amplitude δ for $\varepsilon = 0.001$ $\rightarrow$ reconstructions for $\delta = 0$, $\delta = 0.05$ and $\delta = 0.1$

Extensions

Easy extensions:

- More general parabolic operators
- Replace $\|u\|_{H^1(\Sigma_0)} + \|\partial_n u\|_{L^2(\Sigma_0)}$ by $\|u\|_{L^2(0,T;H^1(\omega))}$ for $\omega \Subset \Omega$

$\longrightarrow$ application to approximate controllability (cost of the control)

More delicate extensions:

- Case when $\overline{\Gamma_0} \cap \overline{\Gamma_1} \neq \emptyset$ (see Phung, 2003)
- Case of Lipschitz domains Ω (for Hölder-type functions, see Bourgeois & Dardé, 2010)
- Initial condition as an additional data: how the estimate is improved?
- Wave operator $\partial_t^2 - \Delta$ instead of heat operator $\partial_t - \Delta$ on time interval $(-T, T)$: estimate u in the uniqueness domain

$$Q_{\mathrm{u}} = \{(x,t) \in \Omega \times (-T,T),\ d_\Omega(x,\Gamma_0) < T - |t|\}$$

Thank you for your attention!