Observation estimate at one point in time for parabolic equations

Kim Dang PHUNG

Université d'Orléans.

April 2017, Paris

The parabolic equations

Ω smooth bounded domain in $\mathbb{R}^n$, $T > 0$,
$A \in C^2\left(\overline{\Omega} \times [0,T]\right)$ $n \times n$ symmetric positive-definite matrix,
$b = (b_0, b_1) \in \left(L^\infty\left(\Omega \times (0,T)\right)\right)^{n+1}$,

$$\begin{cases} \partial_t u - \text{div}\left(A\nabla u\right) + b_1 \cdot \nabla u + b_0 u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u\left(\cdot, 0\right) = u_0 & \in L^2\left(\Omega\right) \ , \end{cases}$$

$$\boxed{\left\|u\left(\cdot,t\right)\right\|_{L^2(\Omega)} \leq e^{-C_A t} e^{C_b t} \left\|u_0\right\|_{L^2(\Omega)}}$$

Observation at one point in time

Ω smooth bounded domain in $\mathbb{R}^n$, $T > 0$,
$A \in C^2\left(\overline{\Omega} \times [0,T]\right)$ $n \times n$ symmetric positive-definite matrix,
$C_{b,T} = 1 + e^{C_b T}$, ω non-empty open subset of Ω,

$$\begin{cases} \partial_t u - \text{div}\left(A\nabla u\right) + b_1 \cdot \nabla u + b_0 u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u\left(\cdot,0\right) = u_0 & \in L^2\left(\Omega\right) \ , \end{cases}$$

$$\left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)} \leq \left(Ce^{\frac{C}{T}}\left\|u\left(\cdot,T\right)\right\|_{L^2(\omega)}\right)^{\theta}\left(C_{b,T}\left\|u_0\right\|_{L^2(\Omega)}\right)^{1-\theta}$$

Plan of the talk

I. Applications to control problems and inverse problems

II. Links with known inequalities

III. Proof by Log Convexity and Frequency Function

IV. Proof by Carleman Commutator

References

Observability : Fursikov-Imanuvilov, Lebeau-Robbiano,...
Measurable sets : Apraiz-Escauriaza-Wang-Zhang, Vessella
Carleman commutator : Escauriaza-Kenig-Ponce-Vega(JEMS 08)
Backward estimate : Bardos-Tartar (ARMA 1973)
Inverse problem : Seidman (SINUM 1996)

Collaboration with :
Claude Bardos, Nhat Vo, Gengsheng Wang, Yashan Xu

Strategy for applications

To have in mind that for the model wave equation

$$\boxed{\quad Observability\ estimate \quad \Longleftrightarrow \quad Controllability \quad}$$

$$\boxed{\quad \Longleftrightarrow \quad Stabilization \quad \Longrightarrow \quad Inverse\ Source \quad}$$

Application 1 : Pulse Control and Cost

$\forall T_1 < T_2 < T_3 \quad \forall \varepsilon > 0 \quad \forall y_e \in L^2(\Omega) \quad \exists f \in L^2(\Omega)$

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (T_1, T_3) \setminus \{T_2\} \ , \\ y = 0 & \text{on } \partial\Omega \times (T_1, T_3) \ , \\ y(\cdot, T_1) = y_e & \text{in } \Omega \ , \\ y(\cdot, T_2) = y(\cdot, T_2 -) + \chi_\omega f & \text{in } \Omega \ , \\ y_d = y(\cdot, T_3) & \text{in } \Omega \ , \end{cases}$$

$\|f\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon^\gamma} \|y_e\|_{L^2(\Omega)}$ and $\|y_d\|_{L^2(\Omega)} \leq \varepsilon \|y_e\|_{L^2(\Omega)}$

Minimize $J(\varphi_0) = \frac{cost}{2} \int_\omega |\varphi(x, T_3 + T_1 - T_2)|^2 dx$
$+ \frac{\varepsilon}{2} \int_\Omega |\varphi_0(x)|^2 dx - \int_\Omega y_e(x) \varphi(x, T_3) dx$

Application 2 : Inverse Source problem

Let $f \in L^2(\omega)$ and $0 < \delta < 1$ be such that

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \times (0,T) \ , \\ u = 0 & \text{on } \partial\Omega \times (0,T) \ , \\ u(\cdot, 0) & \in H_0^1(\Omega) \ , \end{cases}$$

$$\boxed{\|u(\cdot,T) - f\|_{L^2(\omega)} \leq \delta \|u(\cdot,0)\|_{L^2(\Omega)}}$$

$$\Longrightarrow \exists g \in L^2(\Omega)$$

$$\boxed{\|u(\cdot,0) - g\|_{L^2(\Omega)} \leq C \frac{1}{\sqrt{\ln \frac{1}{\delta}}} \|u(\cdot,0)\|_{H_0^1(\Omega)}}$$

Application 3 : Fast Impulse Stabilization

$$\forall \gamma > 0 \quad \exists \mathcal{F} : L^2(\omega_1) \to L^2(\omega) \quad \forall y_0 \in L^2(\Omega)$$

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } \Omega \times (0,+\infty) \setminus \{T + T\mathbb{N}\} , \\ y = 0 & \text{on } \partial\Omega \times (0,+\infty) , \\ y(\cdot,0) = y_0 & \text{in } \Omega , \\ \forall n \in \mathbb{N} & \\ y(\cdot,(n+1)T) = y(\cdot,(n+1)T-) + \chi_\omega \mathcal{F}\left(\chi_{\omega_1} y\left(\cdot,\left(n+\frac{1}{2}\right)T\right)\right) & \end{cases}$$

$$\|y(\cdot,t)\|_{L^2(\Omega)} \leq Ce^{-\gamma t} \|y_0\|_{L^2(\Omega)} \quad \text{and} \quad \|\mathcal{F}\| \leq Ce^{C\gamma}$$

Observability

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|u(\cdot,0)\|_{L^2(\Omega)} \qquad \forall t \geq 0$$

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \|u(\cdot,T)\|_{L^2(\omega)} \right)^\theta \left(\|u(\cdot,0)\|_{L^2(\Omega)} \right)^{1-\theta}$$

$$\Longrightarrow$$

$$\|u(\cdot,T)\|_{L^2(\Omega)} \leq \frac{C_1}{C_2} e^{C_\theta \frac{C_2}{T^\beta}} \int_0^T \|u(\cdot,t)\|_{L^2(\omega)} \, dt$$

Proof of observability and cost

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\| \right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$\Longrightarrow \|u(\cdot,T)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\| + \varepsilon \|u(\cdot,0)\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \dfrac{T}{\mathsf{z}^m}$ and $\mathsf{z} > 1$

$$\Longrightarrow \|u(\cdot,T_m)\| \leq \|u(\cdot,t)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot,t)_{|\omega} \right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

Next, integrate over (T_{m+1}, T_m)

Proof of observability and cost

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$\Longrightarrow \|u(\cdot,T)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\| + \varepsilon \|u(\cdot,0)\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \dfrac{T}{z^m}$ and z> 1

$$\Longrightarrow \|u(\cdot,T_m)\| \leq \|u(\cdot,t)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\|u(\cdot,t)_{|\omega}\right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

Next, integrate over (T_{m+1}, T_m)

Proof of observability and cost

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\| \right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$\Longrightarrow \|u(\cdot,T)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\| + \varepsilon \|u(\cdot,0)\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \dfrac{T}{\mathrm{z}^m}$ and z> 1

$$\Longrightarrow \|u(\cdot,T_m)\| \leq \|u(\cdot,t)\| \leq$$
$$\frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot,t)_{|\omega} \right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

Next, integrate over (T_{m+1}, T_m)

Proof of observability and cost

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$\Longrightarrow \|u(\cdot,T)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\| + \varepsilon \|u(\cdot,0)\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \dfrac{T}{\mathrm{z}^m}$ and z> 1

$$\Longrightarrow \|u(\cdot,T_m)\| \leq \|u(\cdot,t)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\|u(\cdot,t)_{|\omega}\right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

Next, integrate over (T_{m+1}, T_m)

Proof of observability and cost

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$
$$\Longrightarrow \|u(\cdot,T)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\| + \varepsilon \|u(\cdot,0)\|$$

$$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T \text{ with } T_m = \frac{T}{\mathsf{z}^m} \text{ and } \mathsf{z} > 1$$

$$\Longrightarrow \|u(\cdot,T_m)\| \leq \|u(\cdot,t)\| \leq$$
$$\frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\|u(\cdot,t)_{|\omega}\right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

Next, integrate over (T_{m+1}, T_m)

$$\|u(\cdot, T_m)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot, t)_{|\omega} \right\| + \varepsilon \|u(\cdot, T_{m+2})\|$$

$$\Longrightarrow \int_{T_{m+1}}^{T_m} \|u(\cdot, T_m)\| \, dt$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 \int_{T_{m+1}}^{T_m} e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot, t)_{|\omega} \right\| dt + \varepsilon \int_{T_{m+1}}^{T_m} \|u(\cdot, T_{m+2})\| \, dt$$

$$\Longrightarrow \|u(\cdot, T_m)\|$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{T_m - T_{m+1}} e^{\frac{C_2}{(T_{m+1}-T_{m+2})^\beta}} \int_{T_{m+1}}^{T_m} \left\| u_{|\omega} \right\| dt + \varepsilon \|u(\cdot, T_{m+2})\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \dfrac{T}{\mathsf{z}^m}$ and $\mathsf{z} > 1$

$$\Longrightarrow \|u(\cdot, T_m)\| \leq$$

$$\frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{\mathsf{z}^\beta C_2} e^{\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta} \mathsf{z}^{\beta(m+2)}} \int_{T_{m+1}}^{T_m} \left\| u_{|\omega} \right\| dt + \varepsilon \|u(\cdot, T_{m+2})\|$$

$$\|u(\cdot,T_m)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\|u(\cdot,t)_{|\omega}\right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

$$\Longrightarrow \int_{T_{m+1}}^{T_m} \|u(\cdot,T_m)\| \, dt$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 \int_{T_{m+1}}^{T_m} e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\|u(\cdot,t)_{|\omega}\right\| dt + \varepsilon \int_{T_{m+1}}^{T_m} \|u(\cdot,T_{m+2})\| \, dt$$

$$\Longrightarrow \|u(\cdot,T_m)\|$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{T_m - T_{m+1}} e^{\frac{C_2}{(T_{m+1}-T_{m+2})^\beta}} \int_{T_{m+1}}^{T_m} \left\|u_{|\omega}\right\| dt + \varepsilon \|u(\cdot,T_{m+2})\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \dfrac{T}{\mathsf{z}^m}$ and $\mathsf{z} > 1$

$$\Longrightarrow \|u(\cdot,T_m)\| \leq$$

$$\frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{\mathsf{z}^\beta C_2} e^{\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta} \mathsf{z}^{\beta(m+2)}} \int_{T_{m+1}}^{T_m} \left\|u_{|\omega}\right\| dt + \varepsilon \|u(\cdot,T_{m+2})\|$$

$$\|u(\cdot,T_m)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot,t)_{|\omega} \right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

$$\Longrightarrow \int_{T_{m+1}}^{T_m} \|u(\cdot,T_m)\| \, dt$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 \int_{T_{m+1}}^{T_m} e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot,t)_{|\omega} \right\| dt + \varepsilon \int_{T_{m+1}}^{T_m} \|u(\cdot,T_{m+2})\| \, dt$$

$$\Longrightarrow \|u(\cdot,T_m)\|$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{T_m - T_{m+1}} e^{\frac{C_2}{(T_{m+1}-T_{m+2})^\beta}} \int_{T_{m+1}}^{T_m} \left\| u_{|\omega} \right\| dt + \varepsilon \|u(\cdot,T_{m+2})\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \frac{T}{\mathsf{z}^m}$ and $\mathsf{z} > 1$

$$\Longrightarrow \|u(\cdot,T_m)\| \leq$$

$$\frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{\mathsf{z}^\beta C_2} e^{\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta} \mathsf{z}^{\beta(m+2)}} \int_{T_{m+1}}^{T_m} \left\| u_{|\omega} \right\| dt + \varepsilon \|u(\cdot,T_{m+2})\|$$

$$\|u(\cdot,T_m)\| \leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot,t)_{|\omega} \right\| + \varepsilon \|u(\cdot,T_{m+2})\|$$

$$\Longrightarrow \int_{T_{m+1}}^{T_m} \|u(\cdot,T_m)\| \, dt$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} C_1 \int_{T_{m+1}}^{T_m} e^{\frac{C_2}{(t-T_{m+2})^\beta}} \left\| u(\cdot,t)_{|\omega} \right\| dt + \varepsilon \int_{T_{m+1}}^{T_m} \|u(\cdot,T_{m+2})\| \, dt$$

$$\Longrightarrow \|u(\cdot,T_m)\|$$

$$\leq \frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{T_m - T_{m+1}} e^{\frac{C_2}{(T_{m+1}-T_{m+2})^\beta}} \int_{T_{m+1}}^{T_m} \left\| u_{|\omega} \right\| dt + \varepsilon \|u(\cdot,T_{m+2})\|$$

$0 < T_{m+2} < T_{m+1} \leq t < T_m < \cdots < T$ with $T_m = \dfrac{T}{\mathsf{z}^m}$ and $\mathsf{z} > 1$

$$\Longrightarrow \|u(\cdot,T_m)\| \leq$$

$$\frac{1}{\varepsilon^{\frac{1-\theta}{\theta}}} \frac{C_1}{\mathsf{z}^\beta C_2} e^{\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta} \mathsf{z}^{\beta(m+2)}} \int_{T_{m+1}}^{T_m} \left\| u_{|\omega} \right\| dt + \varepsilon \|u(\cdot,T_{m+2})\|$$

$$\varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(z-1)^\beta T^\beta} z^{\beta(m+2)}} \left\| u\left(\cdot, T_m\right) \right\|$$
$$-\varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(z-1)^\beta T^\beta} z^{\beta(m+2)}} \left\| u\left(T_{m+2}\right) \right\| \leq \frac{C_1}{z^\beta C_2} \int_{T_{m+1}}^{T_m} \left\| u_{|\omega} \right\| dt$$

$$\Longrightarrow \varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(z-1)^\beta T^\beta} z^{\beta(2m+2)}} \left\| u\left(T_{2m}\right) \right\|$$
$$-\varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(z-1)^\beta T^\beta} z^{\beta(2m+2)}} \left\| u\left(T_{2m+2}\right) \right\| \leq \frac{C_1}{z^\beta C_2} \int_{T_{2m+1}}^{T_{2m}} \left\| u_{|\omega} \right\| dt$$

$\varepsilon = e^{-\frac{2C_2}{(z-1)^\beta T^\beta} z^{\beta(2m+2)}}$ and $A_m = e^{-\frac{2C_2}{(z-1)^\beta T^\beta} z^{2\beta m}} \Longrightarrow$

$$\overbrace{\left(A_m\right)^{z^{2\beta}\frac{1}{\theta}} \left\| u\left(\cdot, T_{2m}\right) \right\|}^{X_m} - \overbrace{\left(A_{m+1}\right)^{\frac{1}{\theta}+1} \left\| u\left(\cdot, T_{2(m+1)}\right) \right\|}^{X_{m+1}}$$
$$\leq \frac{C_1}{z^\beta C_2} \int_{T_{2m+1}}^{T_{2m}} \left\| u\left(\cdot, t\right)_{|\omega} \right\| dt$$

Next, take $z^{2\beta}\frac{1}{\theta} = \frac{1}{\theta} + 1$ and sum the telescoping series from $m = 0$ to $+\infty$ to complete the proof.

$$\varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\| u\left(\cdot, T_m\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\| u\left(T_{m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{m+1}}^{T_m} \left\| u_{|\omega}\right\| dt$$

$$\Longrightarrow \varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\| u\left(T_{2m}\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\| u\left(T_{2m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\| u_{|\omega}\right\| dt$$

$\varepsilon = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}}$ and $A_m = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{2\beta m}} \Longrightarrow$

$$\overbrace{\left(A_m\right)^{\mathsf{z}^{2\beta}\frac{1}{\theta}} \left\| u\left(\cdot, T_{2m}\right)\right\|}^{X_m} - \overbrace{\left(A_{m+1}\right)^{\frac{1}{\theta}+1} \left\| u\left(\cdot, T_{2(m+1)}\right)\right\|}^{X_{m+1}} \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\| u\left(\cdot, t\right)_{|\omega}\right\| dt$$

Next, take $\mathsf{z}^{2\beta}\frac{1}{\theta} = \frac{1}{\theta} + 1$ and sum the telescoping series from $m = 0$ to $+\infty$ to complete the proof.

$$\varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\| u\left(\cdot, T_m\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\| u\left(T_{m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{m+1}}^{T_m} \left\| u_{|\omega}\right\| dt$$

$$\Longrightarrow \varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\| u\left(T_{2m}\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\| u\left(T_{2m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\| u_{|\omega}\right\| dt$$

$\varepsilon = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}}$ and $A_m = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{2\beta m}} \Longrightarrow$

$$\overbrace{\left(A_m\right)^{\mathsf{z}^{2\beta}\frac{1}{\theta}} \left\| u\left(\cdot, T_{2m}\right)\right\|}^{X_m} - \overbrace{\left(A_{m+1}\right)^{\frac{1}{\theta}+1} \left\| u\left(\cdot, T_{2(m+1)}\right)\right\|}^{X_{m+1}} \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\| u\left(\cdot, t\right)_{|\omega}\right\| dt$$

Next, take $\mathsf{z}^{2\beta}\frac{1}{\theta} = \frac{1}{\theta} + 1$ and sum the telescoping series from $m = 0$ to $+\infty$ to complete the proof.

$$\varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\|u\left(\cdot, T_m\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\|u\left(T_{m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{m+1}}^{T_m} \left\|u_{|\omega}\right\| dt$$

$$\Longrightarrow \varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\|u\left(T_{2m}\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\|u\left(T_{2m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\|u_{|\omega}\right\| dt$$

$\varepsilon = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}}$ and $A_m = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{2\beta m}} \Longrightarrow$

$$\overbrace{\left(A_m\right)^{\mathsf{z}^{2\beta}\frac{1}{\theta}} \left\|u\left(\cdot, T_{2m}\right)\right\|}^{X_m} - \overbrace{\left(A_{m+1}\right)^{\frac{1}{\theta}+1} \left\|u\left(\cdot, T_{2(m+1)}\right)\right\|}^{X_{m+1}} \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\|u\left(\cdot, t\right)_{|\omega}\right\| dt$$

Next, take $\mathsf{z}^{2\beta}\frac{1}{\theta} = \frac{1}{\theta} + 1$ and sum the telescoping series from $m = 0$ to $+\infty$ to complete the proof.

$$\varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\|u\left(\cdot, T_m\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(m+2)}} \left\|u\left(T_{m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{m+1}}^{T_m} \left\|u_{|\omega}\right\| dt$$

$$\Longrightarrow \varepsilon^{\frac{1-\theta}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\|u\left(T_{2m}\right)\right\| - \varepsilon^{\frac{1}{\theta}} e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \left\|u\left(T_{2m+2}\right)\right\| \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\|u_{|\omega}\right\| dt$$

$$\varepsilon = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{\beta(2m+2)}} \text{ and } A_m = e^{-\frac{2C_2}{(\mathsf{z}-1)^\beta T^\beta}\mathsf{z}^{2\beta m}} \Longrightarrow$$

$$\overbrace{\left(A_m\right)^{\mathsf{z}^{2\beta}\frac{1}{\theta}} \left\|u\left(\cdot, T_{2m}\right)\right\|}^{X_m} - \overbrace{\left(A_{m+1}\right)^{\frac{1}{\theta}+1} \left\|u\left(\cdot, T_{2(m+1)}\right)\right\|}^{X_{m+1}} \leq \frac{C_1}{\mathsf{z}^\beta C_2}\int_{T_{2m+1}}^{T_{2m}} \left\|u\left(\cdot, t\right)_{|\omega}\right\| dt$$

Next, take $\mathsf{z}^{2\beta}\frac{1}{\theta} = \frac{1}{\theta} + 1$ and sum the telescoping series from $m = 0$ to $+\infty$ to complete the proof with $C_\theta = \frac{2(1+\theta)}{\theta\left[(1+\theta)^{\frac{1}{2\beta}} - 1\right]^\beta}$.

Lebeau-Robbiano sum of eigenfunctions

$$\begin{cases} -\Delta e_i = \lambda_i e_i \quad \text{in } \Omega \ , \\ e_i \in H_0^1(\Omega) \ , \quad 0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \ \lambda_i \to +\infty \end{cases} .$$

$$\sum_{\lambda_i \le \lambda} |a_i|^2 \le C e^{C\sqrt{\lambda}} \int_\omega \left| \sum_{\lambda_i \le \lambda} a_i e_i(x) \right|^2 dx$$

Reference: Jerison-Lebeau (1996)

Observation at one point in time implies LR

$$\left\|u\left(\cdot,T\right)\right\|_{L^2(\Omega)} \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u\left(\cdot,T\right)\right\|_{L^2(\omega)}\right)^{\theta} \left(\left\|u\left(\cdot,0\right)\right\|_{L^2(\Omega)}\right)^{1-\theta}$$

$$\Longrightarrow$$

$$\sum_{\lambda_i \leq \lambda} \left|a_i\right|^2 \leq C_1^2 e^{4C_2^{\frac{1}{1+\beta}}\left(\frac{1-\theta}{\theta}\right)^{\frac{\beta}{1+\beta}}\lambda^{\frac{\beta}{1+\beta}}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i\left(x\right)\right|^2 dx$$

Example : $\beta = 1/3$ for the bilaplacian.

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} |a_i|^2 = \|u(\cdot,T)\|^2 \leq$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|^2\right)^\theta \left(\|u(\cdot,0)\|^2\right)^{1-\theta}$$

$$= \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta \left(\sum_{\lambda_i \leq \lambda} \left|a_i e^{\lambda_j T}\right|^2\right)^{1-\theta}$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta e^{2\lambda T(1-\theta)} \left(\sum_{\lambda_i \leq \lambda} |a_i|^2\right)^{1-\theta}$$

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\implies \sum_{\lambda_i \leq \lambda} |a_i|^2 = \|u(\cdot,T)\|^2 \leq$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|^2\right)^\theta \left(\|u(\cdot,0)\|^2\right)^{1-\theta}$$

$$= \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta \left(\sum_{\lambda_i \leq \lambda} \left|a_i e^{\lambda_j T}\right|^2\right)^{1-\theta}$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta e^{2\lambda T(1-\theta)} \left(\sum_{\lambda_i \leq \lambda} |a_i|^2\right)^{1-\theta}$$

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} |a_i|^2 = \|u(\cdot,T)\|^2 \leq$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|^2\right)^\theta \left(\|u(\cdot,0)\|^2\right)^{1-\theta}$$

$$= \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta \left(\sum_{\lambda_i \leq \lambda} \left|a_i e^{\lambda_j T}\right|^2\right)^{1-\theta}$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta e^{2\lambda T(1-\theta)} \left(\sum_{\lambda_i \leq \lambda} |a_i|^2\right)^{1-\theta}$$

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\| \right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} |a_i|^2 = \|u(\cdot,T)\|^2 \leq$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\|^2 \right)^\theta \left(\|u(\cdot,0)\|^2 \right)^{1-\theta}$$

$$= \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx \right)^\theta \left(\sum_{\lambda_i \leq \lambda} \left| a_i e^{\lambda_j T} \right|^2 \right)^{1-\theta}$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx \right)^\theta e^{2\lambda T(1-\theta)} \left(\sum_{\lambda_i \leq \lambda} |a_i|^2 \right)^{1-\theta}$$

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} |a_i|^2 = \|u(\cdot,T)\|^2 \leq$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|^2\right)^\theta \left(\|u(\cdot,0)\|^2\right)^{1-\theta}$$

$$= \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta \left(\sum_{\lambda_i \leq \lambda} \left|a_i e^{\lambda_j T}\right|^2\right)^{1-\theta}$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx\right)^\theta e^{2\lambda T(1-\theta)} \left(\sum_{\lambda_i \leq \lambda} |a_i|^2\right)^{1-\theta}$$

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\| \right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} |a_i|^2 = \|u(\cdot,T)\|^2 \leq$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\|^2 \right)^\theta \left(\|u(\cdot,0)\|^2 \right)^{1-\theta}$$

$$= \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx \right)^\theta \left(\sum_{\lambda_i \leq \lambda} \left| a_i e^{\lambda_j T} \right|^2 \right)^{1-\theta}$$

$$\leq \left(C_1^2 e^{\frac{2C_2}{T^\beta}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx \right)^\theta e^{2\lambda T(1-\theta)} \left(\sum_{\lambda_i \leq \lambda} |a_i|^2 \right)^{1-\theta}$$

Proof of Lebeau-Robbiano

$$\left\| u\left(\cdot, T\right) \right\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\| u\left(\cdot, T\right)_{|\omega} \right\| \right)^\theta \left\| u\left(\cdot, 0\right) \right\|^{1-\theta}$$

$$u\left(\cdot, t\right) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i\left(x\right)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} \left| a_i \right|^2 \leq C_1^2 e^{\frac{2C_2}{T^\beta}} e^{2\frac{1-\theta}{\theta}\lambda T} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i\left(x\right) \right|^2 dx$$

Next, optimize w.r.t. T by choosing $T = \left(\frac{\theta}{1-\theta} \frac{C_2}{\lambda} \right)^{\frac{1}{1+\beta}}$ to conclude

$$\sum_{\lambda_i \leq \lambda} \left| a_i \right|^2 \leq C_1^2 e^{4C_2^{\frac{1}{1+\beta}} \left(\frac{1-\theta}{\theta}\right)^{\frac{\beta}{1+\beta}} \lambda^{\frac{\beta}{1+\beta}}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i\left(x\right) \right|^2 dx \; .$$

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\| u(\cdot,T)_{|\omega} \right\| \right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C_1^2 e^{\frac{2C_2}{T^\beta}} e^{2\frac{1-\theta}{\theta}\lambda T} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx$$

Next, optimize w.r.t. T by choosing $T = \left(\frac{\theta}{1-\theta} \frac{C_2}{\lambda} \right)^{\frac{1}{1+\beta}}$ to conclude

$$\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C_1^2 e^{4C_2^{\frac{1}{1+\beta}} \left(\frac{1-\theta}{\theta}\right)^{\frac{\beta}{1+\beta}} \lambda^{\frac{\beta}{1+\beta}}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx \,.$$

Proof of Lebeau-Robbiano

$$\|u(\cdot,T)\| \leq \left(C_1 e^{\frac{C_2}{T^\beta}} \left\|u(\cdot,T)_{|\omega}\right\|\right)^\theta \|u(\cdot,0)\|^{1-\theta}$$

$$u(\cdot,t) = \sum_{\lambda_i \leq \lambda} a_i e^{\lambda_j (T-t)} e_i(x)$$

$$\Longrightarrow \sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C_1^2 e^{\frac{2C_2}{T^\beta}} e^{2\frac{1-\theta}{\theta}\lambda T} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx$$

Next, optimize w.r.t. T by choosing $T = \left(\frac{\theta}{1-\theta}\frac{C_2}{\lambda}\right)^{\frac{1}{1+\beta}}$ to conclude

$$\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C_1^2 e^{4C_2^{\frac{1}{1+\beta}} \left(\frac{1-\theta}{\theta}\right)^{\frac{\beta}{1+\beta}} \lambda^{\frac{\beta}{1+\beta}}} \int_\omega \left|\sum_{\lambda_i \leq \lambda} a_i e_i(x)\right|^2 dx \; .$$

Conversely

$$\sum_{\lambda_j \leq \lambda} |a_j|^2 \leq e^{C_2(1+\lambda^\alpha)} \int_\omega \left| \sum_{\lambda_j \leq \lambda} a_j e_i(x) \right|^2 dx$$

$$\Longrightarrow$$

$\forall \theta \in (0,1)$

$$\|u(\cdot,T)\| \leq e^{C\left(1+\left(\frac{1}{\theta T}\right)^{\frac{\alpha}{1-\alpha}}\right)} \|u(\cdot,0)\|^\theta \left\| u(\cdot,T)_{|\omega} \right\|^{1-\theta}$$

$$\Longrightarrow$$

$$\|u(\cdot,T)\| \leq e^{C_4\left(1+\left(\frac{1}{T}\right)^{\frac{\alpha}{1-\alpha}}\right)} e^{\left(\frac{C_4}{T} \ln\left(\frac{\|u(\cdot,0)\|}{\|u(\cdot,T)\|}\right)\right)^\alpha} \left\| u(\cdot,T)_{|\omega} \right\|$$

See Yubiao Zhang (CRAS 2016) for the Kolmogorov equation.

Estimate with weights implies desired observation

Let $0 \leq t_1 < t_2 < t_3 \leq T$. Denote $M = \frac{-\ln(T-t_3+\hbar)+\ln(T-t_2+\hbar)}{-\ln(T-t_2+\hbar)+\ln(T-t_1+\hbar)}$ and $\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

$$\left(\int_\Omega |u(x,t_2)|^2 e^{\Phi(x,t_2)} dx\right)^{1+M} \leq \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)} dx \left(\int_\Omega |u(x,t_1)|^2 e^{\Phi(x,t_1)} dx\right)^{M}$$

$$\Longrightarrow$$

$$\|u(\cdot,T)\| \leq e^{C\left(1+\frac{1}{T}\right)} \left\|u(\cdot,T)_{|\omega}\right\|^{\theta} \|u(\cdot,0)\|^{1-\theta}$$

Proof of "weights imply observation"

Let $0 \le t_1 < t_2 < t_3 \le T$. Denote $M = \frac{-\ln(T-t_3+\hbar)+\ln(T-t_2+\hbar)}{-\ln(T-t_2+\hbar)+\ln(T-t_1+\hbar)}$
and $\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

$$\left(\int_\Omega |u(x,t_2)|^2 e^{\Phi(x,t_2)} dx\right)^{1+M}$$
$$\le \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)} dx \left(\int_\Omega |u(x,t_1)|^2 e^{\Phi(x,t_1)} dx\right)^{M}$$

Choose $t_3 = T$, $t_2 = T - \ell\hbar$, $t_1 = T - 2\ell\hbar$, then $M_\ell = \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)}$ and

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell}$$
$$\le \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Let $0 \leq t_1 < t_2 < t_3 \leq T$. Denote $M = \frac{-\ln(T-t_3+\hbar)+\ln(T-t_2+\hbar)}{-\ln(T-t_2+\hbar)+\ln(T-t_1+\hbar)}$

and $\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

$$\left(\int_\Omega |u(x,t_2)|^2 e^{\Phi(x,t_2)} dx\right)^{1+M}$$

$$\leq \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)} dx \left(\int_\Omega |u(x,t_1)|^2 e^{\Phi(x,t_1)} dx\right)^M$$

Choose $t_3 = T$, $t_2 = T - \ell\hbar$, $t_1 = T - 2\ell\hbar$, then $M_\ell = \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)}$ and

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell}$$

$$\leq \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Let $0 \leq t_1 < t_2 < t_3 \leq T$. Denote $M = \frac{-\ln(T-t_3+\hbar)+\ln(T-t_2+\hbar)}{-\ln(T-t_2+\hbar)+\ln(T-t_1+\hbar)}$
and $\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

$$\left(\int_\Omega |u(x,t_2)|^2 e^{\Phi(x,t_2)} dx\right)^{1+M} \leq \int_\Omega |u(x,t_3)|^2 e^{\Phi(x,t_3)} dx \left(\int_\Omega |u(x,t_1)|^2 e^{\Phi(x,t_1)} dx\right)^{M}$$

Choose $t_3 = T$, $t_2 = T - \ell\hbar$, $t_1 = T - 2\ell\hbar$, then $M_\ell = \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)}$ and

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell} \leq \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and
$\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$
$$\Longrightarrow$$
$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\frac{-|x-x_0|^2}{4(\ell+1)\hbar}} dx\right)^{1+M_\ell} \frac{1}{(\ell+1)^{n/2}}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\frac{-|x-x_0|^2}{4(2\ell+1)\hbar}} dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and
$\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$
$$\Longrightarrow$$
$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\frac{-|x-x_0|^2}{4(\ell+1)\hbar}} dx\right)^{1+M_\ell} \frac{1}{(\ell+1)^{n/2}}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\frac{-|x-x_0|^2}{4(2\ell+1)\hbar}} dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and
$\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$
$$\Longrightarrow$$
$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\frac{-|x-x_0|^2}{4(\ell+1)\hbar}} dx\right)^{1+M_\ell} \frac{1}{(\ell+1)^{n/2}}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and

$\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$ and $D = \max_{x\in\Omega} |x - x_0|$

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$
$$\Longrightarrow$$
$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 dx\right)^{1+M_\ell} e^{\frac{-D}{4(\ell+1)\hbar}(1+M_\ell)} \frac{1}{(\ell+1)^{n/2}}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and
$\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$ and $D = \max_{x\in\Omega}|x-x_0|$

$$\left(\int_\Omega |u(x,T-\ell\hbar)|^2 e^{\Phi(x,T-\ell\hbar)} dx\right)^{1+M_\ell}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\Phi(x,T)} dx \left(\int_\Omega |u(x,T-2\ell\hbar)|^2 e^{\Phi(x,T-2\ell\hbar)} dx\right)^{M_\ell}$$
$\Longrightarrow$
$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} e^{\frac{-D}{4(\ell+1)\hbar}(1+M_\ell)} \frac{1}{(\ell+1)^{n/2}}$$
$$\leq \int_\Omega |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and $D = \max_{x\in\Omega} |x - x_0|$

$$\int_\Omega |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx$$

$$= \int_{|x-x_0|^2 \leq r} + \int_{|x-x_0|^2 \geq r} |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx$$

$\Longrightarrow$

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} e^{\frac{-D}{4(\ell+1)\hbar}(1+M_\ell)} \frac{1}{(\ell+1)^{n/2}}$$

$$\leq \left(\int_{|x-x_0|^2 \leq r} |u(x,T)|^2 dx + e^{\frac{-r}{4\hbar}} \int_\Omega |u(x,0)|^2 dx\right)$$

$$\times \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and $D = \max_{x\in\Omega} |x - x_0|$

$$\int_\Omega |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx$$

$$= \int_{|x-x_0|^2 \le r} + \int_{|x-x_0|^2 \ge r} |u(x,T)|^2 e^{\frac{-|x-x_0|^2}{4\hbar}} dx$$

$\Longrightarrow$

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} e^{\frac{-D}{4(\ell+1)\hbar}(1+M_\ell)} \frac{1}{(\ell+1)^{n/2}}$$

$$\le \left(\int_{|x-x_0|^2 \le r} |u(x,T)|^2 dx + e^{\frac{-r}{4\hbar}} \int_\Omega |u(x,0)|^2 dx\right)$$

$$\times \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and $D = \max_{x \in \Omega} |x - x_0|$

$$\left(\int_\Omega |u(x,T)|^2 \, dx \right)^{1+M_\ell} \leq (\ell+1)^{n/2} \left(e^{\frac{D}{4(\ell+1)\hbar}(1+M_\ell)} \left\| u(\cdot,T)_{|\omega} \right\|^2 + e^{\frac{D}{4(\ell+1)\hbar}(1+M_\ell)} e^{\frac{-r}{4\hbar}} \|u(\cdot,0)\|^2 \right) \times \left(\int_\Omega |u(x,0)|^2 \, dx \right)^{M_\ell}$$

Choose $\ell > 1$ such that $\frac{D}{4(\ell+1)\hbar}(1+M_\ell) \simeq \frac{D}{4(\ell+1)\hbar}(1+\ln \ell) = \frac{r}{8\hbar}$,

$$\left(\int_\Omega |u(x,T)|^2 \, dx \right)^{1+M_\ell} \leq (\ell+1)^{n/2} \left(e^{\frac{r}{8\hbar}} \left\| u(\cdot,T)_{|\omega} \right\|^2 + e^{\frac{-r}{8\hbar}} \|u(\cdot,0)\|^2 \right) \left(\int_\Omega |u(x,0)|^2 \, dx \right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and $D = \max_{x\in\Omega} |x - x_0|$

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} \leq (\ell+1)^{n/2} \left(e^{\frac{D}{4(\ell+1)\hbar}(1+M_\ell)} \left\| u(\cdot,T)_{|\omega} \right\|^2 + e^{\frac{D}{4(\ell+1)\hbar}(1+M_\ell)} e^{\frac{-r}{4\hbar}} \|u(\cdot,0)\|^2 \right) \times \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Choose $\ell > 1$ such that $\frac{D}{4(\ell+1)\hbar}(1+M_\ell) \simeq \frac{D}{4(\ell+1)\hbar}(1+\ln \ell) = \frac{r}{8\hbar}$,

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} \leq (\ell+1)^{n/2} \left(e^{\frac{r}{8\hbar}} \left\| u(\cdot,T)_{|\omega} \right\|^2 + e^{\frac{-r}{8\hbar}} \|u(\cdot,0)\|^2 \right) \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $\ell > 1$, $2\ell\hbar < T$, $M_\ell \simeq \ln \ell$ and $D = \max_{x\in\Omega} |x - x_0|$

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} \leq (\ell+1)^{n/2} \left(e^{\frac{D}{4(\ell+1)\hbar}(1+M_\ell)} \left\| u(\cdot,T)_{|\omega} \right\|^2 + e^{\frac{D}{4(\ell+1)\hbar}(1+M_\ell)} e^{\frac{-r}{4\hbar}} \|u(\cdot,0)\|^2\right) \times \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Choose $\ell > 1$ such that $\frac{D}{4(\ell+1)\hbar}(1+M_\ell) \simeq \frac{D}{4(\ell+1)\hbar}(1+\ln\ell) = \frac{r}{8\hbar}$,

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+M_\ell} \leq (\ell+1)^{n/2} \left(e^{\frac{r}{8\hbar}} \left\| u(\cdot,T)_{|\omega} \right\|^2 + e^{\frac{-r}{8\hbar}} \|u(\cdot,0)\|^2\right) \left(\int_\Omega |u(x,0)|^2 dx\right)^{M_\ell}$$

Proof of "weights imply observation"

Here $\hbar < 1$, $2\ell\hbar < T$,

$$\left(\|u(\cdot,T)\|^2\right)^{1+K} \leq C\left(e^{\frac{r}{8\hbar}}\left\|u(\cdot,T)_{|\omega}\right\|^2 + e^{\frac{-r}{8\hbar}}\|u(\cdot,0)\|^2\right)\left(\|u(\cdot,0)\|^2\right)^K$$

Also true for $\hbar \geq \min(1, T/(2\ell))$ with $C = Ke^{K/T}$.

Optimize w.r.t $\hbar > 0$ such that

$$\left(\|u(\cdot,T)\|^2\right)^{1+K} \leq Ke^{K/T}\left(2\left\|u(\cdot,T)_{|\omega}\right\|\|u(\cdot,0)\|\right)\left(\|u(\cdot,0)\|^2\right)^K$$

This completes the proof.

Proof of "weights imply observation"

Here $\hbar < 1$, $2\ell\hbar < T$,

$$\left(\|u(\cdot,T)\|^2\right)^{1+K} \leq C\left(e^{\frac{r}{8\hbar}}\left\|u(\cdot,T)_{|\omega}\right\|^2 + e^{\frac{-r}{8\hbar}}\|u(\cdot,0)\|^2\right)\left(\|u(\cdot,0)\|^2\right)^K$$

Also true for $\hbar \geq \min(1, T/(2\ell))$ with $C = Ke^{K/T}$.

Optimize w.r.t $\hbar > 0$ such that

$$\left(\|u(\cdot,T)\|^2\right)^{1+K} \leq Ke^{K/T}\left(2\left\|u(\cdot,T)_{|\omega}\right\|\|u(\cdot,0)\|\right)\left(\|u(\cdot,0)\|^2\right)^K$$

This completes the proof.

Proof of "weights imply observation"

Here $\hbar < 1$, $2\ell\hbar < T$,

$$\left(\left\|u\left(\cdot,T\right)\right\|^2\right)^{1+K}$$
$$\leq C\left(e^{\frac{r}{8\hbar}}\left\|u\left(\cdot,T\right)_{|\omega}\right\|^2 + e^{\frac{-r}{8\hbar}}\left\|u\left(\cdot,0\right)\right\|^2\right)\left(\left\|u\left(\cdot,0\right)\right\|^2\right)^K$$

Also true for $\hbar \geq$min$(1, T/\left(2\ell\right))$ with $C = Ke^{K/T}$.

Optimize w.r.t $\hbar > 0$ such that

$$\left(\left\|u\left(\cdot,T\right)\right\|^2\right)^{1+K}$$
$$\leq Ke^{K/T}\left(2\left\|u\left(\cdot,T\right)_{|\omega}\right\|\left\|u\left(\cdot,0\right)\right\|\right)\left(\left\|u\left(\cdot,0\right)\right\|^2\right)^K$$

This completes the proof.

ODE implies Estimate with weights

$$\begin{cases} \frac{1}{2} y'(t) + N(t)\, y(t) = 0 \\ N'(t) \leq \frac{1}{T - t + \hbar} N(t) \end{cases}$$

$[y(t_2)]^{1+M} \leq y(t_3)\, [y(t_1)]^M$ with $M = \frac{-\ln(T - t_3 + \hbar) + \ln(T - t_2 + \hbar)}{-\ln(T - t_2 + \hbar) + \ln(T - t_1 + \hbar)}$

TRUE with $y(t) = \int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx$

and $N(t) = \frac{\int_\Omega |\nabla u(x,t)|^2 e^{\Phi(x,t)} dx}{\int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx}$ when Ω convex

Resume

When Ω convex, take $e^{\Phi(x,t)} = \frac{1}{(T-t+\hbar)^{n/2}} e^{\frac{-|x-x_0|^2}{4(T-t+\hbar)}}$

then $y(t) = \int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx$

and $N(t) = \frac{\int_\Omega |\nabla u(x,t)|^2 e^{\Phi(x,t)} dx}{\int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx}$ solve the ODE

$$\begin{cases} \frac{1}{2} y'(t) + N(t) y(t) = 0 \\ N'(t) \leq \frac{1}{T-t+\hbar} N(t) \end{cases}$$

which implies the "estimate with weights"

which implies observation in one point in time

then, Lebeau-Robbiano inequality and observability

General case

When $\Omega \in C^2$, localize with $\chi \in C_0^\infty\left(B_{x_0,R}\right)$, χu solves a parabolic equation with a second member g .
Replace $|x - x_0|$ by the geodesic distance $d\left(x, x_0\right)$.

Then $y\left(t\right) = \int_\Omega \left|\chi u\left(x,t\right)\right|^2 e^{\Phi(x,t)} dx$

and $N\left(t\right) = \dfrac{\int_\Omega \left|\nabla\left(\chi u\right)\left(x,t\right)\right|^2 e^{\Phi(x,t)} dx}{\int_\Omega \left|\chi u\left(x,t\right)\right|^2 e^{\Phi(x,t)} dx}$ solve an ODE

$$\begin{cases} \left|\frac{1}{2}y'\left(t\right) + N\left(t\right)y\left(t\right)\right| \leq \left(\dfrac{C}{T-t+\hbar} + C\right) y\left(t\right) + \left|\int_\Omega \left(\chi u\right) g e^\Phi dx\right| \\ N'\left(t\right) \leq \left(\dfrac{1+CR}{T-t+\hbar} + C\right) N\left(t\right) + \dfrac{\int_\Omega |g|^2 e^\Phi dx}{y\left(t\right)} \end{cases}$$

which implies the "estimate with weights" (ok if $CR < 1$).
Next propagate the balls to conclude the observation at one point in time.

Carleman commutator

Another way to get the ODE for Log convexity :

Write $y(t) = \int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx = \int_\Omega |f(x,t)|^2 dx$.

$f = ue^{\Phi/2}$ solves

$$\partial_t f - \Delta f - \frac{1}{2} f \left(\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 \right) + \nabla \Phi \cdot \nabla f + \frac{1}{2} \Delta \Phi f = 0$$

Carleman commutator

Another way to get the ODE for Log convexity :

Write $y(t) = \int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx = \int_\Omega |f(x,t)|^2 dx$.

$f = ue^{\Phi/2}$ solves

$$\partial_t f \underbrace{-\Delta f - \frac{1}{2} f \left(\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 \right)}_{-Sf \text{ with } \langle Sf,g\rangle = \langle Sg,f\rangle + \int_{\partial\Omega} \partial_n f g d\sigma} + \underbrace{\nabla\Phi \cdot \nabla f + \frac{1}{2} \Delta\Phi f}_{-Af \text{ with } \langle Af,g\rangle = -\langle Ag,f\rangle} = 0$$

Carleman commutator

Another way to get the ODE for Log convexity :

Write $y(t) = \int_\Omega |u(x,t)|^2 e^{\Phi(x,t)} dx = \int_\Omega |f(x,t)|^2 dx$.

$f = ue^{\Phi/2}$ solves

$$\underbrace{\partial_t f - \Delta f - \frac{1}{2} f \left(\partial_t \Phi + \frac{1}{2} |\nabla \Phi|^2 \right)}_{-Sf \text{ with } \langle Sf,g \rangle = \langle Sg,f \rangle + \int_{\partial\Omega} \partial_n f g d\sigma} + \underbrace{\nabla \Phi \cdot \nabla f + \frac{1}{2} \Delta \Phi f}_{-Af \text{ with } \langle Af,g \rangle = -\langle Ag,f \rangle} = 0$$

Computations done by Escauriaza-Kenig-Ponce-Vega (JEMS 08)

$\Longrightarrow$

$$\begin{cases} \frac{1}{2} y'(t) - \langle Sf, f \rangle = 0 \\ N(t)\, y(t) = -\langle Sf, f \rangle \\ N'(t) \leq \frac{1}{y(t)} \langle -(S' + [S, A]) f, f \rangle - \frac{\int_{\partial\Omega} \partial_n f A f d\sigma}{y(t)} \end{cases}$$

Applications

For Ω convex, by Log convexity, it works
with $\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar)$

in order to have $-(S' + [S,A]) f = \dfrac{1}{T-t+\hbar}(-Sf)$

and $N'(t) \leq \dfrac{1}{T-t+\hbar} N(t)$.

But it also works with $\Phi(x,t) = \frac{-|x-x_0|^2}{4(T-t+\hbar)}$

Lebeau-Robbiano for convex domain

$$\begin{cases} -\Delta e_i = \lambda_i e_i & \text{in } \Omega \ , \\ e_i \in H_0^1(\Omega) \ , & 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \ \lambda_i \to +\infty \end{cases} .$$

$$\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq 4 e^{C_\varepsilon \sqrt{\lambda} \frac{1}{r^{\varepsilon + o(\varepsilon)}}} \int_{|x - x_0| < r} \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx$$

Lebeau-Robbiano for Schroedinger operator

$$\begin{cases} -\Delta e_i + V(x) e_i = \lambda_i e_i \quad \text{in } \Omega \,, \\ e_i \in H_0^1(\Omega) \,, \quad \lambda_1 \leq \lambda_2 \leq \cdot\cdot \leq \lambda_N \leq 0 < \lambda_{N+1} \leq \cdots \,. \end{cases}$$

$$\boxed{\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C e^{C\left(\|V\|_\infty^{2/3} + \sqrt{\lambda}\right)} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx}$$

Lebeau-Robbiano for Schroedinger operator

$$\begin{cases} -\Delta e_i - \frac{\mu}{|x|^2} e_i = \lambda_i e_i \quad \text{in } \Omega \text{ , } \mu \leq \frac{(n-2)^2}{4} \text{ , } n \geq 3, \\ e_i \in H_0^1(\Omega) \text{ , } \quad \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_i \to +\infty \text{ .} \end{cases}$$

$$\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq C e^{C\sqrt{\lambda}} \int_\omega \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 dx$$

Thank You