Lie brackets and interpolation for controllability.

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Small-time local controllability (STLC)

Let $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}^n$, real-analytic, with $f_0(0) = 0$. Consider:

$$\dot{x} = f_0(x) + u(t) f_1(x). \tag{$\star$}$$

Definition

We say that ($\star$) is STLC when, for every $T, \eta > 0$, there exists $\delta > 0$ such that, for every $x^* \in \mathbb{R}^n$ with $|x^*| \leq \delta$, there exists $u \in L^\infty((0,T);\mathbb{R})$ such that $x(T; u, 0) = x^*$ and $\|u\|_\infty \leq \eta$.

= Local surjectivity at $(0, 0)$ of the input-output map

$$\left| \begin{array}{llllll} \mathcal{F}: & \mathbb{R} & \times & L^\infty & \to & \mathbb{R}^n \\ & (T & , & u) & \mapsto & x(T; u, 0) \end{array} \right.$$

Goal: Find conditions on f_0 and f_1 for ($\star$) to be STLC or not.

Some examples

Linear theory (Kalman rank condition):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \end{cases}$$

Quadratic theory (looks bad):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2. \end{cases}$$

Cubic theory (looks good):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^3. \end{cases}$$

Why Lie brackets? (# 1)

Lie brackets measure the lack of commutativity between motions. For vector fields $f, g \in C^\omega(\mathbb{R}^n;\mathbb{R}^n)$, $[f,g]$ is the vector field

$$[f,g](x) := Dg(x)\cdot f(x) - Df(x)\cdot g(x).$$

Example: If $\dot{x} = f_0(x) + u(t) f_1(x)$, $x(0) = 0$ and one uses

$$\begin{cases} u(t) = +\eta & \text{for } t \in (0,\tau), \\ u(t) = -\eta & \text{for } t \in (\tau, 2\tau), \end{cases}$$

then

$$x(2\tau; u, 0) = \tau^2 \eta [f_1, f_0](0) + \mathcal{O}(\tau^3).$$

For **all** systems, one can move towards **both** $\pm[f_1, f_0](0) \in \mathbb{R}^n$.
The underlying "abstract" Lie bracket $[X_1, X_0]$ is "good".

Notations for brackets

- Let $X := \{X_0, X_1\}$ be non-commutative **indeterminates**
- Let $\mathcal{A}(X)$ be the **free algebra** over X, i.e. the vector space of non-commutative polynomials, e.g. $7X_0^2 + 3X_1X_0 + 2X_0X_1$
- Let $\mathcal{L}(X)$ the **free Lie algebra** over X, i.e. the smallest vector subspace of $\mathcal{A}(X)$ containing X_0, X_1, and stable by the Lie bracket (commutator) operation $[a, b] := ab - ba$
- One can "**evaluate**" (although not injective)

$$b \in \mathcal{L}(X) \hookrightarrow f_b \in C^\omega(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow f_b(0) \in \mathbb{R}^n$$

$$[X_1, X_0] = X_1X_0 - X_0X_1 \to [f_1, f_0] = (Df_0)f_1 - (Df_1)f_0 \to [f_1, f_0](0)$$

The Lie algebra rank condition

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Hermann 1963, Nagano 1966)

If $(\star)$ is STLC, then it satisfies

$$\mathit{Lie}(f_0, f_1)(0) := \operatorname{span}\{f_b(0);\ \ b \in \mathcal{L}(X)\} = \mathbb{R}^n. \qquad \text{(LARC)}$$

For non-zero drift $f_0 \neq 0$, (LARC) is not sufficient.

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2, \end{cases}$$

has $f_{X_1}(0) = f_1(0) = e_1$ and $f_{W_1}(0) = [f_1, [f_1, f_0]](0) = 2e_2$.

The quadratic Lie bracket $W_1 := [X_1, [X_1, X_0]]$ looks like a "bad" bracket, associated with a signed motion in an oriented direction.

Why Lie brackets? (#2)

Consider

$$\dot{x} = f_0(x) + u(t) f_1(x) \quad \text{with} \quad f_0(0) = 0$$

$$\dot{y} = g_0(y) + u(t) g_1(y) \quad \text{with} \quad g_0(0) = 0.$$

Theorem (Nagano 1968, Krener 1973, Sussmann 1974, 1985)

*The two systems are diffeomorphic **iff** same vectorial structure:*

$$\{b \in \mathcal{L}(X); f_b(0) = 0\} = \{b \in \mathcal{L}(X); g_b(0) = 0\}.$$

Hence, **the vectors** $f_b(0)$ **contain all the information for STLC**.

Goal of this talk

$$\dot{x} = f_0(x) + u(t) f_1(x) \qquad (\star)$$

- Prove sufficient/necessary conditions of STLC formulated in terms of Lie brackets of f_0 and f_1 evaluated at 0
- With a new strategy :
  - to go further on ODEs
  - to prepare the transfer to PDEs

Definition ($m \in [\![-1, \infty[\![$)

$(\star)$ is $W^{m,\infty}$-STLC when, $\forall T, \eta > 0$, $\exists \delta > 0$ st $\forall x^* \in \mathbb{R}^n$ with $|x^*| \leq \delta$, $\exists u \in W^{m,\infty}(0,T)$ st $x(T; u, 0) = x^*$ and $\|u\|_{W^{m,\infty}} \leq \eta$.

$$(W^{m,\infty}\text{-STLC}) \Rightarrow (L^{\infty}\text{-STLC}) \Rightarrow (W^{-1,\infty}\text{-STLC}) = (\text{small-state STLC})$$

Computing the state using Lie brackets

$$\dot{x} = f_0(x) + u(t) f_1(x) \qquad\qquad x(0) = 0$$

Theorem (Beauchard, Le Borgne, Marbach 2020)

$$x(t;u) = \sum_b \eta_b(t,u) f_b(0) + O(\text{“}\textit{remainders}\text{”}) + o(x(t;u)).$$

The sum

- ranges over elements b of a basis of $\mathcal{L}(X)$
- involves system-dependent vectors $f_b(0) \in \mathbb{R}^n$
- universal functionals $\eta_b(t,u)$ homogeneous:
  $\eta_b(t, \epsilon u) = \epsilon^{n_1(b)} \eta_b(t,u) \qquad \eta_b(\epsilon, u(\frac{\cdot}{\epsilon})) = \epsilon^{|b|} \eta_b(1,u) \qquad \ldots$

Caution: The full sum does not converge, even with analyticity. One has to consider (possibly infinite) truncations (wrt t, or u, or a parameter). And well chosen bases of $\mathcal{L}(X)$. **This is not a Taylor expansion, but a csq of a Magnus-type formula.**

State-of-the-art about sufficient conditions

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Known sufficient conditions for STLC share a common structure:

Theorem

Assume (LARC) *and that, for every* $b \in \mathfrak{B}$*,*

$$f_b(0) \in \operatorname{span}\{f_g(0);\ \omega(g) < \omega(b)\}.$$

Then $(\star)$ *is STLC.*

- $\mathfrak{B} \subset \mathcal{L}(X)$ is a set of "potentially bad" brackets, which you do not know how to use with your current technology
- $\omega : \mathcal{L}(X) \to \mathbb{R}$ is a "weight" which sorts the brackets according to a small-parameter limit you are considering

Version #1: Linear test

An example:

$$\begin{cases} \dot{x}_1 = u + x_1^3, \\ \dot{x}_2 = x_1 + x_1^2 + x_3^5, \\ \dot{x}_3 = x_2 + x_2^4. \end{cases}$$

Version #1: Linear test

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Kalman 1960)

The result holds with

- $\mathfrak{B} := \{b \in \mathcal{L}(X);\ \ n_1(b) \geq 2\}$
- $\omega(b) := n_1(b)$

Indeed, use a control of the form $u(t) = \varepsilon \bar{u}(t)$, then

$$x(t; u, 0) \approx \sum_b \varepsilon^{n_1(b)} \eta_b(t, \bar{u}) f_b(0)$$

$$\approx \varepsilon \left(\sum_{n_1(b)=1} \eta_b(t, \bar{u}) f_b(0) + \varepsilon \sum_{n_1(b) \geq 2} \cdots \right)$$

When $n_1(b) = 1$, $b = \pm \operatorname{ad}^k_{X_0}(X_1)$ and $f_b(0) = \pm (Df_0(0))^k f_1(0)$.

Version #2: Hermes condition

An example:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^3 + x_1^4, \\ \dot{x}_3 = x_2^5 + x_1^{16}. \end{cases}$$

Version #2: Hermes condition

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Sussmann 1983)

The result holds with

- $\mathfrak{B} := \{b \in \mathcal{L}(X);\ \ n_1(b)$ *is even*$\}$
- $\omega(b) := n_1(b)$

Indeed, use a control of the form $u(t) = \varepsilon \bar{u}(t)$, then

$$x(t; u, 0) \approx \sum_b \varepsilon^{n_1(b)} \eta_b(t, \bar{u}) f_b(0)$$

$$\approx \sum_{j \geq 0} \varepsilon^{2j+1} \left(\sum_{n_1(b)=2j+1} \eta_b(t, \bar{u}) f_b(0) + \varepsilon \sum_{n_1(b)=2j+2} \eta_b(t, \bar{u}) f_b(0) \right)$$

Key point: Odd terms aren't signed since $\eta_b(t, -u) = -\eta_b(t, u)$.
Back to the local inversion thm thanks to Γ-controls N-normal.

Version #3: $\mathcal{S}(\theta)$ condition

An example (Jakubczyk, Sussmann 1983):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_2^2 + x_1^3. \end{cases}$$

If u is oscillating (contains high frequencies), then x_1 and x_2 also.
Since $x_1 = \dot{x}_2$, one can have $|x_1^3| \gg x_2^2$.
Involves $W_2 = \mathrm{ad}^2_{[X_1,X_0]}(X_0)$ and $\mathrm{ad}^3_{X_1}(X_0)$.

Another (Stefani, 1985):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 x_2. \end{cases}$$

The last one is quartic ... but good!
By time-reversal $\check{u}(t) = u(T-t)$ then $x_3(T) \leftarrow -x_3(T)$

Version #3: $\mathcal{S}(\theta)$ condition

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Sussmann 1987)

Let $\theta \in [0,1]$. The result holds with

- $\mathfrak{B} := \{b \in \mathcal{L}(X);\ n_1(b)$ *is even and* $n_0(b)$ *is odd*$\}$
- $\omega(b) := n_1(b) + \theta n_0(b)$

For controls of the form $u(t) = \varepsilon^{1-\theta}\bar{u}(t/\varepsilon^{\theta})$, then

$$x(t; u, 0) \approx \sum_b \varepsilon^{\omega(b)} \eta_b(t, \bar{u}) f_b(0)$$

Key point: Terms with $n_1(b)$ even and $n_0(b)$ even aren't signed since, by time reversal, $\eta_b(t, \check{u}) \approx -\eta_b(t, u)$.

Version #4: Agrachev Gamkrelidze 1993, Krastanov 2009

- much stronger sufficient condition for STLC
- smaller set $\mathfrak{B}$ of potentially bad brakets
- richer control variations
- new weight

Version #5: A new condition for quartic terms

An example:

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = x_2 \\ \dot{x}_4 = x_3 + x_1^2 \\ \dot{x}_5 = x_4 + x_1^4 \\ \dot{x}_6 = x_1^3 + x_1^2 x_2^2 \\ \dot{x}_7 = x_1^3 x_2 \\ \dot{x}_8 = x_1^3 x_3 \\ \dot{x}_9 = x_3^3 x_5 \end{cases}$$

+ terms of order higher than the green one on each line

The last 2 lines correspond to brackets of type (even, odd) but the associated coordinates are simultaneously surjective.

Version #5: A new condition for quartic terms

Theorem (KB-Marbach 2023)

$$S_{[\![1,4]\!]}(f)(0) = \mathbb{R}^d$$
$$\forall b \in \mathcal{B}_2^\star, \quad f_b(0) \in S_1(f)(0),$$
$$\forall b \in \mathcal{B}_{4,bad}^\star, \quad f_b(0) \in S_{[\![1,3]\!]}(f)(0).$$

$\Rightarrow \quad \dot{x} = f_0(x) + u(t) f_1(x)$ *is smoothly-STLC.*

$\mathcal{B}_1^\star$: $M_\nu := X_1 0^\nu$ $\qquad u_{\nu+1}(t) = \int_0^t \frac{(t-\tau)^\nu}{\nu!} u(\tau) d\tau$

$\mathcal{B}_2^\star$: $W_{j,\nu} := (M_{j-1}, M_j)0^\nu$ $\qquad \int_0^t \frac{(t-\tau)^\nu}{\nu!} u_j(\tau)^2 d\tau$

$\mathcal{B}_3^\star$: $P_{j,k,\nu} := (M_{k-1}, W_{j,0})0^\nu$ $\qquad \int_0^t \frac{(t-\tau)^\nu}{\nu!} u_j(\tau)^2 u_k(\tau) d\tau$

$\mathcal{B}_4^\star$: $Q_{j,k,l,\nu} := (M_{l-1}, P_{j,k,0})0^\nu$ $\qquad \int_0^t \frac{(t-\tau)^\nu}{\nu!} u_j(\tau)^2 u_k(\tau) u_l(\tau) d\tau$

$Q^\sharp_{j,\mu,k,\nu} := (W_{j,\mu}, W_k)0^\nu$ $\qquad \int_0^t \int_0^\tau \frac{(\tau-s)^\mu}{\mu!} u_j(s)^2 ds\, u_k(\tau)^2 d\tau$

$Q^\flat_{j,\mu,\nu} := (W_{j,\mu}, W_{j,\mu+1})0^\nu$ $\qquad \int_0^t \left(\int_0^\tau \frac{\tau-s)^\mu}{\mu!} u_j(s)^2 ds \right)^2 d\tau$

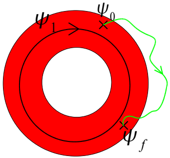
$$\mathcal{B}^\star_{4,\mathrm{bad}} := \{Q_{j,k,k,\nu};\ j \le k\} \cup \{Q^\sharp_{j,\mu,k,\nu};\ j < k\} \cup \{Q^\flat_{j,\mu,\nu}\}$$

$\mathcal{B}^\star_{4,\mathrm{good}} := \{Q_{j,k,l,\nu};\ j \le k < l\} \longrightarrow$ simult. surjective coordinates

Example of transfer to a PDE

$$i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi \qquad \psi(t,0) = \psi(t,1) = 0$$

Ground state:
$\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}$



Depending on the assumption on μ:

- linear test + smoothing effect [KB-Laurent 2010]
- 1 direction lost on the linearized syst and [Bournissou 2022]
  - quadratic obstruction to H^3-STLC
  - H^2-STLC : $\int_0^T u_3^2 dt + \int_0^T u_1^2 u_2$
    by adaptation of Sussmann's condition $S(\theta)$ in ∞ dim

This is the first positive STLC result for a PDE with a nonlinear competition.

Sufficient conditions: conclusion

$$\dot{x} = f_0(x) + u(t) f_1(x)$$

We proposed a new methodology to prove SC for STLC

- approximate representation of the state from the $f_b(0)$
- simultaneously surjective coordinates ξ_b

that gives new results and a work plan for PDEs.

The brackets of $\mathcal{L}(X) \setminus \mathfrak{B}$ are good: we know how to use them for controllability.

What about the brackets of $\mathfrak{B}$? Are they really bad? Is some kind of compensation condition indeed **necessary** for STLC?

Necessary #1

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Sussmann 1983, Stefani 1986)

If $(\star)$ is $W^{-1,\infty}$-STLC, then, for each $\ell \in \mathbb{N}^*$,

$$f_{\mathrm{ad}_{X_1}^{2\ell}(X_0)}(0) \in \operatorname{span}\{f_b(0); n_1(b) \leq 2\ell - 1\}.$$

Idea:

$$x(t;u,0) \approx \sum_{n_1(b)<2\ell} \eta_b(t,u) f_b(0) + \frac{1}{(2\ell)!} \overbrace{\left(\int_0^t u_1^{2\ell}\right)}^{\text{coercive}} f_{\mathrm{ad}_{X_1}^{2\ell}(X_0)}(0)$$
$$+ \mathcal{O}\left(t \int_0^t u_1^{2\ell} + \int_0^t |u_1|^{2\ell+1}\right).$$

where $u_1(s) := \int_0^s u$.

A naive strategy to prove obstructions

$$x(T;u,0) = \sum_{b \in \mathcal{B}_{[1,M]}} \eta_b(T,u) f_b(0) + O\left(\|u\|_{W^{-1,\infty}}^{M+1}\right) + o\left(|x(T;u)|\right)$$

- Choose $B \in \mathcal{B}$ st the functionnal $\eta_B(T,.)$ is signed for T small.
- Find $M \in \mathbb{N}$ st $\|u\|_{W^{-1,\infty}}^{M+1} = o(\eta_B(T,u))$ when $(T, \|u\|_{W^{m,\infty}}) \to 0$.
- Determine $\mathcal{N} = \{b \in \mathcal{B}_{[1,M]}; \eta_b(T,u) \neq o(\eta_B(T,u))\}$.

Then a necessary condition for STLC is

$$f_B(0) \in \text{Span}\left\{f_b(0); b \in \mathcal{N}\right\}$$

because otherwise, $x(T;u,0)$ drifts along $f_B(0)$.

Drawback: The expression of the coordinates η_b is not simple :

- a principal part ξ_b ("coordinates of the second kind") : easily computable by recursion, nice for $\mathcal{B}^\star$: obvious signs
- a pollution : products of other $\xi_{b'}$

Example: $\eta_{W_1}(t,u) = \int_0^t u_1^2 - u_1(t)u_2(t)$ is not signed

Our unified approach for obstructions to STLC

$$x(T;u) = \sum_{b \in \mathcal{B}_{[1,M]}} \xi_b(T,u) f_b(0) + \text{pollution} + O\left(\|u\|_{W^{-1,\infty}}^{M+1}\right) + o\left(|x(T;u)|\right)$$

- Choose $B \in \mathcal{B}$ such that the functionnal $\xi_B(T,.)$ is signed.
- Find $M \in \mathbb{N}$ st $\|u\|_{W^{-1,\infty}}^{M+1} = o(\xi_B(T,u))$ when $(T, \|u\|_{W^{m,\infty}}) \to 0$
  using interpolation inequality.
- Determine $\mathcal{N} = \{b \in \mathcal{B}_{[1,M]}; \xi_b(T,u) \neq o(\xi_B(T,u))\}$
  using interpolation inequality.
- Prove pollution $= o(\xi_B(T,u)) + o\left(|x(T;u)|\right)$ when

$$f_B(0) \notin \text{Span}\,\{f_b(0); b \in \mathcal{N}\} \qquad (\star)$$

  using closed loop estimates + interpolation

If $(\star)$ and T, $\|u\|_{W^{m,\infty}}$ are small enough then $x(T;u)$ drifts along $f_B(0)$.

Thus a NC for $W^{m,\infty}$-STLC is $\quad f_B(0) \in \text{Span}\,\{f_b(0); b \in \mathcal{N}\}$.

Necessary #2 : on $W_j := \mathrm{ad}^2_{\mathrm{ad}^{j-1}_{X_0}(X_1)}(X_0)$

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (KB-Marbach 2022)

Let $m \in [\![-1, \infty[\![$. If $(\star)$ is $W^{m,\infty}$-STLC, then, for each $j \in \mathbb{N}$

$$f_{W_j}(0) \in \operatorname{span}\{f_b(0); 2 \neq n_1(b) \leq \pi(j,m)\}.$$

where $\pi(j,m) = 1 + \left\lceil \frac{2(j-1)}{m+1} \right\rceil$.

$\xi_{W_j}(t,u) = \int_0^t u_j^2$ is the square of an L^2-norm thus Gagliardo-Nirenberg inequalities give the conclusion. For instance

$$\int_0^t |u_1|^{\pi+1} \leq C_t \|u\|^{\pi-1}_{W^{m,\infty}} \int_0^t u_j^2$$

Necessary #3: on $Q_{j,k} = \mathrm{ad}^2_{\mathrm{ad}^{k-1}_{X_0}(X_1)} \mathrm{ad}^2_{\mathrm{ad}^{j-1}_{X_0}(X_1)}(X_0)$

Theorem (KB-Marbach 2023)

Let $j \leq k \in \mathbb{N}^*$ and $m \in [\![-1, \infty[\![$.
We assume $\dot{x} = f_0(x) + u(t) f_1(x)$ *is $W^{m,\infty}$-STLC and*

- *either $k \leq (2j+m)$,*
- *or $(2j+m) < k$ and $f_{W_j}(0) \in$ Span$\{f_b(0); b \in \mathcal{B}^\star_{[\![1,\pi]\!] \setminus \{2\}} \setminus \{M_{k-1}\}\}$.*

Then

$$f_{Q_{j,k}}(0) \in \mathit{Span}\{f_b(0); b \in \mathcal{B}^\star, b \neq Q_{j,k}, n_1(b) \leq M\}$$

where $M = M(j,k,m) := 3 + \left\lceil \frac{2(k+j-2)}{m+1} \right\rceil$.

For $j \neq k$, $\xi_{Q_{j,k}}(t,u) = \int_0^t u_j^2 u_k^2$ is **not** the exponent 4 of a norm, thus new interpolation inequalities are required. For instance

$$\int_0^t |u_1|^{M+1} \leq C_t \|u\|^{M-3}_{W^{m,\infty}} \int_0^t u_j^2 u_k^2$$

[Marbach 2023]

Necessary #3: on $Q_{j,k}$

In the low regularity case $(2j+m) < k$, the absence of component of $f_{W_j}(0)$ along $f_{M_{k-1}}(0)$ is necessary for the obstruction to hold.

Example: For every $\lambda \in \mathbb{R}^*$, the nilpotent system

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 + \lambda x_1^2 \\ \dot{x}_3 = 2x_1^2 x_2 \\ \dot{x}_4 = x_1^2 x_2^2 + \lambda x_1^2 x_3 \end{cases} \tag{1}$$

is $W^{-1,\infty}$-STLC despite satisfying $f_{Q_{1,2,2}(0)} \notin \mathcal{N}^{\infty}_{1,2}(f)(0)$.

Ccl: The splitting pb is better posed in smooth-STLC.

Necessary #3: on $Q_{j,k}$ for $m=-1$

$$x(T;u)=\sum_{b\in\mathcal{B}_{[1,M]}}\xi_b(T,u)f_b(0)+\text{pollution}+O\left(\|u\|_{W^{-1,\infty}}^{M+1}\right)+o\left(|x(T;u)|\right)$$

In this representation formula, the rest $\int_0^T |u_1|^{M+1}$ cannot be absorbed by the drift $\int_0^T u_j^2 u_k^2$ when $\|u_1\|_\infty$ is small.

For nilpotent systems, this rest does not appear, thus our approach proves the drift.

We use an argument of "embedded nilpotent system" to deduce the drift for non-nilpotent systems.

Necessary conditions: conclusion, perspectives

$$\dot{x} = f_0(x) + u(t) f_1(x)$$

We have :

(1) proposed methodology ingredients to prove NC for STLC:

- approximate formula for the state from the $f_b(0)$,
- interpolation inequalities to absorb the remainder by the coercive signed drift and the smallness of the control
- or embedded nilpotent systems

(2) solved the long-standing problem of "splitting" $\mathcal{L}(X)$ between good and bad brackets, at the level of $\{n_1 = 4\}$:

- a new Hall basis $\mathcal{B}^\star$ of $\mathcal{L}(X)$, designed for this purpose,
- new interpolation inequalities

Perspectives: multi-input systems [Gherdaoui]