Some results about the stability and the controllability of the KdV equation

Lucie Baudouin, Eduardo Cerpa, Emmanuelle Crépeau and Julie Valein

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Motivation

Korteweg-de Vries equation 1895 (Russell 1834, Boussinesq 1877, Bona-Winther 1983)

\[ y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) = 0 \]

Model water waves propagating along a shallow canal.
The stability problem

The goal is to study the stability of the following non-linear KdV equation with a boundary feedback term on a bounded domain

\[
\begin{align*}
    y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) &= 0, \quad x \in (0, L), \ t > 0, \\
y(0, t) &= y(L, t) = 0, \quad t > 0, \\
y_x(L, t) &= \alpha y_x(0, t), \quad t > 0, \\
y(x, 0) &= y_0(x), \quad x \in (0, L),
\end{align*}
\]

(1)

In the above equations:

- \( y(x, t) \): amplitude of the water wave at position \( x \) at time \( t \);
- \( L > 0 \) is the length of the spatial domain;
- \( \alpha \) is a real constant parameter;
- \( y_0 \in L^2(0, L) \).
Known results with boundary feedback

\[
\begin{aligned}
&y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) \\
&\quad + y(x, t)y_x(x, t) = 0, \quad x \in (0, L), \ t > 0, \\
&y(0, t) = y(L, t) = 0, \quad t > 0, \\
&y_x(L, t) = \alpha y_x(0, t), \quad t > 0, \\
&y(x, 0) = y_0(x), \quad x \in (0, L).
\end{aligned}
\]

Stability result [Zhang 1994 ($L = 1$), Perla Menzala, Vasconcellos, Zuazua 2002]

For $L \notin \mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, \ k, l \in \mathbb{N}^* \right\}$ and $|\alpha| < 1$, local exponential stability result (i.e. for small initial data).

Remark

If $L = 2\pi$, there exists a solution $(y(x, t) = 1 - \cos x)$ of the linearized system around 0 which has a constant energy.
Known results with internal feedback

\[
\left\{ \begin{array}{l}
y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + a(x)y(x, t) \\
+ y(x, t)y_x(x, t) = 0, \\
y(0, t) = y(L, t) = y_x(L, t) = 0, \\
y(x, 0) = y_0(x),
\end{array} \right. \quad x \in (0, L), \ t > 0,
\]

where \( a \) is a nonnegative function in \( L^\infty(0, L) \) such that \( a(x) \geq a_0 > 0 \) a.e. in an open nonempty subset \( \omega \) of \( (0, L) \).

**Stability result [Perla Menzala, Vasconcellos, Zuazua 2002, Pazoto 2005]**

For any \( L > 0 \), local exponential stability result (i.e. for small initial data) and semi-global stability result (i.e. for any initial data in a given ball).
The controllability problem with one control

The second question is to study the controllability of the following non-linear KdV equation with a boundary control on a bounded domain

\[
\begin{aligned}
\begin{cases}
y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) = 0, & x \in (0, L), t > 0, \\
y(0, t) = y(L, t) = 0, & t > 0, \\
y_x(L, t) = u(t), & t > 0, \\
y(x, 0) = y_0(x), & x \in (0, L),
\end{cases}
\end{aligned}
\]

(2)

In the above equations:

- \(y(x, t)\): amplitude of the water wave at position \(x\) at time \(t\);
- \(L > 0\) is the length of the spacial domain;
- \(u(t)\) is the control in \(L^2(0, T)\);
- \(y_0 \in L^2(0, L)\).
Known results with one control

Theorem (Rosier 1997)

- KdV equation linearized around 0 is exactly controllable in $L^2(0, L)$ iff $L \notin \{2\pi \sqrt{\frac{k^2+l^2+kl}{3}}, \ k, l \in \mathbb{N}^*\}$.

- If $L \notin \{2\pi \sqrt{\frac{k^2+l^2+kl}{3}}, \ k, l \in \mathbb{N}^*\}$, then KdV equation is exactly controllable around 0.
Controllability results with one control in critical cases
[Coron-Crépeau 2004, Cerpa 2007, Cerpa-Crépeau 2009]

\[ \begin{cases} 
  y_t + y_x + y_{xxx} + yy_x = 0, \\
  y(t, 0) = 0, \ y(t, L) = 0, \\
  y_x(t, L) = u(t), \\
  y(0, x) = y_0(x). 
\end{cases} \]

- To avoid the critical lengths the solution is to study the nonlinear problem directly.
- Only a finite space of missed directions $M$ for linear equation.
- The linearized equation is controllable in $M^\perp \subset L^2(0, L)$.
- The missed directions are attainable with a development of KdV at order 2 or 3.

**Main theorem (Cerpa-Crépeau 2009)**

For all $L$, KdV equation is exactly controllable around 0 at a time $T$ large enough.
Related works

- **Saturating distributed control:** Marx, Cerpa, Prieur, Andrieu 2017 (global stabilization),
- **Built of boundary feedback laws:** Cerpa, Crépeau 2009 (Gramian approach), Krstic, Smyshlyaev 2008, Cerpa, Coron 2013 (Backstepping approach), Coron, Lü 2014,
- **Locally asymptotically stable:** Chu, Coron, Shang 2015, Tang, Chu, Shang, Coron 2018,
- ...
1. Stability with boundary delayed feedback
   - Well-posedness and regularity results
   - Lyapunov approach for a first stability result
   - Second stabilization result - Observability approach

2. Internal feedbacks with delay

3. Boundary controllability on a tree
Outline

1. Stability with boundary delayed feedback
   - Well-posedness and regularity results
   - Lyapunov approach for a first stability result
   - Second stabilization result - Observability approach

2. Internal feedbacks with delay

3. Boundary controllability on a tree
The first main goal is to study the stability of the following non-linear KdV equation with a boundary feedback delayed term

\[
\begin{align*}
  y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) \\
  + y(x, t)y_x(x, t) &= 0, \quad x \in (0, L), \ t > 0, \ \\
  y(0, t) = y(L, t) &= 0, \quad t > 0, \ \\
  y_x(L, t) &= \alpha y_x(0, t) + \beta y_x(0, t - h), \quad t > 0, \\
  y_x(0, t) &= z_0(t), \quad t \in (-h, 0), \\
  y(x, 0) &= y_0(x), \quad x \in (0, L), \ \\
\end{align*}
\]  

(3)

- \( y(x, t) \): amplitude of the water wave at position \( x \) at time \( t \);
- \( h > 0 \) is the delay;
- \( L > 0 \) is the length of the spacial domain;
- \( \alpha \) and \( \beta \neq 0 \) are real constant parameters;
- \( y_0 \in L^2(0, L) \) and \( z_0 \in L^2(-h, 0) \).
Let us choose the following definition of the energy of system (3):

$$E(t) = \int_0^L y^2(x,t) dx + |\beta| h \int_0^1 y_x^2(0,t-h\rho) d\rho.$$  

Moreover, we will assume, that the parameters $\alpha$ and $\beta$ satisfy the following limitation:

$$|\alpha| + |\beta| < 1.$$  

Goal

- Long-time behavior of the energy $E(t)$
- Exponential stability: $E(t) \leq Ce^{-\nu t} E(0), \forall t > 0$
- Robustness with respect to the delay
Consider, for instance, the wave equation with boundary feedback delay:

\[
\begin{align*}
&u_{tt}(x, t) - u_{xx}(x, t) = 0 & x \in (0, L), t > 0, \\
&u(0, t) = 0, & t > 0, \\
&u_x(L, t) = -\alpha u_t(L, t) - \beta u_t(L, t - h), & t > 0, \\
&u_t(L, t) = z_0(t), & t \in (-h, 0), \\
&u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & x \in (0, L).
\end{align*}
\]

Assumption

\[0 \leq \beta < \alpha\]

If it is not the case, it can be shown that instabilities may appear:

- Datko 1988, Datko, Lagnese, Polis 1986 with \(\alpha = 0\)
- Nicaise, Pignotti 2006 in the more general case for the wave equation (see also Nicaise, V. 2010).
Strategy for the well-posedness

Ideas [Rosier 1997]

- Well-posedness result of the linear equation, with a priori estimates and regularity of the solutions,
- KdV linear equation with a right hand side,
- Well-posedness result of the nonlinear equation by a fixed point argument.
The linear KdV equation

We begin by proving the well-posedness of the KdV equation linearized around 0, that writes

\[ \begin{align*}
  y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) &= 0, & x \in (0, L), & t > 0, \\
y(0, t) &= y(L, t) = 0, & t > 0, \\
y_x(L, t) &= \alpha y_x(0, t) + \beta y_x(0, t - h), & t > 0, \\
y_x(0, t) &= z_0(t), & t \in (-h, 0), \\
y(x, 0) &= y_0(x), & x \in (0, L).
\end{align*} \]
Taking into consideration of the delay

Following Nicaise and Pignotti 2006, we set

\[ z(\rho, t) = y_x(0, t - \rho h) \]

for any \( \rho \in (0, 1) \) and \( t > 0 \). Then \( z \) satisfies the transport equation

\[
\begin{align*}
hz_t(\rho, t) + z_\rho(\rho, t) &= 0, & \rho \in (0, 1), t > 0, \\
z(0, t) &= y_x(0, t), & t > 0, \\
z(\rho, 0) &= z_0(-\rho h), & \rho \in (0, 1).
\end{align*}
\]

Consequently, (4) can be written as

\[
\begin{align*}
y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) &= 0, & x \in (0, L), t > 0, \\
hz_t(\rho, t) + z_\rho(\rho, t) &= 0, & \rho \in (0, 1), t > 0, \\
y(0, t) &= y(L, t) = 0, & t > 0, \\
z(0, t) &= y_x(0, t), & t > 0, \\
y_x(L, t) &= \alpha y_x(0, t) + \beta z(1, t), & t > 0, \\
z(\rho, 0) &= z_0(-\rho h), & \rho \in (0, 1), \\
y(x, 0) &= y_0(x), & x \in (0, L).
\end{align*}
\]
We introduce the Hilbert space $H = L^2(0, L) \times L^2(0, 1)$ equipped with the inner product
\[
\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^L y \tilde{y} \, dx + |\beta| h \int_0^1 z \tilde{z} \, d\rho.
\]

We denote by $\| \cdot \|_H$ the associated norm and this new norm is equivalent to the usual norm on $H$.

We then rewrite (4) as a first order system:
\[
\begin{cases}
  U_t(t) = \mathcal{A} U(t), & t > 0, \\
  U(0) = U_0 \in H,
\end{cases}
\]
where $U = \begin{pmatrix} y \\ z \end{pmatrix}$, $U_0 = \begin{pmatrix} y_0 \\ z_0(\cdot - h \cdot) \end{pmatrix}$,
and where the operator $\mathcal{A}$ is defined by
\[
\mathcal{A} = \begin{pmatrix}
  -\partial_{xxx} - \partial_x & 0 \\
  0 & -\frac{1}{h} \partial_\rho
\end{pmatrix},
\]
with domain
\[
\mathcal{D}(\mathcal{A}) = \left\{ (y, z) \in H^3(0, L) \times H^1(0, 1) | y(0) = y(L) = 0, z(0) = y_x(0), y_x(L) = \alpha y_x(0) + \beta z(1) \right\}.
\]
Well-posedness result

We define the space

\[ \mathcal{B} := C([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L)) \]

endowed with the norm

\[
\|y\|_{\mathcal{B}} = \max_{t \in [0, T]} \|y(t)\|_{L^2(0, L)} + \left( \int_0^T \|y\|^2_{H^1(0, L)} \, dt \right)^{1/2}.
\]

To prove the well-posedness result of the non-linear KdV equation, we exactly follow Coron-Crêpeau 2004 (see also Cerpa 2014).

**Proposition**

Assume \(|\alpha| + |\beta| < 1\). There exist \(r > 0\) and \(C > 0\) such that for every \((y_0, z_0(-h \cdot)) \in H\) such that

\[
\|(y_0, z_0(-h \cdot))\|_H \leq r,
\]

there exists a unique solution of (3) which satisfies

\[
\|y\|_{\mathcal{B}} \leq C \|(y_0, z_0(-h \cdot))\|_H.
\]
Decay of the energy

Proposition

Let $|\alpha| + |\beta| < 1$. Then, for any regular solution of (3) the energy $E$ is non-increasing and satisfies

$$E'(t) = (\alpha^2 - 1 + |\beta|) y_x^2(0, t) + (\beta^2 - |\beta|) y_x^2(0, t - h) + 2\alpha\beta y_x(0, t)y_x(0, t - h) \leq 0.$$ 

Remark

We deduce that the energy $E$ is decreasing as long as $y_x(0, t)$ does not vanish.
Why $E$ is non-increasing?

Differentiating $E$ and using the system, we obtain

$$
\frac{d}{dt} E(t) = -2 \int_0^L y(x, t)(y_{xxx} + y_x + yy_x)(x, t)dx \\
- 2|\beta| \int_0^1 y_x(0, t - h\rho)\partial_{\rho} y_x(0, t - h\rho) d\rho
$$

$$
= y_x^2(L, t) - y_x^2(0, t) - |\beta| y_x^2(0, t - h) + |\beta| y_x^2(0, t)
$$

$$
= (\alpha^2 - 1 + |\beta|) y_x^2(0, t) + (\beta^2 - |\beta|) y_x^2(0, t - h) \\
+ 2\alpha\beta y_x(0, t)y_x(0, t - h)
$$

$$
= (MX(t), X(t)),
$$

where

$$
X(t) = \begin{bmatrix} y_x(0, t) \\ y_x(0, t - h) \end{bmatrix}
$$

and

$$
M = \begin{bmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 - |\beta| \end{bmatrix}.
$$
Why $E$ is non-increasing?

One can verify that $M$ is definite negative. Indeed, the trace of $M$ satisfies

$$\text{tr}M = \alpha^2 + \beta^2 - 1 < 0$$

if and only if $\alpha^2 + \beta^2 < 1$, and when we calculate the determinant of $M$, one gets

$$\det M = |\beta|((|\beta| - 1)^2 - \alpha^2),$$

so that $M$ is definite negative iff

$$\alpha^2 + \beta^2 < 1 \text{ and } (|\beta| - 1)^2 - |\alpha|^2 > 0,$$

which is equivalent to hypothesis

$$|\alpha| + |\beta| < 1.$$
We choose now the following Lyapunov functionnal

\[ V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t), \]

where \( \mu_1 \) and \( \mu_2 \in (0, 1) \) are positive constants that will be fixed small enough later on, \( V_1 \) is defined by

\[ V_1(t) = \int_0^L x y^2(x, t)dx, \]

and \( V_2 \) is defined by

\[ V_2(t) = h \int_0^1 (1 - \rho) y_x^2(0, t - h \rho) d\rho. \]

It is clear that the two energies \( E \) and \( V \) are equivalent, in the sense that

\[ E(t) \leq V(t) \leq \left( 1 + \max \left\{ L\mu_1, \frac{\mu_2}{|\beta|} \right\} \right) E(t). \]
Theorem

Assume $|\alpha| + |\beta| < 1$ and assume that the length $L$ fulfills

$$L < \pi \sqrt{3}.$$ 

Then, there exist $r > 0$ sufficiently small, such that for every $(y_0, z_0) \in L^2(0, L) \times L^2(-h, 0)$ satisfying

$$\|(y_0, z_0)\|_{L^2(0, L) \times L^2(-h, 0)} \leq r,$$

the energy $E$ of system (3) decays exponentially. More precisely, there exist two positive constants $\gamma$ and $\kappa$ such that

$$E(t) \leq \kappa E(0) e^{-2\gamma t}, \quad t > 0,$$

where for $\mu_1 > 0$ and $\mu_2 \in (0, 1)$ sufficiently small

$$\gamma \leq \min \left\{ \frac{(9\pi^2 - 3L^2 - 2L^3/2r\pi^2)\mu_1}{6L^2(1 + L\mu_1)}, \frac{\mu_2}{2(\mu_2 + |\beta|)h} \right\},$$
Remarks

- When the delay $h$ becomes larger, then the decay rate $\gamma$ is slower.

- In the case without delay (i.e. $\beta = 0$), adapting the previous results, the exponential stability is satisfied if and only if

$$|\alpha| < 1,$$


- In the case where $\alpha = 0$, the exponential stability is satisfied if and only if

$$|\beta| < 1.$$

- Even if $\alpha = \beta = 0$, the system (3) is exponentially stable in the case where $0 < L < \sqrt{3}\pi$. We recall that the main goal of this paper is to show that a delay does not destabilize the system, which may be the case in many other delayed systems.
Idea of the proof

Let \( y \) be a regular solution of (3). For any \( \gamma > 0 \), we have

\[
\frac{d}{dt} V(t) + 2\gamma V(t) \leq \left( \widetilde{M} X(t), X(t) \right) - 3\mu_1 \int_0^L y_x^2(x, t) dx + (2\gamma h(\mu_2 + |\beta|) - \mu_2) \int_0^1 y_x^2(0, t - h\rho) d\rho \\
+ (2\gamma (1 + L\mu_1) + \mu_1) \int_0^L y^2(x, t) dx + \frac{2}{3} \mu_1 \int_0^L y^3(x, t) dx,
\]

where

\[
X(t) = \begin{bmatrix} y_x(0, t) \\ y_x(0, t - h) \end{bmatrix}
\]

and

\[
\widetilde{M} = \begin{bmatrix} (1 + L\mu_1)\alpha^2 - 1 + |\beta| + \mu_2 & \alpha \beta (1 + L\mu_1) \\
\alpha \beta (1 + L\mu_1) & (1 + L\mu_1)\beta^2 - |\beta| \end{bmatrix}.
\]
Idea of the proof

Thus we have,

\[ \widetilde{M} = M + \mu_1 L \begin{pmatrix} \alpha^2 & \alpha \beta \\ \alpha \beta & \beta^2 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]

where \( M \) is defined as previously. As \( M \) is definite negative, we easily prove that for \( \mu_1 > 0 \) and \( \mu_2 > 0 \) sufficiently small the matrix \( \widetilde{M} \) is definite negative, by continuity of the applications Trace and Determinant.

Finally, for \( \mu_1 \) and \( \mu_2 \) sufficiently small, using Poincaré inequality, we obtain that

\[
\frac{d}{dt} V(t) + 2\gamma V(t) \leq \left( \frac{L^2 (2\gamma (1 + L\mu_1) + \mu_1)}{\pi^2} - 3\mu_1 \right) \int_0^L y_x^2(x, t) dx \\
+ (2\gamma h(\mu_2 + |\beta|) - \mu_2) \int_0^1 y_x^2(0, t - h\rho) d\rho \\
+ \frac{2}{3} \mu_1 \int_0^L y^3(x, t) dx.
\]
Idea of the proof

Moreover, using Cauchy-Schwarz inequality and since $H^1(0, L)$ embeds in $L^\infty(0, L)$, we have:

\[
\int_0^L y^3(x, t) \, dx \leq \|y\|^2_{L^\infty(0,L)} \int_0^L y(x, t) \, dx \\
\leq L \sqrt{L} \|y\|^2_{H^1(0,L)} \|y\|_{L^2(0,L)} \\
\leq L^{3/2} \| (y_0, z_0(-h \cdot)) \|_H \|y\|^2_{H^1(0,L)} \\
\leq L^{3/2} r \|y\|^2_{H^1(0,L)}.
\]

Consequently, we have

\[
\frac{d}{dt} V(t) + 2\gamma V(t) \\
\leq \left( \frac{L^2 (2\gamma (1 + L\mu_1) + \mu_1)}{\pi^2} - 3\mu_1 + \frac{2L^{3/2} r\mu_1}{3} \right) \int_0^L y_x^2(x, t) \, dx \\
+ (2\gamma h(\mu_2 + |\beta|) - \mu_2) \int_0^1 y_x^2(0, t - h\rho) \, d\rho.
\]
It is then sufficient to choose $r$ small enough such that

$$r < \frac{3(3\pi^2 - L^2)}{2L^{3/2}\pi^2}$$

(which is possible due to $L < \sqrt{3\pi}$) and $\gamma > 0$ such that

$$\gamma \leq \min \left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2}r\pi^2)\mu_1}{6L^2(1 + L\mu_1)}, \frac{\mu_2}{2(\mu_2 + |\beta|)h} \right\}.$$

to have

$$\frac{d}{dt} V(t) + 2\gamma V(t) \leq 0,$$

which is equivalent to $V(t) \leq V(0)e^{-2\gamma t}$ for any $t > 0$. Using the equivalence between $E$ and $V$, we obtain that

$$E(t) \leq \left( 1 + \max \left\{ L\mu_1, \frac{\mu_2}{|\beta|} \right\} \right) E(0)e^{-2\gamma t}, \quad t > 0.$$

By density of $\mathcal{D}(A)$ in $H$, the results extend to arbitrary $(y_0, z_0) \in L^2(0, L) \times L^2(-h, 0)$. 
We have seen that $\mu_1$ and $\mu_2$ are chosen small enough such that $\tilde{M}$ is definite negative. Simple calculations show that taking $\mu_1 > 0$ and $\mu_2 \in (0, 1)$ such that

$$\mu_2 < \min \left\{ 1 - \alpha^2 - \beta^2, \frac{(|\beta| - 1)^2 - \alpha^2}{1 - |\beta|}, \frac{\alpha^2 - \beta^2 + |\beta|}{|\beta|} \right\},$$

$$\mu_1 < \min \left\{ \frac{1 - \mu_2 - (\alpha^2 + \beta^2)}{L(\alpha^2 + \beta^2)}, \frac{(|\beta| - 1)^2 - \alpha^2 - \mu_2(1 - |\beta|)}{L(\alpha^2 - \beta^2 + |\beta| (1 - \mu_2))} \right\}$$

implies that $\tilde{M}$ is definite negative. It is then sufficient to take $r > 0$ such that $r < \frac{3(3\pi^2 - L^2)}{2L^{3/2} \pi^2}$ to have the exponential decay of the energy $E$ with the given decay rate $\gamma$.

Our hypothesis $L < \sqrt{3\pi}$ eliminates the set of critical lengths $N$ (see Rosier 1997). In particular, in the case $L = 2\pi$, it is obvious that the well-known solution $y(x, t) = 1 - \cos x$ of (4) has a constant energy.
Observability result of the linear equation

**Theorem**

Assume that $|\alpha| + |\beta| < 1$ is satisfied. Let $L \in (0, +\infty) \setminus \mathcal{N}$, where

$$
\mathcal{N} = \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, \ k, l \in \mathbb{N}^* \right\}
$$

and $T > h$. Then there exists $C > 0$ such that for all $(y_0, z_0(-h\cdot)) \in H$, we have

$$
\int_0^L y_0^2(x) \, dx + |\beta| h \int_0^1 z_0^2(-h\rho) \, d\rho \leq C \int_0^T \left( y_x^2(0, t) + z^2(1, t) \right) \, dt
$$

where $(y, z) = S(\cdot)(y_0, z_0(-h\cdot))$.

**Ideas of the proof**

- Contradiction argument (as in Rosier 1997);
Stability result for the linear KdV equation

Theorem
Assume that

\[ L \in (0, +\infty) \setminus \mathcal{N} \quad \text{and} \quad |\alpha| + |\beta| < 1. \]

Then, for every \((y_0, z_0) \in L^2(0, L) \times L^2(-h, 0)\), the energy of the linear system (4) decays exponentially. More precisely, there exist two positive constants \(\gamma\) and \(\kappa\) such that

\[ E(t) \leq \kappa E(0)e^{-\nu t}, \quad t > 0. \]

Ideas of the proof
Combine

- the observability inequality,
- the decay of the energy,
- the fact that the system is invariant by translation in time.

Remark: The value of the decay rate can not be estimated in this approach.
Exponential decay of small amplitude solutions of the non-linear KdV equation [Baudouin, Crépeau, V. 2019]

Theorem
Assume that

\[ L \in (0, +\infty) \setminus \mathcal{N} \quad \text{and} \quad |\alpha| + |\beta| < 1. \]

Then, there exists \( r > 0 \) such that for \( (y_0, z_0) \in L^2(0, L) \times L^2(-h, 0) \) st

\[ \| (y_0, z_0) \|_{L^2(0, L) \times L^2(-h, 0)} \leq r, \]

the energy of the non-linear system (3) decays exponentially. More precisely, there exist two positive constants \( \gamma \) and \( \kappa \) such that

\[ E(t) \leq \kappa E(0)e^{-2\gamma t}, \quad t > 0. \]

Idea of the proof: follows Cerpa 2014

- To decompose the solution as the solution of the linear system and the solution of the linear system with trivial initial data and right hand side;
- Use the exponential stability of the linear system.
Numerical simulations: $t \mapsto \ln(E(t))$ for different values of $\alpha, \beta$ and $h$ (adaptation of [Colin, Gisclon, 2001])

$T = 1$, $L = 1$ and $y_0(x) = 1 - \cos(2\pi x)$ and $z_0(\rho) = 0.1 \sin(-2\pi \rho h)$
Outline

1. Stability with boundary delayed feedback
   - Well-posedness and regularity results
   - Lyapunov approach for a first stability result
   - Second stabilization result - Observability approach

2. Internal feedbacks with delay

3. Boundary controllability on a tree
First case: \( \text{supp } b \subset \text{supp } a \)

\[
\begin{align*}
&\begin{cases}
  y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + a(x)y(x, t) \\
  \quad + b(x)y(x, t - h) + y(x, t)y_x(x, t) = 0, & x \in (0, L), \ t > 0, \\
  y(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\
  y(x, 0) = y_0(x), & x \in (0, L), \\
  y(x, t) = z_0(x, t), & x \in \omega, \ t \in (-h, 0),
\end{cases}
\end{align*}
\]

where \( a = a(x) \) and \( b = b(x) \) are nonnegative functions belonging to \( L^\infty(0, L) \). We will also assume that \( \text{supp } b = \omega, \ b(x) \geq b_0 > 0 \) a.e. in an open, nonempty subset \( \omega \) of \((0, L)\). We first assume that

\[\exists c_0 > 0, \quad b(x) + c_0 \leq a(x), \quad \text{a.e. in } \omega.\]

We define the energy as

\[E(t) = \int_0^L y^2(x, t)dx + h \int_\omega \int_0^1 \xi(x)y^2(x, t - h\rho)d\rho dx,\]

where \( \xi \in L^\infty(0, L) \) is chosen such that \( \text{supp } \xi = \text{supp } b = \omega \) and

\[b(x) + c_0 \leq \xi(x) \leq 2a(x) - b(x) - c_0, \quad x \in \omega.\]
First case: $\text{supp } b \subset \text{supp } a$

\[
\frac{d}{dt} E(t) = -y_x^2(0, t) - 2 \int_0^L a(x)y^2(x, t)dx \\
-2 \int_0^L b(x)y(x, t)y(x, t-h)dx + \int_\omega \xi(x)y^2(x, t)dx \\
- \int_\omega \xi(x)y^2(x, t-h)dx
\]

\[
\leq -y_x^2(0, t) + \int_\omega (-2a(x) + b(x) + \xi(x))y^2(x, t)dx \\
-2 \int_{(0, L) \setminus \omega} a(x)y^2(x, t)dx + \int_\omega (b(x) - \xi(x))y^2(x, t-h)dx.
\]

Results [V. 2020]:

- Local exponential stability result with a Lyapunov approach for $L < \pi \sqrt{3}$,
- Local exponential stability result for any $L > 0$,
- Semi-global stability result for any $L > 0$. 

Second case: $\text{supp } b \not\subset \text{supp } a$

In this case, the derivative of the energy $E$ satisfies

$$\frac{d}{dt} E(t) = -y_x^2(0,t) - 2 \int_{\text{supp } a} a(x)y^2(x,t)dx - 2 \int_{\omega} b(x)y(x,t)y(x, t-h)dx + \int_{\omega} \xi(x)y^2(x,t)dx - \int_{\omega} \xi(x)y^2(x, t-h)dx$$

$$\leq -y_x^2(0,t) - 2 \int_{\text{supp } a} a(x)y^2(x,t)dx + \int_{\omega} (b(x) + \xi(x))y^2(x,t)dx + \int_{\omega} (b(x) - \xi(x))y^2(x, t-h)dx,$$

and so the energy is not decreasing in general due to the term $b(x) + \xi(x) > 0$ on $\omega$. 
Following Nicaise Pignotti 2014, we consider the next auxiliary problem, which is "close" to (5) but whose the energy is decreasing:

\[
\begin{aligned}
\ddot{y}_t(x, t) + \dddot{y}(x, t) + \ddot{y}_x(x, t) + a(x)\ddot{y}(x, t) + b(x)\ddot{y}(x, t - h) \\
\quad + \xi b(x)\ddot{y}(x, t) &= 0, \\
\ddot{y}(0, t) = \ddot{y}(L, t) = \dddot{y}_x(L, t) &= 0, \\
\ddot{y}(x, 0) &= y_0(x), \\
\ddot{y}(x, t) &= z_0(x, t),
\end{aligned}
\]

where \(\xi\) is a positive constant. Then the derivative of the energy \(E\) defined by

\[
E(t) = \int_0^L \ddot{y}^2(x, t)dx + h\xi \int_{\omega} \int_0^1 b(x)\ddot{y}^2(x, t - h\rho)d\rho dx,
\]

with \(\xi > 1\) satisfies

\[
\frac{d}{dt} E(t) = -\dddot{y}^2_x(0, t) - 2 \int_{\text{supp } a} a(x)\dddot{y}^2(x, t)dx - 2 \int_{\omega} b(x)\ddot{y}(x, t)\ddot{y}(x, t - h)dx \\
- 2\xi \int_{\omega} b(x)\ddot{y}^2(x, t)dx + \xi \int_{\omega} b(x)\ddot{y}^2(x, t)dx - \xi \int_{\omega} b(x)\ddot{y}^2(x, t - h)dx \\
\leq -\dddot{y}^2_x(0, t) - 2 \int_{\text{supp } a} a(x)\dddot{y}^2(x, t)dx + \int_{\omega} (b(x) - \xi b(x))\ddot{y}^2(x, t)dx \\
+ \int_{\omega} (b(x) - \xi b(x))\ddot{y}^2(x, t - h)dx \leq 0.
\]
Second case: \( \text{supp} \, b \notin \text{supp} \, a \)

We would like to use the classical perturbation result of Pazy:

**Theorem (Pazy)**

Let \( X \) be a Banach space and let \( A \) be the infinitesimal generator of a \( C_0 \) semigroup \( T(t) \) on \( X \) satisfying \( \|T(t)\| \leq Me^{\omega t} \). If \( B \) is a bounded linear operator on \( X \), then \( A + B \) is the infinitesimal generator of a \( C_0 \) semigroup \( S(t) \) on \( X \) satisfying \( \|S(t)\| \leq Me^{(\omega + M\|B\|)t} \).

**Strategy:**

1. Exponential stability for (6) **linearized around** \( 0 \) by the Lyapunov approach for all \( L < \sqrt{3\pi} \);

2. Exponential stability for (5) **linearized around** \( 0 \) using the perturbation theorem of Pazy for all \( L < \sqrt{3\pi} \) and for \( \|b\|_{L^\infty(0,L)} \) small enough \( (-\alpha + \sqrt{\beta} \xi \|b\|_{L^\infty(0,L)} < 0) \);

3. **Local** exponential stability for the **nonlinear** system (3) for all \( L < \sqrt{3\pi} \) and for \( \|b\|_{L^\infty(0,L)} \) small enough using the same proof as previously.
Theorem

Let $L < \sqrt{3}\pi$ and $\xi > 1$. Then there exist $\delta > 0$ (depending on $\xi$, $L$, $h$) and $r > 0$ sufficiently small such that if

$$\|b\|_{L^\infty(0,L)} \leq \delta,$$

for every $(y_0, z_0) \in \mathcal{H}$ satisfying

$$\|(y_0, z_0)\|_{\mathcal{H}} \leq r,$$

the energy decays exponentially.

Remarks:
- we can take $a = 0$,
- if $h$ is large, the choice of $b$ is such that $\|b\|_{L^\infty(0,L)}$ is small.
Numerical simulations: $t \mapsto \ln(E(t))$ for different values of $a$ and $b$

$T = 10$, $L = 3$, $h = 2$ and $y_0(x) = 1 - \cos(2\pi x)$ and $z_0(x, \rho) = (1 - \cos(2\pi x)) \cos(2\pi \rho h)$ with $\text{supp} \ a = \text{supp} \ b = (0, L/5)$
Mixed internal and boundary dampings with delay: first case

\[
\begin{cases}
    y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) + a(x)y(x, t) = 0, \\
    x \in (0, L), \
    t > 0, \\
    y(0, t) = y(L, t) = 0, \\
    y_x(L, t) = \beta y_x(0, t - h), \\
    y(x, 0) = y_0(x), \\
    y_x(0, t) = z_0(t),
\end{cases}
\]

where \(|\beta| < 1\), and where \(a\) is a nonnegative function in \(L^\infty(0, L)\) such that \(a(x) \geq a_0 > 0\) a.e. in \(\omega\) an open nonempty subset of \((0, L)\). In this case, it is sufficient to use [Pazoto 2005] and [Baudouin, Crépeau, V. 2019] to obtain the local exponential stability result for every non critical length (i.e.

\(L \notin \mathcal{L} = \left\{ 2\pi \sqrt{\frac{k^2+kl+l^2}{3}}, \ k, l \in \mathbb{N}^* \right\} \)).
Mixed internal and boundary dampings with delay: second case

\[
\begin{aligned}
    &y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + y(x, t)y_x(x, t) + b(x)y(x, t - h) = 0, \\
    &x \in (0, L), \ t > 0, \\
    &y(0, t) = y(L, t) = 0, \\
    &y_x(L, t) = \alpha y_x(0, t), \\
    &y(x, 0) = y_0(x), \\
    &y(x, t) = z_0(x, t),
\end{aligned}
\]

where $|\alpha| < 1$, $b$ is a nonnegative function in $L^\infty(0, L)$ such that $b(x) \geq b_0 > 0$ a.e. in $\text{supp} \ b = \omega$ and where $\|b\|_{L^\infty(0, L)}$ is small enough. Then we can follow the previous method by introducing a "close" exponentially stable system, to obtain the local exponential stability result for every $L < \sqrt{3}\pi$.

Note that we can take $\alpha = 0$ here.
Some open problems

- Improve some previous results, for instance remove $L < \sqrt{3\pi}$, with an appropriate Lyapunov functional;
- Time-delay on the nonlinear term;
- Time-varying delay;
- Rapid stabilization with delay;
- Saturating control with delay;
- ...

Outline

1 Stability with boundary delayed feedback
   - Well-posedness and regularity results
   - Lyapunov approach for a first stability result
   - Second stabilization result - Observability approach

2 Internal feedbacks with delay

3 Boundary controllability on a tree
A tree-shaped network

We consider a tree-shaped network $\mathcal{R}$ of $(N + 1)$ edges $e_i$, of lengths $l_i > 0$, $i \in \{1, ..., N + 1\}$, connected at one vertex that we assume to be 0 for all the edges. We assume that $e_1$ is parametrized on the interval $I_1 := (-l_1, 0)$ and the $N$ other edges $e_i$ are parametrized on the interval $I_i := (0, l_i)$. 

![Diagram of a tree-shaped network with vertices labeled 0, -l_1, l_2, and l_3, and edges indicating no control and control.]
The nonlinear KdV equation on a tree

\[\begin{align*}
(y_i, t + y_i, x + y_i, xxx + y_i y_i, x)(x, t) &= 0, \quad i \in \{1, \cdots, N + 1\}, \quad x \in I_i, \quad t > 0, \\
y_1(-l_1, t) &= 0, \\
y_i(l_i, t) &= 0, \\
y_i, x(l_i, t) &= h_i(t), \\
y_1(0, t) &= \alpha_i y_i(0, t), \\
\forall i \in \{2, \cdots, N + 1\}, \quad t > 0, \\
y_1, x(0, t) &= \sum_{i=2}^{N+1} \beta_i y_i, x(0, t), \\
y_1, xx(0, t) &= \sum_{i=2}^{N+1} \frac{1}{\alpha_i} y_i, xx(0, t), \\
y_i(x, 0) &= y_{i0}(x), \\
\forall i \in \{1, \cdots, N + 1\}, \quad x \in I_i,
\end{align*}\]

- \(y_i(x, t)\): amplitude of the water wave on \(e_i\) at \(x \in I_i\) at \(t\),
- \(h_i = h_i(t)\) is the control on the edge \(e_i\) belonging to \(L^2(0, T)\),
- \(\alpha_i\) and \(\beta_i\) are positive constants, \(y_{i0} \in L^2(I_i)\),
- the transmission conditions at the central node 0 are inspired by [Mugnolo, Noja, Seifert 2018] and [Cavalcante 2018].
Notation

\[ L^2(\mathcal{R}) = \{ f : \mathcal{R} \rightarrow \mathbb{R}, f_i \in L^2(I_i), \forall i \in \{1, \ldots, N + 1\} \} \]

For shortness, for \( f \in L^1(\mathcal{R}) = \{ f : \mathcal{R} \rightarrow \mathbb{R}, f_i \in L^1(I_i), \forall i \in \{1, \ldots, N + 1\} \} \) we often write,

\[
\int_{\mathcal{R}} f \, dx = \int_{-l_1}^{0} f_1(x) \, dx + \sum_{i=2}^{N+1} l_i \int_{0}^{l_i} f_i(x) \, dx.
\]

Then the norm of the Hilbert space \( L^2(\mathcal{R}) \) is defined by

\[
\|f\|_{L^2(\mathcal{R})}^2 = \int_{\mathcal{R}} |f|^2 \, dx.
\]

Goal: exact controllability

For any \( T > 0, l_i > 0, y_0 \in L^2(\mathcal{R}) \) and \( y_T \in L^2(\mathcal{R}) \), is it possible to find \( N \) Neumann boundary controls \( h_i \in L^2(0, T) \) such that the solution \( y \) on the tree shaped network of \( N + 1 \) edges satisfies \( y(\cdot, 0) = y_0 \) and \( y(\cdot, T) = y_T \)?
Known results about the controllability of the KdV equation on a network

Known results: star-shaped network

- Ammari, Crépeau 2018: \( N + 1 \) controls for \( N \) edges,
- Cerpa, Crépeau, Moreno 2019: \( N \) controls for \( N \) edges.

Main differences

- the sense of the propagation of the water wave on the first edge,
- the transmission conditions at the central node,
- the fact that we improve the previous results having one control less here.
Proposition

Let $T > 0$, $l_i > 0$ and assume

$$\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} \leq 1 \quad \text{and} \quad \sum_{i=2}^{N+1} \beta_i^2 \leq 1.$$ 

Then, there exist $r > 0$ and $C' > 0$ such that for every $y_0 \in L^2(\mathcal{R})$ and $h_i \in L^2(0, T)$ verifying

$$\|y_0\|_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0, T)} \leq r,$$

there exists a unique $y \in \mathcal{B} = C([0, T], L^2(\mathcal{R})) \cap L^2(0, T, H^1_0(\mathcal{R}))$ which satisfies

$$\|y\|_{\mathcal{B}} \leq C \left(\|y_0\|_{L^2(\mathcal{R})} + \sum_{i=2}^{N+1} \|h_i\|_{L^2(0, T)}\right).$$
The proof of the well-posedness result

Ideas of the proof

- **Linear** equation with no control, then with regular initial data and controls,
- **Linear** equation with less regularity on the data using density and multiplier arguments,
- **Linear** equation with a source term,
- **Nonlinear** equation by fixed point argument.
Theorem

Let \( l_i > 0 \) satisfying

\[
L := \max_{i=1,\ldots,N+1} l_i < \sqrt{3\pi} \left( \frac{\min(1, \frac{\alpha_i}{N \beta_i})}{\max(1, \frac{\alpha_i}{N \beta_i})} \right)^{1/2} \sqrt{\frac{1}{2\pi^2 \left( 1 - \sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} \right) + 1}}
\]  

(7)

and assume that

\[
\sum_{i=2}^{N+1} \frac{1}{\alpha_i^2} \leq 1 \quad \text{and} \quad \sum_{i=2}^{N+1} \beta_i^2 = 1.
\]

(8)

There exists \( T_{\text{min}} > 0 \) such that the system is locally exactly controllable in any time \( T > T_{\text{min}} \): there exists \( r > 0 \) sufficiently small such that for any \( y_0 \in L^2(\mathcal{R}) \) and \( y_T \in L^2(\mathcal{R}) \) with

\[
\|y_0\|_{L^2(\mathcal{R})} < r \quad \text{and} \quad \|y_T\|_{L^2(\mathcal{R})} < r,
\]

there exist \( N \) Neumann boundary controls \( h_i \in L^2(0, T) \) such that \( y \) satisfies \( y(\cdot, 0) = y_0 \) and \( y(\cdot, T) = y_T \) for \( T > T_{\text{min}} \).
Remarks

If \( \sum_{i=2}^{N+1} \frac{1}{\alpha_{i}^2} = 1 \), then (7) becomes \( L < \sqrt{3\pi} \left( \frac{\min(1, \frac{\alpha_{i}}{N\beta_{i}})}{\max(1, \frac{\alpha_{i}}{N\beta_{i}})} \right)^{1/2} \).

If \( \alpha_{i} = \sqrt{N} \) and \( \beta_{i} = \frac{1}{\sqrt{N}} \), then (7) becomes \( L < \sqrt{3\pi} \).

Ideas of the proof

- Linearize the system around a stationary solution (here 0),
- Show the exact controllability result of the linear KdV equation (by linearity, we can take \( y_{0} = 0 \)) with an observability inequality of the linear backward adjoint system obtained by the multiplier method,
- Apply a fixed point theorem to have the local exact controllability result of the nonlinear equation.
Remarks

- Drawback of this method: we do not obtain sharp conditions on the lengths $l_i$ and on the time of control $T_{\text{min}}$.
- Advantage: we get an explicit constant of observability.

Comment

A same type of result can be obtained for a general tree with $N + 1$ external vertices, we get the controllability result with only $N$ Neumann controls.

Open questions

- Is it possible to reduce the number of controls at the external vertices and still having a control result?
- Network with a circuit.
- Observability inequality with a contradiction argument, case of critical lengths.
Time-delay on the nonlinear term

\[ \begin{aligned}
    y_t(x, t) + y_{xxx}(x, t) + y_x(x, t) + a(x)y(x, t) \\
    + y(x, t - h)y_x(x, t) = 0, \\
    y(0, t) = y(L, t) = y_x(L, t) = 0, \\
    y(x, 0) = y_0(x), \\
    y(x, t) = z_0(x, t),
\end{aligned} \]

\( x \in (0, L), \ t > 0, \)
\( t > 0, \)
\( x \in (0, L), \)
\( x \in \omega, \ t \in (-h, 0), \)

The energy defined by

\[ E(t) = \int_0^L y^2(x, t)\,dx \]

satisfies

\[ \frac{d}{dt} E(t) = -y_x^2(0, t) - 2 \int_0^L a(x)y^2(x, t)\,dx - 2 \int_0^L y(x, t)y(x, t - h)y_x(x, t)\,dx \]

and

\[ \frac{d}{dt} \int_0^L \int_0^1 y^2(x, t-h\rho)\,d\rho dx = -\frac{1}{h} \int_0^L y^2(x, t-h)\,dx + \frac{1}{h} \int_0^L y^2(x, t)\,dx \ldots \]