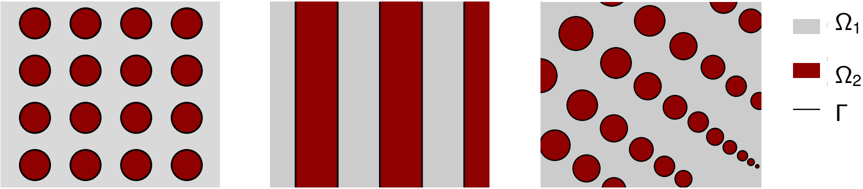
# Transmission conditions for HJB equations and optimal control problems on stratified domains

Hasnaa Zidani

Groupe de Travail "Contrôle", Laboratoire Jacques-Louis Lions




$$\begin{cases} \partial_t u_i(t,x) + H_i(x, D_x u_i(t,x)) = 0 & t \in (0,T),\ x \in \Omega_i, \\ u_i(0,x) = \varphi(x) & x \in \mathbb{R}^d, \end{cases}$$

$$u = u_1, u_2$$

$$\partial_t u(t,x) + H_\Gamma(x, D_x u_1(t,x), D_x u_2(t,x)) = 0 \quad \text{on } \Gamma$$

# Outline

# Example 1: A navigation problem in presence of an observer

- Consider a controlled system:

$$\dot{y}(s) = f(y(s), \alpha(s)), \quad \dot{z}(s) = \mathbf{1}_{\Omega_1}(y(s))$$
$$y(0) = x_0, \quad z(0) = 0,$$

with the control input $\alpha(s) \in A$ for a.e $s \geq 0$.

- Goal: finding the fastest path to the "*Target*" subject to restriction on a maximum allowable *visibility* time:

$$z(s) \leq b.$$

- The state variable $z(\cdot)$ has a discontinuous dynamics.

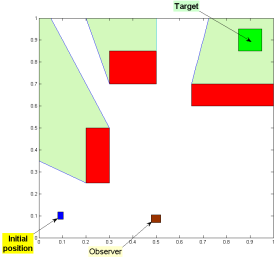
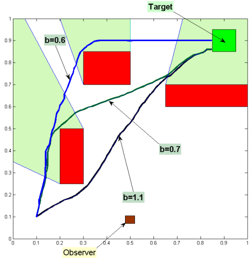



Figure: Navigation problem with an observer



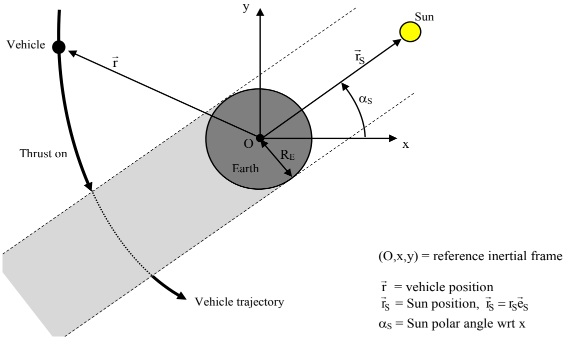
# Coplanar orbit transfer with eclipse constraints (Ref. Haberkorn-Trélat'11)

➢ Consider a coplanar orbit transfer of a satellite around the Earth:

$$\ddot{r}(s) = -\frac{\mu}{|r(s)|^3} r(s) + \frac{T_{\max}}{m(s)} u(s) \mathbf{1}_{\Omega_2}(r(s)), \quad \dot{m}(s) = -\beta T_{\max} |u(s)|,$$

where the control input $u(s) \in \mathbb{B}(0,1)$, and $T_{\max}$ is the maximal allowed thrust.

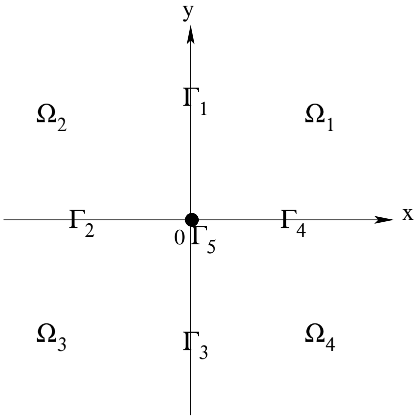
- $\Omega_1$: *the shadow zone*,
- and $\Omega_2 := \mathbb{R}^2 \setminus \overline{\Omega}_1$.



# Position of the problem: junction conditions on multi-domains

- $\mathbb{R}^d$ is divided into several open disjoint sub-domains $(\Omega_i)_{i=1,\dots,m}$ with

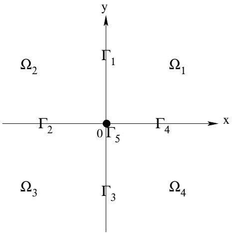
$$\mathbb{R}^d = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_m.$$



# Transmission conditions on multi-domains

- $\mathbb{R}^d$ is divided into several open disjoint sub-domains $(\Omega_i)_{i=1,\dots,m}$ with

$$\mathbb{R}^d = \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_m.$$




- Consider a dynamical system given by:

$$\begin{cases} \dot{y}(s) \in F_i(y(s)) & \text{if } y(s) \in \Omega_i, \\ y(0) = x, \end{cases} \qquad \text{(SI)}$$

where $F_i$ is a Lipschitz set-valued function on $\overline{\Omega}_i$ and consider $\mathcal{S}_{[0,t]}(x)$ the set of all trajectories *satisfying* (SI).

- Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function, and consider the control problem

$$v(t,x) := \inf \left\{ \varphi(y(t)) : y \in \mathcal{S}_{[0,t]}(x) \right\}.$$

➢ We can expect the value function $v$ to satisfy the system of HJB equations:

$$\begin{cases} \partial_t v(t,x) + H_i(x, D_x v(t,x)) = 0 & t \in (0,T),\ x \in \Omega_i, \\ v(0,x) = \varphi(x) & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $T > 0$ is a fixed final time and $\varphi$ is a given regular function.

➢ Each $H_i$ is a Lipschitz continuous Hamiltonian of Bellman form:

$$H_i(x,p) := \sup_{q \in F_i(x)} \{-q \cdot p\}.$$

### Transmission conditions

Question 1: what conditions of transition between the subdomains $\Omega_i$ should be considered to ensure the existence and *uniqueness* of solution for the system (1)?

Question 2: What controllability conditions are needed to ensure that the value function $v$ is Lipschitz continuous?

# Mayer's optimal control problems

- Assume: $\mathbb{R}^d = \Omega$
- Let $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ be a Lipschitz continuous multi-function with closed convex images. For an initial data $(t, x) \in [0, +\infty) \times \mathbb{R}^d$, consider the differential inclusion:

$$\begin{cases} \dot{y}(s) \in F(y(s)) & s \in (0, t), \\ y(0) = x. \end{cases} \tag{2}$$

The set $\mathcal{S}_{[0,t]}(x)$ of all trajectories satisfying (2) is a compact set in $W^{1,\infty}(0, t)$.

- Consider $\varphi : \mathbb{R}^d \to \mathbb{R}$ a Lipschitz function
- And the value function $v : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ is defined by

$$v(t, x) := \inf \left\{ \varphi(y(t)) : y \in \mathcal{S}_{[0,t]}(x) \right\}.$$

# Classical Dynamical Programming Principle

For any $t \in [0, T],\ x \in \mathbb{R}^d$, $h \in [0, t]$, we have

$$v(t,x) = \min_{y \in \mathcal{S}_{[0,t]}(x)} \{v(t-h, y(h))\}.$$

More precisely,

- The super-optimality: There exists an optimal trajectory $\bar{y} \in \mathcal{S}_{[0,t]}(x)$ such that

$$v(t,x) \geq \ (=)\, v(t-h, \bar{y}(h));$$

- The sub-optimality: For any trajectory $y \in \mathcal{S}_{[0,t]}(x)$, we have:

$$v(t,x) \leq v(t-h, y(h)).$$

# Characterization

In the classical case where $F$ is Lipschitz continuous, consider the Hamiltonian

$$H_F(x,p) = \sup_{q \in F(x)} \{-q \cdot p\}.$$

For any Lipschitz continuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, we have

- $u$ satisfies the super-optimality $\Leftrightarrow$ $u$ satisfies

$$\partial_t u(t,x) + H_F(x, Du(t,x)) \geq 0;$$

- $u$ satisfies the sub-optimality $\Leftrightarrow$ $u$ satisfies

$$\partial_t u(t,x) + H_F(x, Du(t,x)) \leq 0.$$

## Classical case

The value function $v$ defined by:

$$v(t,x) := \min_{y \in \mathcal{S}_{[0,t]}(x)} \varphi(y(t)),$$

is the unique continuous viscosity solution of

$$\begin{aligned} &\partial_t v(t,x) + H_F(x, Dv(t,x)) = 0 \\ &v(0,x) = \varphi(x). \end{aligned}$$

A proof of uniqueness: Let $u_1, u_2$ be continuous functions with $u_1(0,x) = u_2(0,x) = \varphi(x)$.

- $u_1$ is subsolution $\Longrightarrow$ $u_1$ satisfies the sub-optimality $\Longrightarrow$ $u_1 \leq v$;
- $u_2$ is supersolution $\Longrightarrow$ $u_2$ satisfies the super-optimality $\Longrightarrow$ $u_2 \geq v$;

# Consider the simplest situation of two-domains

- Let $g : \mathbb{R}^d \to \mathbb{R}$ be a $C^2$ function, and assume:

$$\Omega_1 := \{x \in \mathbb{R}^d \mid g(x) < 0\},$$
$$\Omega_2 := \{x \in \mathbb{R}^d \mid g(x) > 0\},$$
$$\Gamma := \{x \in \mathbb{R}^d \mid g(x) = 0\}.$$

- Define a global dynamics by:

$$F(x) := F_i(x) \quad \text{when } x \in \Omega_i, \qquad F(x) = F_1(x) \cup F_2(x) \quad \text{when } x \in \Gamma,$$

where $F_i : \overline{\Omega}_i \rightsquigarrow \mathbb{R}^2$ are multi-applications satisfying the following: for any $x \in \overline{\Omega}_i$,

  - $F_i(x)$ is a non-empty, convex and compact set;
  - $F_i$ is Lipschitz continuous on $\overline{\Omega}_i$.

# Filippov regularization for the dynamics

To define the global dynamics, we consider the following Filippov regularization of the multi-functions $(F_i)_{i=1,2}$:

$$FC(x) := \begin{cases} F_i(x) & \text{if } x \in \Omega_i, \\ \overline{co}(F_i(x) : x \in \overline{\Omega}_i) & \text{otherwise.} \end{cases}$$

Then consider the differential inclusion

$$\begin{cases} \dot{y}(s) \in FC(y(s)) & s \in (0,t), \\ y(0) = x. \end{cases} \tag{3}$$

Remark: *FC* is the smallest upper semi-continuous envelope of $(F_i)_{i=1,2}$. Moreover,

$$FC(x) = \bigcap_{\varepsilon > 0} \overline{co} \left\{ F_i(y) : \|y - x\| < \varepsilon, y \in \overline{\Omega}_i \right\}.$$

# Global optimal control problem

For any $t \in [0, T]$, $x \in \mathbb{R}^2$, we set

$$S_{[0,t]}(x) := \{y(\cdot) | y(t) = x, \ \ \dot{y}(s) \in FC(y(s)) \text{ for } s \in (0,t)\}$$

as the set of trajectories. Consider the value function

$$v(t,x) := \inf_{y(\cdot)\in S_{[0,t]}(x)} \{\varphi(y(t))\}.$$

The properties of $FC$ imply that $S_{[0,t]}(x)$ is a compact set, then the "inf" is in fact "min".

Remark: The optimal control problem involves a hybrid dynamical system: $\Longrightarrow$ Zeno effect

# Regularity of the value function - Controllability assumptions

- (H1) - There exists $\delta > 0$ such that $B(0, \delta) \subset F_i(x)$.
- (H2) - Either for any $x \in \Gamma$, $FC(x) \cap \mathcal{T}_\Gamma(x) = \emptyset$ or there exists $\delta > 0$ such that $B(0, \delta) \subset FC(x)$

- Under assumption (H1), the value function is Lipschitz continuous
- Under assumption (H2), the value function is lsc.

# Dynamical Programming Principle

For any $t \in [0, T],\ x \in \mathbb{R}^2$, $h \in [0, t]$, we have

$$v(t,x) = \min_{y(\cdot)\in S_{[t,T]}(x)} \{v(t-h, y(h))\}.$$

More precisely,

- The super-optimality: $\exists\, \bar{y} \in S_{[t,T]}(x)$ such that

$$v(t,x) \geq\ (=)\, v(t-h, \bar{y}(h));$$

- The sub-optimality: $\forall\, y \in S_{[t,T]}(x)$ such that

$$v(t,x) \leq v(t-h, y(h)).$$

# Some references on non-Lipschitz Hamiltonians

- Hamilton-Jacobi equations with discontinuous Hamiltonians: Ishii'89.
- Hamilton-Jacobi-Bellman equations with discontinuous Lagrangians: Soravia'02.
- Hamilton-Jacobi equations with measurable Hamiltonians: Camilli-Siconolfi'03.
- Optimal control on multi-domains: Bressan-Hong'07; Barles-Briani-Chasseigne'13-14; Rao-HZ'13; Rao-Siconolfi-HZ'14, Barles-Chasseigne'18; Rao-Ghili-HZ'19, ...
- Hamilton-Jacobi approach to junction problems on network: Achdou-Camilli-Cutri-Tchou'12, Achdou-Tchou'15, Imbert-Monneau-HZ'13, Lions-Souganidis'17, Graber-Hermosilla-HZ'17, Dao'18, ...

# Characterization of the the value function

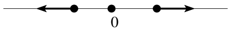
$FC$ is only upper semi-continuous, then the characterization of the sub-optimality fails, i.e. $v$ does not satisfy

$$\partial_t v(t,x) + \sup_{p \in FC(x)} \{-p \cdot Dv(t,x)\} \leq 0.$$

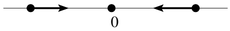
From a point view of trajectories: the set of dynamics $FC(x)$ is too large and may contain some useless dynamics which are never used by the trajectories.

Examples in dimension 1 with one interface $\{0\}$:

- The case "pull-pull":
  $F_1(x) = [-1, \frac{1}{2}]$ for $x < 0$, $F_2(x) = [-\frac{1}{2}, 1]\}$ for $x > 0$. The convexification on 0 is $FC(0) = [-1, 1]$.




- The case "push-push":
  $F_1(x) = \{1\}$ for $x < 0$, $F_2(x) = \{-1\}$ for $x > 0$. The convexification on 0 is also $FC(0) = [-1, 1]$ which is not the reasonable set of dynamics.




We need to consider only the "useful" dynamics!

# Essential dynamics

Some notations:

- $\mathcal{T}_{\overline{\Omega}_i}(x)$ is the tangent cone of $\overline{\Omega}_i$ on $x$,
- $\mathcal{T}_\Gamma$ the tangent space of $\Gamma$ on $x$.

Introduce the **essential dynamic** multifunction $F^E : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ (ref. Barnard-Wolenski'13):

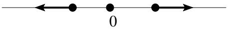
- For $x \in \Omega_i$, $F^E(x) := F_i(x)$;
- For $x \in \Gamma$,

$$F^E(x) = (FC(x) \cap \mathcal{T}_\Gamma(x)) \bigcup \bigcup_{i=1,2} \left( F_i(x) \cap \mathcal{T}_{\overline{\Omega}_i}(x) \right).$$

  - $F_i(x) \cap \mathcal{T}_{\overline{\Omega}_i}(x)$: the dynamics in $F_i(x)$ are inward to $\overline{\Omega}_i$,
  - $F(x) \cap \mathcal{T}_\Gamma(x)$: the dynamics in $F(x)$ are tangent to $\Gamma$.

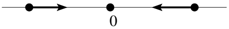
# Examples in dimension 1

- Example of "pull-pull":




$$F_1(0) \cap \mathcal{T}_{\overline{\Omega}_1}(0) = \{-1\},\ F_2(0) \cap \mathcal{T}_{\overline{\Omega}_2}(0) = \{1\},\ F(0) \cap \mathcal{T}_\Gamma(0) = \{0\},$$
$$F^E(0) = \{-1, 0, 1\}.$$

- Example of "push-push":




$$F_1(0) \cap \mathcal{T}_{\overline{\Omega}_1}(0) = F_2(0) \cap \mathcal{T}_{\overline{\Omega}_2}(0) = \emptyset,\ F(0) \cap \mathcal{T}_\Gamma(0) = \{0\},$$
$$F^E(0) = \{0\}.$$

# Examples in dimension 2

$\Omega_1 = \{(x,y) | x < 0, y \in \mathbb{R}\}$, $\Omega_2 = \{(x,y) | x > 0, y \in \mathbb{R}\}$, $\Gamma = \{(0,y) | y \in \mathbb{R}\}$.

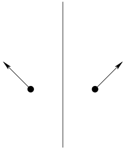


Figure: *

Original dynamics:
$F_1 = \{(-1,1)\}$,
$F_2 = \{(1,1)\}$.

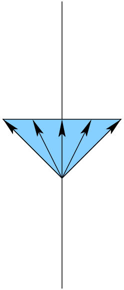


Figure: *

Convexified dynamics

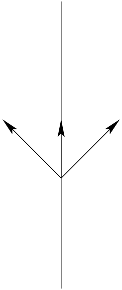


Figure: *

Essential dynamics

# Properties of $F^E$

- $F^E(x) \subset FC(x)$.
- $F^E(x)$ doesn't have good properties: no convexity, no continuity, But ...

- the vectorfield $F^E$ provides the same set of trajectories than $FC$:

$$\{y \mid \dot{y}(s) \in FC(y(s)); y(t) = x\} = \left\{y \mid \dot{y}(s) \in F^E(y(s)); y(t) = x\right\}.$$

- For any $p \in F^E(x)$, there exists $y \in S_{[t;T]}(x)$ such that $y(t) = x$ and $\dot{y}(t) = p$.

# Equations on the interface

Consider the new Essential Hamiltonian

$$H^E(x,q) := \sup_{p \in F^E(x)} \{-p \cdot q\}.$$

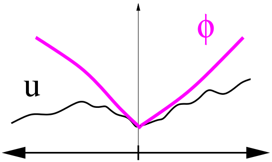
And here is the existence result:

**Theorem** (Rao-HZ'12, Rao-Ghili-HZ'19)

*- If (H1) holds and $\varphi$ is Lipschitz continuous, then t $v$ is Lipschitz continuous and is the viscosity solution of the system of HJB equations:*

$$\begin{cases} \partial_t u(t,x) + H_i(x, Du(t,x)) = 0 & t \in (0,T),\ x \in \Omega_i, \\ \partial_t u(t,x) + H^E(x, Du(t,x)) = 0 & t \in (0,T),\ x \in \Gamma, \\ u(0,x) = \varphi(x) & x \in \mathbb{R}^d. \end{cases}$$

*- If (H2) holds and $\varphi$ is l.s.c, then $v$ is lsc and is the Bilateral-viscosity solution of the above system of HJB equations.*




## Definition: viscosity solution

A continuous function u is sub-solution (resp. super-solution) if for every
$\Phi \in C^0((0,T)\times\mathbb{R}^d)$ such that
$\Phi \in C^1((0,T)\times\mathcal{M})$, for $\mathcal{M}=\Omega_i$ or $\mathcal{M}=\Gamma$,
$u \leq \Phi$ (resp. $u \geq \Phi$) with equality at $(t_o,x_o)\in\Omega$, then:

$$\partial_t\Phi(t_o,x_o) + \sup_{k\in\mathbb{I}(x_o)} H_k(x_o, D_x^k\Phi(t_o,x_o)) \leq 0 \quad \text{(resp. } \geq 0\text{)},$$

where $\mathbb{I}(x_o) := \{i \mid x\in\overline{\Omega}_i\}\cup\{E \mid x\in\Gamma\}$.

# Super-solution: Idea of the proof (I)

The value function $v$ is viscosity super-solution of:

$$\begin{cases} \partial_t u(t,x) + H^E(x, Du(t,x)) \geq 0 & t \in (0,T),\ x \in \mathbb{R}^d, \\ u(0,x) \geq \varphi(x) & x \in \mathbb{R}^d. \end{cases}$$

- for every $(t,x) \in (0,T) \times \mathbb{R}^d$, there exists $\bar{y}_x \in \mathcal{S}_{[0,t]}(x)$:

$$v(t,x) \geq v(t-h, \bar{y}(h)) \quad \forall h \in (0,t).$$

- if $v \geq \Phi$ with equality at $(t_o, x_o)$, then

$$\Phi(t_o, x_o) \geq \Phi(t_o - h, \bar{y}_{x_o}(h)) \quad \forall h \in (0, t_o).$$

# Super-solution: Idea of the proof (II)

- On the other hand, for any $x \in \mathbb{R}^d$, and every $y \in \mathcal{S}_{[0,t]}(x)$, we have (whenever the limit exists):

$$\lim_n \frac{y(h_n) - x}{h_n} \in \mathrm{co}F^E(x).$$

- Therefore

$$\partial_t \Phi(t_o, x_o) + sup_{p \in \mathrm{co}F^E(x_o)}(-p \cdot D\Phi(t_o, x_o)) \geq 0.$$

- By separation Theorem, we get:

$$\partial_t \Phi(t_o, x_o) + H^E(x_o, D\Phi(t_o, x_o)) \geq 0.$$

# Sub-solution:Idea of the proof

## Theorem

*The value function v is viscosity sub-solution of the system of HJB equations:*

$$\begin{cases} \partial_t u(t,x) + H^E(x, Du(t,x)) \le 0 & t \in (0,T),\ x \in \mathbb{R}^d, \\ u(0,x) \le \varphi(x) & x \in \mathbb{R}^d. \end{cases}$$

- for every $(t,x) \in (0,T) \times \mathbb{R}^d$, for every $\bar{y}_x \in \mathcal{S}_{[0,t]}(x)$:

$$v(t,x) \le v(t-h, \bar{y}(h)) \quad \forall h \in (0,t).$$

- if $v \le \Phi$ with equality at $(t_o, x_o)$, then

$$\Phi(t_o, x_o) \le \Phi(t_o - h, \bar{y}_{x_o}(h)) \quad \forall h \in (0, t_o).$$

- Therefore

$$\partial_t \Phi(t_o, x_o) + H^E(x_o, D\Phi(t_o, x_o)) \le 0.$$

# The uniqueness result

A strong comparison principle:

**Theorem**

*Let $u_1$ and $u_2$ be respectively a Lipschitz continuous sub-solution and super-solution of* (2)*, then we have*

$$u_1(t,x) \leq u_2(t,x), \ \textit{for } t \in [0,T], x \in \mathbb{R}^d.$$

Main steps of the proof:

- The subsolution $u_1$ satisfies the sub-optimality $\Rightarrow u_1 \leq v$;
- The supersolution $u_2$ satisfies the super-optimality $\Rightarrow u_2 \geq v$;

# First step: proof for the characterization of super-optimality

Since $F^E(x) \subset FC(x)$, the super-solution inequation

$$\partial_t u(t,x) + sup_{p\in F^E(x)}(p \cdot Du(t,x)) \geq 0$$

implies that

$$\partial_t u(t,x) + sup_{p\in FC(x)}(p \cdot Du(t,x)) \geq 0$$

Then the following of the proof is classical since $FC$ is upper semi-continuous.

# Second step: The characterization of the sub-optimality is more technical ...

- For a trajectory starting in $\Omega_i$ and living in the same manifold : NO Problem
- When a trajectory reaches an interface, it can
  - cross the interface and pass to another manifold: continuity of the function
  - take a sliding mode and fellow the same interface for a while: Junction conditions include the behavior of the trajectories on each interface
- The set of trajectories contains also some trajectories with more complicated behavior: Crucial difficulty

# An explicit example in 1d.

- The multi-domains: $\Omega_1 = \{x : x < 0\}$, $\Omega_2 = \{x : x > 0\}$, $\Gamma = \{0\}$.
- The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
- if the final cost function $\varphi(x) = x$. Then

$$v_1(t,x) := \min\{y_x(t)\} = \begin{cases} x - \frac{1}{2}t & x \leq 0, \\ \frac{1}{2}(x-t) & 0 \leq x \leq t, \\ x - t & x \geq t \end{cases}$$

- At $x = 0$ and for $t > 0$, $\partial_t v_1(t,0) = -\frac{1}{2}$, $Dv_1(t,0^-) = 1$, $Dv_1(t,0^+) = \frac{1}{2}$, $D^+ v_1(t,0) = [\frac{1}{2}, 1]$.

# An explicit example in 1d

The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
The convexification: $FC(0) = [-1, 1]$.
The essential dynamics: $F^E(0) = [-\frac{1}{2}, \frac{1}{2}]$.

$$\partial_t v_1(t,0) + \max_{p \in F^E(0)} \{-p \cdot Dv_1(t,0)\} = 0 \leq 0,$$

while

$$\partial_t v_1(t,0) + \max_{p \in FC(0)} \{-p \cdot D^+ v_1(t,0)\} = \frac{1}{2} > 0.$$

The subsolution property fails for $FC$ which is larger than $F^E$.

# A second explicit example in 1d.

- The multi-domains: $\Omega_1 = \{x : x < 0\}$, $\Omega_2 = \{x : x > 0\}$, $\Gamma = \{0\}$.
- The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
- The final cost function $\varphi(x) = -x$.

$$v_2(t,x) := \min\{-y_x(t)\} = \begin{cases} -[x + \frac{1}{2}t] & x \geq 0, \\ -\frac{1}{2}(x+t) & -t \leq x \leq 0, \\ -[x+t] & x \leq -t. \end{cases}$$

- For $t > 0$ and $x = 0$, $\partial_t v_2(t,0) = -\frac{1}{2}$, $Dv_1(t, 0^+) = -1$, $Dv_1(0, 0^-) = -\frac{1}{2}$, $D^- v_1(0,0) = [-1, -\frac{1}{2}]$.

# An explicit example in 1d

The dynamics: $F_1 = [-\frac{1}{2}, 1]$, $F_2 = [-1, \frac{1}{2}]$.
The tangent dynamics: $F_\Gamma(0) = \{0\}$.
The essential dynamics: $F^E(0) = [-\frac{1}{2}, \frac{1}{2}]$.

$$-\partial_t v_2(0,0) + \max_{p \in F^E(0)} \{-p \cdot D^- v_2(0,0)\} = 0 \geq 0,$$

while

$$-\partial_t v_2(0,0) + \max_{p \in F_\Gamma(0)} \{-p \cdot D^- v_1(0,0)\} = -\frac{1}{2} < 0.$$

The supersolution property fails for $F_\Gamma$ which is smaller than $F^E$.

**Thank you for your attention.**