Obstructions to STLC

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Program

Talk based on:

- arXiv:2012.15653 (KB, JLB, FM)
  a survey of expansions for nonlinear systems,
- arXiv:2111.14398 (KB, JLB, FM)
  bounds on structure constants of Hall bases,
- A unified approach of obstructions to small-time local controllability for scalar-input systems (KB, FM),
  soon on arXiv

An old question

Let $f_0, f_1 \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$, real-analytic, with $f_0(0) = 0$. Consider

$$\dot{x} = f_0(x) + u(t) f_1(x)$$

Notation $x(t; u, p)$ for init. data $p \in \mathbb{R}^n$ and control $u \in L^\infty(0, T)$.

Definition (Small-time local controllability)

For every $T, \eta > 0$, there exists $\delta > 0$ such that, for every $x^* \in \mathbb{R}^n$ with $|x^*| \leq \delta$, there exists $u \in L^\infty(0, T)$ such that $x(T; u, 0) = x^*$ and $\|u\|_\infty \leq \eta$.

Necessary and sufficient conditions on f_0, f_1 for STLC? Still open!

Definition is coordinate-invariant, so the answer must be too!

Two toy examples

Submanifold case:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = (n+1)ux_1^n \end{cases} \tag{1}$$

hence, for all $u \in L^\infty(0,T)$, $x(T;u,0) \in \mathcal{M} := \{(a, a^{n+1}); a \in \mathbb{R}\}$.

Drift case:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^{2n} + x_1^{2n+1} \end{cases} \tag{2}$$

hence, when $T\|u\|_\infty \leq 1$, $x(T;u,0) \in \mathcal{H}_+ := \mathbb{R} \times \mathbb{R}_+$.

We focus on the second case.

Known drift results and goals

- Sussmann, 1983. First quadratic obstruction.
- Stefani, 1986. Strongest obstruction at each even order.
- Kawski, 1986. A conjecture on all quadratic obstructions.
- Kawski, 1987. Second quadratic obstruction.

Despite “easy” drifts, tedious computations and “re-organizations” of the terms of the Chen-Fliess series.

Our goals:

- Give simpler proofs, in a unified framework.
- Prove Kawski’s conjecture.
- Rely on an approach which allows to “easily” conjecture and prove new necessary conditions.

A key tool: linearization principle

Consider $\dot{x} = f_0(x) + u(t) f_1(x)$, which is a nonlinear problem.

Define the zero-order operator

$$L(t) : \begin{cases} C^\infty(\mathbb{R}^n; \mathbb{R}) \to C^\infty(\mathbb{R}^n; \mathbb{R}) \\ \varphi \mapsto (p \mapsto \varphi(x(t; u, p))) \end{cases}$$

Then, $\forall \varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$, $\forall p \in \mathbb{R}^n$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(L(t)\varphi)(p) &= D\varphi(x(t; u, p)) \left(f_0(x(t; u, p)) + u(t) f_1(x(t; u, p)) \right) \\ &= (L(t)(f_0 \cdot \nabla + u(t) f_1 \cdot \nabla)\varphi)(p) \end{aligned}$$

So, in the weak-weak sense, $\dot{L}(t) = L(t)(f_0 \cdot \nabla + u(t) f_1 \cdot \nabla)$.
We are solving a linear differential equation in $\mathrm{Op}(C^\infty(\mathbb{R}^n; \mathbb{R}))$.

Welcome to algebra

Let $X := \{X_0, X_1\}$ be non-commutative indeterminates.

Let $\mathcal{A}(X)$ the free algebra over X, i.e. the vector space of non-commutative polynomials of X_0, X_1.
For example $1 + 7X_0 + (3X_0X_1 + 2X_1X_0) + 42X_1^3 \in \mathcal{A}(X)$.
$\mathcal{A}(X) = \oplus_{n\in\mathbb{N}}\mathcal{A}_n(X)$ (spanned by monomials of degree n).

Let $\widehat{\mathcal{A}}(X)$ the formal series generated by $\mathcal{A}(X)$, i.e. sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathcal{A}_n(X)$. Notation $a = \sum_{n\in\mathbb{N}} a_n$.
For example

$$\left(\sum_{n\in\mathbb{N}} n^n X_0^n + e^{n^3} X_1^n\right) \in \widehat{\mathcal{A}}(X)$$

No convergence issue (here)!

Formal linear differential equations

Let $T > 0$ and $u \in L^\infty(0,T)$.
Consider the formal differential equation in $\widehat{\mathcal{A}}(X)$

$$\begin{cases} \dot{x}(t) = x(t)(X_0 + u(t)X_1), \\ x(0) = 1. \end{cases}$$

Definition

The solution to this formal equation is the formal-series valued function $x : [0,T] \to \widehat{\mathcal{A}}(X)$ whose homogeneous components $x_n : [0,T] \to \mathcal{A}_n(X)$ are given by $x_0(t) = 1$ and

$$x_{n+1}(t) = \int_0^t x_n(s)(X_0 + u(s)X_1)\mathrm{d}s.$$

If $u(t) = 7t$, then $x_0(t) = 1$, $x_1(t) = tX_0 + \frac{7t^2}{2}X_1$, ...
If $u(t) = 0$, then $x(t) = \sum_{n \in \mathbb{N}} \frac{t^n}{n!} X_0^n$

Key observation

Thanks to the linearization principle, any formula obtained at the formal level for the differential equation

$$\dot{x}(t) = x(t)(X_0 + u(t)X_1)$$

will have counterparts for all systems of the form

$$\dot{x}(t) = f_0(x(t)) + u(t)f_1(x(t)).$$

(Provided we are able to handle convergence issues).

The Chen series - The Chen-Fliess expansion

Theorem (Chen, 1954)

$$x(t) = \sum_{n\in\mathbb{N}} \sum_{\sigma\in\{0,1\}^n} c_\sigma(t,u) X_{\sigma_1} \cdots X_{\sigma_n},$$

where

$$c_\sigma(t,u) := \int_{0<\tau_1<\cdots<\tau_n<t} \left(\prod_{i=1}^{n} \sigma_i u(\tau_i) \right) \mathrm{d}\tau.$$

Theorem (Sussmann, 1983)

Let $f_0, f_1 \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ *real-analytic. Let* $u \in L^\infty(0,T)$*. For* $t > 0$ *small enough, the following series converges absolutely*

$$x(t;u,0) = \sum_{n\in\mathbb{N}} \sum_{\sigma\in\{0,1\}^n} c_\sigma(t,u) \left((f_{\sigma_1} \cdot \nabla) \cdots (f_{\sigma_n} \cdot \nabla)\mathrm{Id} \right)(0).$$

Used for all previous results, but some drawbacks.

Brackets, free Lie algebras, Lie brackets of vector fields

Let $\mathrm{Br}(X)$ be the **free magma** over X. One can see $\mathrm{Br}(X)$ as binary trees, or parenthesized words. For example, $X_0 \in \mathrm{Br}(X)$, $(X_0, X_1) \in \mathrm{Br}(X)$, $((X_1, X_1), X_0) \in \mathrm{Br}(X)$.

Let $\mathcal{L}(X)$ the **free Lie-algebra** over X, i.e. the smallest vector subspace of $\mathcal{A}(X)$ containing X_0, X_1, and stable by the Lie bracket operation $[a, b] := ab - ba$.
For example, $1 \notin \mathcal{L}(X)$, $X_1^3 \notin \mathcal{L}(X)$ but
$2X_0X_1 - 2X_1X_0 = 2[X_0, X_1] \in \mathcal{L}(X)$.

For analytic vector fields f_0, f_1, $[f_0, f_1] := (Df_1)f_0 - (Df_0)f_1$.

One can "evaluate" (although not injective)

$$b \in \mathrm{Br}(X) \hookrightarrow \mathrm{E}(b) \in \mathcal{L}(X) \hookrightarrow f_b \in \mathcal{L}(\{f_0, f_1\})$$

Lie brackets contain the answer

Consider $\dot{x} = f_0(x) + u(t) f_1(x)$ with $f_0(0) = 0$ and $x(0) = 0$.
Consider $\dot{y} = g_0(y) + u(t) g_1(y)$ with $g_0(0) = 0$ and $y(0) = 0$.

Theorem (Krener, 1973)

Assume that, for every Lie bracket b,

$$g_b(0) = f_b(0).$$

Then there exists a (local) diffeomorphism ϕ from $\mathbb{R}^n \to \mathbb{R}^n$ with $\phi(0) = 0$ such that, for all $u \in L^\infty$ and t small enough,

$$y(t; u, 0) = \phi(x(t; u, 0)).$$

Hence STLC can only depend on $\{f_b(0)\}$.

Some formal expansions of $x(t)$

The Chen-Fliess expansion, which is not within Lie brackets

$$x(t) = \sum_{n \in \mathbb{N}} \sum_{\sigma \in \{0,1\}^n} c_\sigma(t,u) X_{\sigma_1} \cdots X_{\sigma_n}.$$

Given $\mathcal{B}$ a (Hall) basis of $\mathcal{L}(X)$:

$$x(t) = \exp\left(\sum_{b \in \mathcal{B}} \zeta_b(t,u) b \right) \quad \text{Magnus}$$

$$x(t) = \overleftarrow{\prod_{b \in \mathcal{B}}} \exp\left(\xi_b(t,u) b \right) \quad \text{Sussmann}$$

$$x(t) = \exp(tX_0) \exp\left(\sum_{b \in \mathcal{B} \setminus \{X_0\}} \eta_b(t,u) b \right) \quad \text{Our mixed one}$$

Short wrap-up on these expansions for vector fields

Expansion	Chen	Magnus	Mixed-Magnus	Sussmann
Intrinsic	No	Yes	Yes	Yes
Nice coefs.	Yes	No	No	Yes
Yields $x(t)$	Yes	Yes	Yes	Indirectly
u^M estimate	Yes	No (new)	Yes (new)	Yes (new)

Our choice

Our choice, for SLTC results with $f_0(0) = 0$,

$$x(t; u, 0) = e^{\mathcal{Z}_M(t,f,u)} e^{t f_0}(0) + \mathcal{O}(\|u\|_{W^{-1,\infty}}^{M+1})$$
$$= \mathcal{Z}_M(t, f, u)(0) + o(|x(t; u, 0)|) + \mathcal{O}(\|u\|_{W^{-1,\infty}}^{M+1})$$

where

$$\mathcal{Z}_M(t, f, u)(0) = \sum_{\substack{b \in \mathcal{B} \\ 1 \leq n_1(b) \leq M}} \eta_b(t, u) f_b(0)$$

where the **infinite** sum converges absolutely within Hall bases, thanks to estimates on the growth of their structure constants.

Bases of $\mathcal{L}(X)$

There is a canonical basis of $\mathcal{A}(X)$, made of all monomials, e.g. 1, X_0, X_1^9, $X_0^2X_1^3X_0$. A basis of $\mathcal{A}_n(X)$ is thus $X_{\sigma_1}\cdots X_{\sigma_n}$ with $\sigma \in \{0,1\}^n$. Hence $\dim \mathcal{A}_n(X) = 2^n$.

There is no canonical basis of $\mathcal{L}(X)$.

By Witt's formula

$$\dim \mathcal{L}_n(X) = \frac{1}{n}\sum_{d|n}\mu(d)2^{\frac{n}{d}}$$

where μ is the Möbius function: $\mu(d) = 0$ if d has a squared prime factor, else $\mu(d) = (-1)^p$ when d has p prime factors.

Ex: $\dim \mathcal{L}_1 = 2$, $\dim \mathcal{L}_2 = 1$, $\dim \mathcal{L}_3 = 2$, $\dim \mathcal{L}_4 = 3$, $\dim \mathcal{L}_5 = 6$, ... (vs $2, 4, 8, 16, 32$ ...)

Hall basis of $\mathcal{L}(X)$

A **Hall set** over X is a totally ordered subset $\mathcal{B} \subset \mathrm{Br}(X)$ st.

- $X_0, X_1 \subset \mathcal{B}$
- for all $b_1, b_2 \in \mathrm{Br}(X)$, $(b_1, b_2) \in \mathcal{B}$ iff
  - $b_1, b_2 \in \mathcal{B}$,
  - $b_1 < b_2$,
  - $b_2 \in X$ or $b_2 = (b_2', b_2'')$ where $b_2' \leq b_1$
- for all $b_1, b_2 \in \mathcal{B}$, if $(b_1, b_2) \in \mathcal{B}$, then $b_1 < (b_1, b_2)$.

Theorem (Viennot, 1978)

The evaluation of a Hall set is a basis of $\mathcal{L}(X)$.

Notations: n_0, n_1, $|\cdot|$, λ, μ.

Examples of classical Hall sets

- Historical length-compatible ones

$$a < b \Rightarrow |a| \leq |b|$$

- Chen-Fox-Lyndon basis

$$a < b \Leftrightarrow \text{lexico}(a) < \text{lexico}(b)$$

- Spitzer-Foata

$$a < b \Rightarrow \frac{n_1(a)}{n_0(a)} \leq \frac{n_1(b)}{n_0(b)}$$

Problem: not well suited to reflect the asymmetry between f_0 and $u(t)f_1$, and Taylor expansions wrt. u.

Hard to guess which brackets will be good or bad.

Our Hall basis

We introduce a new Hall set $\mathcal{B}^\star$ over $\{X_0, X_1\}$ defined by the following order

- X_1 is minimal and X_0 is maximal.
  - Thus, for any $b \in \mathcal{B}^\star \setminus \{X_0\}$, and $\nu \in \mathbb{N}$, $b0^\nu := (\cdots(b, X_0), \dots, X_0) \in \mathcal{B}^\star$.
  - When $b = X_1$ or $\mu(b) \neq X_0$, we call b a germ.
- If $a < b$ are germs, then $a0^{\nu_1} < b0^{\nu_2}$ for all $\nu_1, \nu_2 \in \mathbb{N}$.
- If a, b are germs, then $a < b$ if and only if
  - either $n_1(a) < n_1(b)$,
  - or $n_1(a) = n_1(b)$ and $\lambda(a) < \lambda(b)$,
  - or $n_1(a) = n_1(b)$ and $\lambda(a) = \lambda(b)$ and $\mu(a) < \mu(b)$.

First elements of the basis

Proposition

The elements of $\mathcal{B}^\star$ *with* $1 \le n_1(b) \le 4$ *are:*

1. $M_j := X_1 0^j$, $\quad 0 \le j$,
2. $W_{j,\nu} := (M_{j-1}, M_j)0^\nu$, $\quad 1 \le j,\ 0 \le \nu$,
3. $P_{j,k,\nu} := (M_{k-1}, W_{j,0})0^\nu$, $\quad 1 \le j \le k,\ 0 \le \nu$,
4. - $Q_{j,k,l,\nu} := (M_{l-1}, P_{j,k,0})0^\nu$, $\quad j \le k \le l,\ 0 \le \nu$,
   - $Q^\sharp_{j,\mu,k,\nu} := (W_{j,\mu}, W_{k,0})0^\nu$, $\quad j < k,\ 0 \le \nu$,
   - $Q^\flat_{f,\mu,\nu} := (W_{j,\mu}, W_{j,\mu+1})0^\nu$, $\quad 1 \le j,\ 0 \le \mu, \nu$.

Coordinates of the second kind

Let $\mathcal{B} \subset \mathrm{Br}(X)$ a Hall set. For every $b \in \mathcal{B} \setminus X$, there exists a maximal $k \in \mathbb{N}^*$ such that

$$b = \mathrm{ad}_{b_1}^k(b_2)$$

with $b_1 < b_2 \in \mathcal{B}$.

The magical induction formula (Sussmann) is

$$\dot{\xi}_{\mathrm{ad}_{b_1}^k(b_2)}(t) = \frac{(\xi_{b_1}(t))^k}{k!}\dot{\xi}_{b_2}(t)$$

with $\dot{\xi}_{X_0}(t) = 1$ and $\dot{\xi}_{X_1}(t) = u(t)$.

Coordinates of the second kind in $\mathcal{B}^\star$

First, for all $b \in \mathcal{B}^\star$

$$\dot{\xi}_{(b,X_0)} = \xi_b \quad \text{thus} \quad \xi_{b0^\nu}(t,u) = \int_0^t \frac{(t-s)^{\nu-1}}{(\nu-1)!}\xi_b(s,u)\mathrm{d}s.$$

Proposition

1. $\xi_{M_j} = u_{j+1}(t)$, *the* $(j+1)^{th}$ *primitive of* u,
2. $\xi_{W_{j,\nu}} = \int_0^t \frac{(t-s)^\nu}{\nu!} u_j^2(s)\mathrm{d}s$,
3. $\xi_{P_{j,k,\nu}} = \alpha_{j,k} \int_0^t \frac{(t-s)^\nu}{\nu!} u_j^2(s)u_k(s)\mathrm{d}s$,
   with $\alpha_{j,k} = 1/2$ *when* $j < k$ *else* $1/6$.
4.
   - $\xi_{Q_{j,k,l,\nu}} \propto \int_0^t \frac{(t-s)^\nu}{\nu!} u_j(s)^2 u_k(s) u_l(s)\mathrm{d}s$
   - $\xi_{Q^\flat_{j,\mu,\nu}} \propto \int_0^t \frac{(t-s)^\nu}{\nu!} \left(\int_0^s \frac{(s-s')^\mu}{\mu!} u_j^2(s')\mathrm{d}s'\right)^2 \mathrm{d}s$
   - $\xi_{Q^\sharp_{j,\mu,k,\nu}} \propto \int_0^t \frac{(t-s)^\nu}{\nu!} u_k^2(s) \int_0^s \frac{(s-s')^\mu}{\mu!} u_j^2(s')\mathrm{d}s'\mathrm{d}s$

Definitions of STLC

Let $m \geq -1$.

Definition ($W^{m,\infty}$ STLC (KB-FM, *JDE 2018*))

We say that the system $\dot{x} = f_0(x) + u f_1(x)$ is $W^{m,\infty}$ STLC, when: for every $T, \eta > 0$, there exists $\delta > 0$ such that, for every $x^* \in \mathbb{R}^n$ with $|x^*| \leq \delta$, there exists $u \in L^1(0,T)$ such that $x(T; u, 0) = x^*$ and $\|u\|_{W^{m,\infty}} \leq \eta$.

- Usual definition (Coron / Kawski) is $m = 0$, i.e. L^∞-STLC.
- When $m > 0$, $W^{m,\infty}$ STLC is a stronger notion (e.g. when $m = 1$, one asks that $\|\dot{u}\|_\infty \ll 1$).
- When $m = -1$, weaker notion, equivalent to asking that the state remains small (but $\|u\|_{L^1}$ can be large).

Result #1: Strongest at each order

Let

$$S_{[\![1,k]\!]}(X) := \mathrm{span}\{b \in \mathrm{Br}(X); n_1(b) \in [\![1,k]\!]\}.$$

Theorem (Sussmann-Stefani)

Assume that $\dot{x} = f_0(x) + u f_1(x)$ is $W^{-1,\infty}$-STLC.
Then, for all $k \in \mathbb{N}^*$,

$$\mathrm{ad}_{f_1}^{2k}(f_0)(0) \in S_{[\![1,2k-1]\!]}(f)(0).$$

Historical proofs of Sussmann ($k = 1$) and Stefani ($k > 1$) denied bounded L^∞ STLC, a stronger notion.

Result #2: Loose quadratic obstructions

Recall $W_k = (M_{k-1}, M_k)$ for $k \geq 1$.

Theorem (KB-FM 2022)

Let $m \geq 0$. Assume that $\dot{x} = f_0(x) + u f_1(x)$ is $W^{m,\infty}$-STLC. Then, for all $k \in \mathbb{N}^*$,

$$f_{W_k}(0) \in S_{[\![1,\pi(k,m)]\!] \setminus \{2\}}(f)(0)$$

where

$$\pi(k,m) = \left\lceil \frac{2k+m-1}{m+1} \right\rceil.$$

- $m = 0$, $\pi(k,m) = 2k - 1$, is Kawski's 1986 conjecture
- $m = 2k - 3$, $\pi(k,m) = 2$, is our 2018 JDE result
- intermediate cases are new
- $k = 1$ gives $\pi(k,m) = 1$ for any m

Result #3: Tight W_2

Recall $P_{1,1,\nu} = \mathrm{ad}^3_{X_1}(X_0)0^\nu$. Let

$$\mathcal{E}_2(X) := S_1(X) \cup \{P_{1,1,\nu}; \nu \geq 0\}.$$

Theorem (Kawski)

Assume that $\dot{x} = f_0(x) + u f_1(x)$ *is* L^∞*-STLC.*
Then,

$$f_{W_2}(0) \in \mathcal{E}_2(f)(0).$$

Result #4: Tight W_3

Let

$$\mathcal{E}_3(X) := \mathcal{E}_2(X) \cup \Big\{ P_{1,l,\nu}, Q_{1,1,1,\nu}, Q_{1,1,2,\nu}, Q^{\flat}_{1,\mu,\nu}, R_{1,1,1,1,\nu}, R^{\sharp}_{1,1,1,\mu,\nu} \Big\},$$

where the R are explicit brackets of $\mathcal{B}^\star$ with $n_1(R) = 5$.

Theorem (KB-FM 2022)

Assume that $\dot{x} = f_0(x) + u f_1(x)$ is L^∞-STLC.
Then,

$$f_{W_3}(0) \in \mathcal{E}_3(f)(0).$$

Result #5: A sixth-order example

Recall $P_{1,1,0} = \mathrm{ad}^3_{X_1}(X_0) \in \mathcal{B}^\star$. Let $\mathfrak{b} := \mathrm{ad}^2_{P_{1,1,0}}(X_0) \in \mathcal{B}^\star$. Then

$$\xi_{\mathfrak{b}}(t,u) = \frac{1}{2}\int_0^t \left(\int_0^s \frac{u_1^3}{3!} \right)^2 \mathrm{d}s.$$

Theorem (KB-FM 2022)

Assume that $\dot{x} = f_0(x) + u f_1(x)$ is L^∞-STLC.
Then,

$$f_{\mathfrak{b}}(0) \in \mathrm{span}\{b \in \mathcal{B}^\star; n_1(b) \leq 7; b \neq \mathfrak{b}\}(f)(0).$$

Heuristic is

$$\int u_1^8 \lesssim \|u\|_\infty^2 \xi_{\mathfrak{b}}(t,u)$$

A difficulty: coordinates of the pseudo-first kind

We compute them using the relation

$$\eta_b = \xi_b + \sum \text{product of } \{\xi_{b_i}\} \times \text{structure constants of } \mathcal{B}$$

where $|b_i| < |b|$ and $n_1(b_i) < n_1(b)$.

A key argument of the proofs is to obtain "closed-loop" estimates on these "pollutions", of the form

$$\xi_{b_i}(t,u) = x(t;u,0) + \text{higher order terms}$$

which yield

$$\eta_b = \xi_b + o(|x(t;u,0)|) + o(\text{drift}).$$

An example of pollutions

When working on $W_2 = (M_1, M_2)$, $\xi_{W_2}(t,u) = \frac{1}{2}\int_0^t u_2^2$ and

$$\eta_{W_2}(t,u) = \xi_{W_2}(t,u) + \frac{1}{2}u_1(t)u_4(t) - \frac{1}{2}u_2(t)u_3(t)$$

For $j = 3, 4$, one has $|u_j(t)| \lesssim \|u_2\|_{L^2(0,t)}$ by Cauchy-Schwarz.
For $j = 1, 2$, we obtain that

$$\begin{aligned} u_j(t) &= O(|x(t;u,0)| + \|u_1\|_{L^2}^2 + \|u_3\|_{L^2}) \\ &= O(|x(t;u,0)| + \|u\|_\infty \|u_2\|_{L^2} + t\|u_2\|_{L^2}). \end{aligned}$$

(The key point is proving that $f_1(0)$ and $[f_0, f_1](0)$ are free).
So

$$\eta_{W_2}(t,u) = \xi_{W_2}(t,u) + o(|x(t;u,0)|) + o(\xi_{W_2}(t,u))$$

Ingredients of the approach

- Choose $\mathfrak{b} \in \mathcal{B}^\star$ such that $\xi_{\mathfrak{b}}(t,u)$ looks positive definite
  using Sussmann's induction formula
- Find M, m st. $\|u\|_{W^{-1,\infty}}^{M+1} = o(\xi_{\mathfrak{b}}(t,u))$ when $\|u\|_{W^{m,\infty}} \ll 1$
  using Gagliardo-Nirenberg interpolation inequality
- Determine $\mathcal{N} := \{b \in \mathcal{B}^\star; \xi_b(t,u) \neq o(\xi_{\mathfrak{b}}(t,u)), n_1(b) \leq M\}$
  using Gagliardo-Nirenberg interpolation inequality
- For $b \notin \mathcal{N}$, prove a bound on $|\eta_b(t,u) - \xi_b(t,u)|$
  using BCH formula and a closed-loop type estimate
- Write $\mathcal{Z}_M(t,f,u)(0) = \sum \eta_b(t,u) f_b(0)$ where $1 \leq n_1(b) \leq M$
- Write $x(t;u,0) \approx \mathcal{Z}_M(t,f,u)(0) + \mathcal{O}(\|u\|_{W^{-1,\infty}}^{M+1})$
- Prove that $W^{m,\infty}$-STLC implies $f_{\mathfrak{b}}(0) \in \mathcal{N}(f)(0)$.

Thank you for your attention!

Related ongoing works

- first overview of some quartic obstructions and behaviors,
- detailed account of quadratic/cubic competitions,
- investigations on multi-input systems.

Thank you!