# The fast-switching phenomenon in time-optimal control problems

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# The Bang-Bang principle in the linear case

Consider the finite-dimensional control system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n^2}, \quad B \in \mathbb{R}^{n \times m},$$

and $U \subset \mathbb{R}^m$ a bounded polyhedron.

**Problem (Linear time-optimal problem)**

*Given $x_0, x_1 \in \mathbb{R}^n$, find $u \in L^2([0, T], \mathbb{R}^m)$ such that*

- $x(0, x_0; u) = x_0$ *and* $x(T, x_0; u) = x_1$;
- $u(t) \in U$ *for a.e.* $t \in [0, T]$;
- $T$ *be as small as possible.*

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- $u(t) \in U$ *for a.e.* $t \in [0, T]$;
- $T$ *be as small as possible.*

- If there exist any pair $\widetilde{T} > 0$ and $\widetilde{u} \in L^2([0, \widetilde{T}], U)$ that solve the problem, then there exist a minimal $T > 0$ and an optimal control $u^* \in L^2([0, T], U)$ that solve it.

# The Bang-Bang principle in the linear case

- Taking the dual curve $t \mapsto \lambda(t) \in \mathbb{R}^{n^*} \backslash \{0\}$, that satisfies the adjoint equation

$$\dot{\lambda}(t) = -A^T \lambda(t), \quad t \in [0, T],$$

the optimal control $u^*$ solves, for a.e. $t \in [0, T]$, the linear programming problem

$$\langle \lambda(t), Bu^*(t) \rangle = \min_{u \in U} \langle \lambda(t), Bu \rangle, \quad \text{a.e.} \quad t \in [0, T],$$

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- Under appropriate normality assumptions, we find non-overlapping intervals $I_1, \ldots, I_N \subset [0, T]$ such that

$$[0, T] = \bigcup_{i=1}^{N} I_i, \quad u^*(t) = v_i \ \text{ if } \ t \in \mathrm{int}(I_i).$$

and $v_i$ a vertex of $U$. The control bangs around vertexes of $U$.
LaSalle, 1961

# The Bang-Bang principle in infinite dimensions

- The Bang-Bang phenomenon appears if we want to control parabolic PDE's.
- For hyperbolic equations: "There is no true Bang-Bang principle at all" Russell.
- Depending on the initial and on the final state, the controls governing a controlled wave equation

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = g(x)u(t),$$

may possess any degree of differentiability.

# The Bang-Bang principle in infinite dimensions

ZUAZUA, WANG ET AL. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ a $C^2$ boundary, $\omega \subset \Omega$ be an open set. Consider

$$\begin{cases} y_t - \Delta y + a(x,t)y = \chi_\omega u & \text{in } \Omega \times \mathbb{R}^+ \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ y(x,0) = y_0(x) & \text{in } \Omega \end{cases}$$

where

- $a \in L^\infty(\Omega \times \mathbb{R}^+)$ and $y_0 \in L^2(\Omega)\backslash\{0\}$,
- It is of the form $\dot{y} = Ay + Bu$, with
  - $A(x,t)y = \Delta y - a(x,t)y$,
  - $B(x) = \chi_\omega(x)$.

# The Bang-Bang principle in infinite dimensions

- For $M > 0$, we choose the control $u$ in

$$\mathcal{U}^M := \{u : \mathbb{R}^+ \to L^2(\Omega) \mid \|u(t,\cdot)\|_{L^2(\Omega)} \leqslant M \ \text{ a.e. } \ t > 0\}.$$

- We say that a control $u$ belongs to $\mathcal{U}^M_{\text{adm}}$ if $u \in \mathcal{U}^M$ and there exists $t > 0$ such that $y(t,x;u) = 0$.

**Problem (Time-optimal problem)**

$$T(M) := \inf_{u \in \mathcal{U}^M_{adm}} \{t > 0 \mid y(t,x;u) = 0\},$$

# The Bang-Bang principle in infinite dimensions

For $T > 0$ define the norm-optimal control problem $(NP)^T$

$$N(T) := \inf\{\|u\|_{L^\infty([0,T],L^2(\Omega))} \mid y(T,x;u) = 0\}, \quad N := \lim_{T\to\infty} N(T).$$

- $N(T) < +\infty$ because heat equations are null-controllable
- $N$ exists.

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- $N$ exists.

**Theorem (Wang, Xu, Zhang, 2014)**

*Let $y_0 \in L^2(\Omega)\backslash\{0\}$. Then the time-optimal problem admits time-optimal controls iff $M > N$.*

- The crucial point of the proof is the Bang-Bang property (Wang, Phung 2013) for the $(NP)^T$ problem: any optimal control $u^*$ for $(NP)^T$ verifies

$$\|u^*(t,\cdot)\|_{L^2(\Omega)} = N(T), \quad \text{a.e. } t \in [0,T].$$

# The Bang-Bang principle in infinite dimensions

Consider another example, studied by FATTORINI, RUSSELL, EGOROV et al. in the 60's - 70's. For $w = w(x,t)$, consider

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + r(x)w = g(x)u(t), \quad t \geqslant 0, \quad 0 < x < 1,$$

with boundary conditions

$$a_0 w(0,t) + b_0 \frac{\partial w}{\partial x}(0,t) = 0, \quad a_1 w(1,t) + b_1 \frac{\partial w}{\partial x}(1,t) = 0$$
$$a_i^2 + b_i^2 \neq 0, \quad i = 0,1.$$

It is of the form $\dot{w} = Aw + Bu$, with

- $A(x)w = Lw = \frac{\partial^2 w}{\partial x^2} - r(x)w$ a Sturm-Liouville operator,
- $B(x) = g(x)$.

# The Bang-Bang principle in infinite dimensions

**Problem (Time-optimal control problem)**

*Assuming* $w(\cdot, 0) = 0$ *and* $u \in L^\infty(\mathbb{R}_+, [-1, 1])$*, reach in minimal time* $\tau$*, the final state* $w(\cdot, \tau) = w_1$*.*

# The Bang-Bang principle in infinite dimensions

**Problem (Time-optimal control problem)**

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- If $u$ have no bounds, the problem has a solution only if $w_1$ is in the domain of $\exp(A_1(-L)^{1/2})$, $A_1 > 0$ a suitable constant.
- No solution to the time-optimal control problem is granted for $L^2$ controls.
- A bang-bang principle can be established only for states $w_1 \in B$ known to be reachable. The subspace $B \subset L^2([0, 1])$ depends in general on the eigensystem associated to $L$ and the coefficients of $g$ in this basis.

# The Bang-Bang principle in infinite dimensions

**Theorem (Russell, 1978)**

*Suppose that a final state $w_1 \in B$ is reached from $w_0 = 0$ at time $\tau > 0$ with a control $u \in L^\infty([0, \tau], [-1, 1])$. Suppose that $\tau$ is minimal with respect to all $u \in L^\infty(\mathbb{R}_+, [-1, 1])$. Then*

- $|u(t)| = 1$ *for almost every* $t \in [0, \tau]$.
- *There exists* $t \mapsto \eta(t) \in \mathbb{R}$ *such that*
  $u(t) = \mathrm{sgn}(\eta(t)), \quad \text{if} \quad \eta(t) \neq 0.$

*$\eta$ is determined by the data of our problem, is real-analytic on $[0, \tau)$, and has discrete zeros accumulating at $\tau$.*

- Notice that the situation is different between $u \in L^2$ or $u \in L^\infty$, in general.
- Typically we can avoid oscillations with unbounded controls.

# Back to the finite-dimensional case

Fuller, 1963

In $\mathbb{R}^2$, among the trajectories of the differential system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \quad u \in [-1,1], \end{cases}$$

with initial conditions $x_1(0) = \bar{x}_1$, $x_2(0) = \bar{x}_2$, and the origin $(0,0)$ as final state, minimize the cost functional

$$\int_0^\infty |x_1(t)|^2 dt,$$

among all trajectories from $(\bar{x}_1, \bar{x}_2)$ to $(0,0)$.

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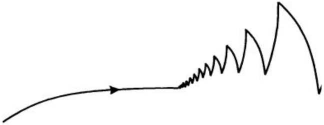
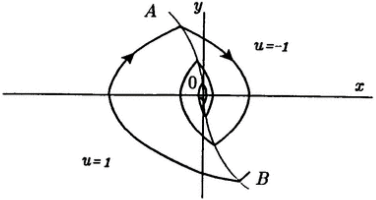
This problem can be reformulated as a (non-linear) time-optimal problem in dimension 3.

It can be generalized to higher dimensions.

# Back to the finite-dimensional case

Optimal trajectories exist. They exhibit this strange behavior...






Two different phenomena

⟶ The trajectory is a join of a singular and a bang bang part. It makes an infinite number of switchings to "escape" from the singular trajectory.

⟶ The optimal control is bang-bang but switches wildly near a singular trajectory.

# Back to the finite-dimensional case

This situation is unpleasant for various reasons:

- Theoretically it is difficult to investigate these trajectories
- Has important drawbacks on the optimal synthesis
- Practically chattering can destroy machines
- There are engineering reports observing that the implementation of optimal controls on a machine may lead to its break down.

# Regularizing the chattering

Attempts have been made to regularize fast-oscillations
Caponigro, Ghezzi, Piccoli, Trélat, 2017

- Works for more general optimal control problems

$$C(u) = \int_0^{t(u)} L(s, x(s), u(s)) ds \to \min,$$

with $u \in \mathcal{U}$ an admissible control and $L \in C^0(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^m)$ a Lagrangian function. Optimal controls $u^*$ exist under standard hypotheses (Lie Algebra condition and STLC at the origin)

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- For $\varepsilon > 0$, consider a new cost $C_\varepsilon(u) := C(u) + \varepsilon TV(u)$. The correction excludes chattering controls ($TV(u) = +\infty$). Optimal controls $u_\varepsilon^*$ exist
- These controls $u_\varepsilon^*$ are quasi-optimal for the original problem, but don't chatter

# Single-input control affine systems and regularity of time-optimal trajectories

$$\dot{q} = f_0(q) + u f_1(q), \qquad u \in [-1, 1]$$

$q \in M$ smooth $n$-dimensional manifold, $f_0, f_1 \in \mathrm{Vec}(M)$ (i.e., smooth vector fields on $M$)

Time optimal problem: $q(0) = q_0, \quad q(T) = q_1, \quad T \to \min$

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Regularity of a time-optimal trajectory $q : [0, T] \to M$ measured in terms of

$$O = \bigcup_{\omega \text{ open, } q|_\omega \text{ smooth}} \omega, \qquad \Sigma = [0, T] \setminus O$$

Is $\Sigma$ empty? finite? countable? of finite measure? of empty interior?

# Genericity

As pointed out by SUSSMANN in 1986, for any $t \mapsto u(t)$ measurable and any $M, q_0$, there exist $f_0, f_1 \in \mathrm{Vec}(M)$ such that the admissible trajectory driven by $u$ and starting at $q_0$ is time-optimal.

The natural reflex is to look for generic properties: properties that hold for all time-optimal trajectories of the single-input control affine system, provided that $(f_0, f_1)$ belongs to a residual set of $\mathrm{Vec}M \times \mathrm{Vec}M$ for the $C^\infty$ Whitney topology. (Open problem 1 in AGRACHEV 2014, G. Stefani et al. editors)

# Time-extremal trajectories and the switching function

By the Pontryagin maximum principle, if $q : [0, T] \to M$ is time-optimal, then there exists an extremal lift $\lambda : [0, T] \to T^*M \setminus \{0\}$ of $q(\cdot)$ such that, for every $X \in \mathrm{Vec} M$,

$$\frac{d}{dt}\langle \lambda(t), X(q(t)) \rangle = \langle \lambda(t), [f_0 + u(t) f_1, X](q(t)) \rangle \quad \text{a.e. } t \in [0, T]$$

and the switching function

$$h_1(t) = \langle \lambda(t), f_1(q(t)) \rangle$$

satisfies $u(t) = \mathrm{sgn}(h_1(t))$ whenever $h_1(t) \neq 0$.

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satisfies $u(t) = \mathrm{sgn}(h_1(t))$ whenever $h_1(t) \neq 0$.
Let $f_\pm = f_0 \pm f_1$ and, for $I = (i_1 \cdots i_d)$ a word with letters in $\{+, -, 0, 1\}^d$,

$$f_I = [f_{i_1}, \ldots, [f_{i_{d-1}}, f_{i_d}] \cdots], \quad h_I(t) = \langle \lambda(t), f_I(q(t)) \rangle.$$

In particular, $\frac{d}{dt} h_1(t) = h_{01}(t)$ for every $t \in [0, T]$, and $\frac{d^2}{dt^2} h_1(t) = h_{001}(t) + u(t) h_{101}(t)$ for a.e. $t \in [0, T]$.

# Previous results

- For $n = 2$, $\Sigma$ is generically finite Lobry 1970, Sussmann 1982, 1987.
- Finiteness of $\Sigma$ close to points at which some suitable non-dependence condition between Lie brackets holds Agrachev, Bressan, Gamkrelidze, Krener, Schättler, Sigalotti, Sussmann,. . . .
- For $n$ large enough time-extremal trajectories of generic systems might exhibit Fuller phenomenon ($\#\Sigma = \infty$) Kupka 1990, Zelikin–Borisov 1994

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- For $n$ large enough time-extremal trajectories of generic systems might exhibit Fuller phenomenon ($\#\Sigma = \infty$) Kupka 1990, Zelikin–Borisov 1994
- Generically, for every extremal trajectory $q : [0, T] \to M$, the set $O$ is open and dense in $[0, T]$ Agrachev 1995
- Generically, for any extremal triple $(q(\cdot), u(\cdot), \lambda(\cdot))$ on $[0, T]$ such that $h_1|_{[0,T]} \equiv 0$, the set $\Omega = \{t \in [0, T] \mid h_{101}(t) \neq 0\}$ is of full measure in $[0, T]$ and $u(t) = -h_{001}(t)/h_{101}(t)$ almost everywhere on $\Omega$ Bonnard–Kupka 1997, Chitour–Jean–Trélat 2008

# Bang and singular arcs

### Definition (Bang and singular arcs)

An arc $\omega$ is a connected component of $O$.

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- bang if $u$ (a.e.) constant and equal to $\pm 1$ on $\omega$,
- singular otherwise.
- Two arcs are concatenated if they share one endpoint.
- The time-instant between two concatenated arcs (i.e., an isolated point of $\Sigma$) is a switching time.

# Fuller times

## Definition (Fuller times)

Let $\Sigma_0$ be the set of isolated points in $\Sigma$ (switching times).
The elements of $\Sigma \backslash \Sigma_0$ are Fuller times.
By recurrence, let

$$\Sigma_k \text{ set of isolated points of } \Sigma \backslash (\cup_{j=0}^{k-1} \Sigma_j), \quad k \in \mathbb{N} \cup \{\infty\}.$$

If $t \in \Sigma_k$ then $t$ is a Fuller time of order $k$.

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- $\Sigma_2$ are the isolated points of $\Sigma \setminus (\Sigma_0 \cup \Sigma_1) \longrightarrow$ every point in $\Sigma_2$ admits a converging sequence in $(\Sigma_0 \cup \Sigma_1)$ but not in $\Sigma \setminus (\Sigma_0 \cup \Sigma_1)$. Actually it admits a converging sequence in $\Sigma_1$, otherwise it would be a point in $\Sigma_1$. And so on...

# Main result

**Theorem (F. B., M. Sigalotti, submitted)**

*There exists $K(n) \in \mathbb{N}$ such that, for a generic pair $(f_0, f_1)$, every extremal trajectory $q(\cdot)$ of the time-optimal control problem*

$$\dot{q} = f_0(q) + u f_1(q), \quad q \in M, \ u \in [-1, 1],$$

*has at most Fuller times of order $K(n)$, i.e.,*

$$\Sigma = \Sigma_0 \cup \cdots \cup \Sigma_{K(n)}.$$

- In particular, $u$ can be taken smooth out of a finite union of discrete sets (hence, out of a countable set).

# Strategy of the proof

- At switching times $h_1 = 0$ (hence by continuity $h_1|_\Sigma \equiv 0$)
- Elements of $\Sigma \backslash \Sigma_0$ are accumulations of switching times and one easily deduces that $\frac{d}{dt} h_1 = h_{01}$ vanishes on $\Sigma \backslash \Sigma_0$ (between two zeros of $h_1$, $\frac{d}{dt} h_1$ vanishes)
- Higher order Fuller times are accumulations of accumulations and new relations between $\lambda(t)$ and the brackets of $f_0(q(t))$ and $f_1(q(t))$ can be derived
  Initialization on $\Sigma \backslash \Sigma_0$
  Recursion $\Sigma \backslash \cup_{j=0}^{k} \Sigma_j \longrightarrow \Sigma \backslash \cup_{j=0}^{k+1} \Sigma_j$
- At high order Fuller times $(\lambda(t), j^N(f_0, f_1)(q(t)))$ belong to a set of large codimension
- The projection of sets of large codimension has large codimension
- Fuller times of too large order can be ruled out by standard transversality arguments

# Initialization: dependence conditions on $\Sigma \backslash \Sigma_0$

**Proposition (No genericity assumption here)**

*Let* $t \in \Sigma \backslash \Sigma_0$. *Then* $h_1(t) = h_{01}(t) = 0$ *and, in addition, either* $h_{+01}(t) = 0$ *or* $h_{-01}(t) = 0$.

- This proposition does not require genericity. Can we derive other conditions without it? This will improve the bounds on $K(n)$ in our main theorem

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- This proposition does not require genericity. Can we derive other conditions without it? This will improve the bounds on $K(n)$ in our main theorem
- Heuristic of the proof: let $t \in \Sigma$. The difficult case is when we can find arbitrarily close to $t$ an infinite sequence of bang arcs. Technical considerations permit to conclude in all the other cases.
- Key point: assuming $h_{\pm 01}(t) \neq 0$ allows to suppose that along all these arcs the second derivative of switching function $h_1(t)$ remains bounded from below in absolute value.

# A technical lemma

## Lemma

*Assume that there exists an infinite sequence of concatenated bang arcs converging to $\tau \in [0, T]$. Then either $h_{+01}(\tau) = 0$ or $h_{-01}(\tau) = 0$.*

Denote the lengths of the subsequent bang arcs by $\{\tau_i\}_{i \in \mathbb{N}}$.
The proof works by contradiction: if $h_{+01}(\tau) \neq 0$ and $h_{-01}(\tau) \neq 0$, then a simple computation shows that $\tau_{i+1} = O(\tau_i)$ and

$$\tau_{i+2} = \tau_i + O(\tau_i^2).$$

Hence

$$\sum_{i=1}^{\infty} \tau_i = +\infty.$$

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REMARK: $I_1 = (1)$ and $I_2 = (01)$ would have not worked as a starting point for the recurrence. We need this third word $I_3 = (\pm 01)$.

# Recursion $\Sigma \setminus \cup_{j=0}^{k} \Sigma_j \longrightarrow \Sigma \setminus \cup_{j=0}^{k+1} \Sigma_j$

Let $I_1$ and $I_2$ be two words with letters in $\{+, -, 0, 1\}^d$, $t_n$ a sequence of times converging to $t$ such that, along $q : [0, T] \to M$

$$h_{I_1}(t_n) = 0 = h_{I_2}(t_n).$$

Up to subsequences

$$\frac{1}{t - t_n} \int_{t_n}^{t} u(s)ds \to \bar{u} \in [-1, 1].$$

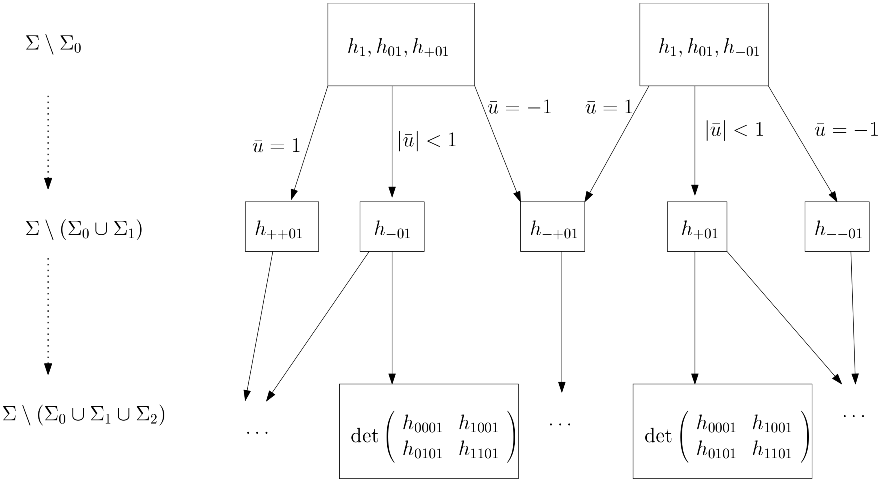
Then, for $j = 1, 2$,

$$0 = \frac{h_{I_j}(t) - h_{I_j}(t_n)}{t - t_n} = \frac{\int_{t_n}^{t} (h_{0I_j}(s) + u(s)h_{1I_j}(s))ds}{t - t_n} \to h_{0I_j}(t) + \bar{u} h_{1I_j}(t).$$

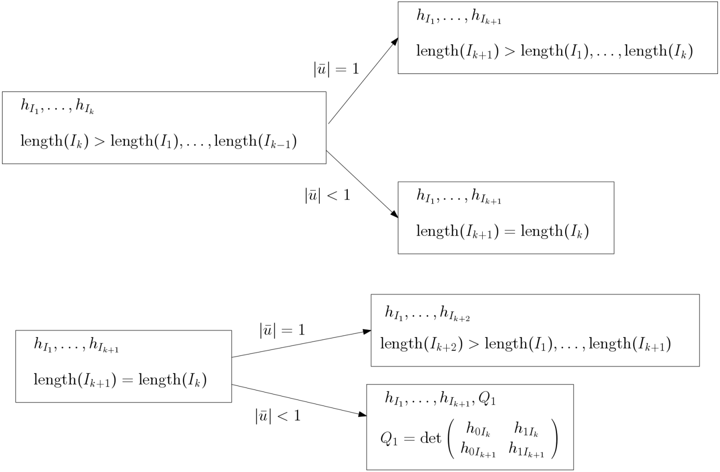
If $|\bar{u}| = 1$ then we get a new word $J$ of longer length such that $h_J(t) = 0$.

In any case $\det \begin{pmatrix} h_{0I_1}(t) & h_{1I_1}(t) \\ h_{0I_2}(t) & h_{1I_2}(t) \end{pmatrix} = 0.$

# Tree of dependence conditions



# Along a branch of the tree



# Along a branch (continued)

$$h_{I_1}, \ldots, h_{I_k}, Q_1, \ldots, Q_l \longrightarrow$$

$$h_{I_1}, \ldots, h_{I_k}, Q_1, \ldots, Q_{l+1}$$

$$Q_{l+1} = \det \begin{pmatrix} h_{0I_k} & h_{1I_k} \\ \{h_0, Q_l\} & \{h_1, Q_l\} \end{pmatrix}$$

# Along a branch (continued) and independence of the conditions

$$h_{I_1}, \dots, h_{I_k}, Q_1, \dots, Q_l \longrightarrow \begin{array}{l} h_{I_1}, \dots, h_{I_k}, Q_1, \dots, Q_{l+1} \\ Q_{l+1} = \det \begin{pmatrix} h_{0I_k} & h_{1I_k} \\ \{h_0, Q_l\} & \{h_1, Q_l\} \end{pmatrix} \end{array}$$

Two reasons for losing independence:

- $h_{0I_k} = 0 = h_{1I_k}$: all determinants $Q_1, \dots, Q_{l+1}$ vanish

$$h_{I_1}, \dots, h_{I_k}, Q_1, \dots, Q_l \longrightarrow \begin{array}{l} h_{I_1}, \dots, h_{I_{k+2}} \\ h_{I_{k+1}} = h_{0I_k},\ h_{I_{k+2}} = h_{1I_k} \end{array}$$

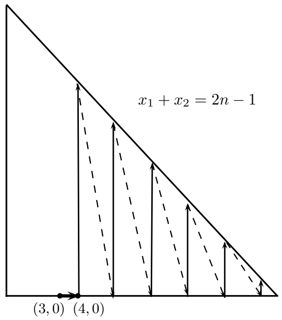
- $f_0 \wedge f_1 = 0$: conditions on lower order jets
  e.g., if $f_0(q) = 0$ then $[f_0, [f_0, \dots, [f_0, f_1] \dots](q)$ only depends on $f_1(q)$ and $Df_0(q)$

# Collecting conditions in $\{f_0 \wedge f_1 \neq 0\}$

If $h_{I_1} = 0, \ldots, h_{I_k} = 0, Q_1 = 0, \ldots, Q_l = 0$ are independent conditions at $\lambda(t)$ then $(\lambda(t), j^N(f_0, f_1)(q(t)))$ is in a codimension $k + l$ set ($N$ large enough).
For $k + l$ larger than $2n - 1$ we get a condition on $j^N(f_0, f_1)$ that is generically nowhere satisfied on $M$.
Any higher Fuller order gives rise to one of the following moves

- $(k, 0) \to (k+1, 0)$
- $(k, l) \to (k, l+1)$
- $(k, l) \to (k+2, 0)$

To find $K(n)$ we compute the longest sequence of moves staying in $\{k + l \leqslant 2n - 1\}$



# Inside the collinearity set $\{f_0 \wedge f_1 = 0\}$

- This set is already of codimension $n-1$
- We need to find just one more condition to conclude
- We may work with accumulation points $q(\bar{t})$ that lie in the set

$$\{f_1 \wedge [f_0, f_1] \neq 0\}.$$

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- The two conditions single out an embedded hypersurface transverse to $f_1$.
- Write, up to subsequences, $\dot{q}(\bar{t}) = f_0(q(\bar{t})) + \bar{u} f_1(q(\bar{t}))$.
- On the right-hand side we are parallel to $f_1(q(\bar{t}))$, on the left we are transverse to it. Both sides are zero.
- The limit $\bar{u}$ exists (does not depend upon subsequences), and is forced by the pair $(f_0, f_1)$.
- We obtained a new smooth field to use in our algorithm also in this case.
- Including estimates for the set $\{f_0 \wedge f_1 = 0\}$, we can show that $K(n) \leqslant (n-1)^2$.

# Better bounds using optimality

- We have not used optimality
- For time-optimal trajectories we have 2nd order conditions that may rule out trajectories with too many concatenated arcs.
- They are usable in low dimensions ($n = 3, 4$). After this they become difficult to handle.

# Better bounds using optimality

- We have not used optimality
- For time-optimal trajectories we have 2nd order conditions that may rule out trajectories with too many concatenated arcs.
- They are usable in low dimensions ($n = 3, 4$). After this they become difficult to handle.
- In dimension 3 we have the following result

**Proposition**

*For a generic pair* $(f_0, f_1) \in \mathrm{Vec}(M)^2$*, time-optimal trajectories have at most Fuller times of order two. Notice that* $K(3) = 2 \leqslant (3-1)^2 = 4$.

This follows collecting previous results of Agrachev, Gamkrelidze, Krener, Schättler, Sigalotti et al.

# Perspectives and open problems

- What is the minimal $K(n)$ (for time-extremal and time-optimal trajectories)? We showed $K(n) \leqslant (n-1)^2$. Can we at least bound $K(n)$ by a sub-quadratic function in $n$?
- What can be said for $M$, $f_0$, $f_1$ analytic?
- What about the multi-input case? Chattering phenomenon for $U$ polytope structurally stable for extremal trajectories ZELIKIN–LOKUTSIEVSKIY–HILDEBRAND, 2012. Switching studied for $U$ ball in AGRACHEV–BIOLO.
- Is optimality of (iterated) Fuller extremals structurally stable? Nobody has proved yet that the extremals found in the example of KUPKA are time-optimal (but everybody believes so).

Thank you for the attention!