Stability and control of Linear Difference Equations and Hyperbolic Partial Differential Equations

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Outline

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Linear Delay Difference Equations

$$\begin{cases} X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} N(-\nu) X(t+\nu) d\nu + f(t) \\ X_0 = \psi \end{cases} \tag{1}$$

with $X \in \mathbb{R}^n$,
A_k $n \times n$ constant matrices,
$N : [0, \tau_M] \mapsto \mathbb{R}^{n \times n}$ a piecewise continuous function
and $0 < \tau_1 < \tau_2 < \ldots < \tau_M$ constant and ordered delays.

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Remarks

- Choosing the distributed delay τ_M as equal to the maximum pointwise delay is not restrictive.
- It is always possible to rewrite the system such that the delays are rationally independent (with a state extension $Z(t) = (X(t), X(t-h), \ldots, X(t-(\ell-1)h)$ if $\tau_i = n_i h$ and $\tau_j = \ell h$ for a certain $h > 0$)

Linear Delay Difference Equations – Motivations

- Usually considered for *neutral* time-delay systems, i.e.,

$$\frac{d}{dt}\left[DX_{[t]}\right] = LX_{[t]} + h(t) \tag{2}$$

with $X_{[t]} : s \in [-\tau_M, 0] \mapsto X(t+s)$, linear operators $D, L : C([-\tau_M, 0], \mathbb{R}^n) \mapsto \mathbb{R}^n$ and

$$D\varphi = \varphi(0) - \sum_{k=1}^{M} A_k \varphi(-\tau_k) - \int_{-\tau_M}^{0} N(-\nu)\varphi(\nu)d\nu$$

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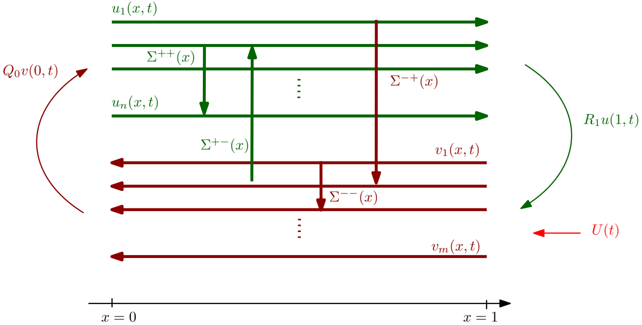
- A necessary condition for the exponential stability of the neutral equation (2) is the exponential stability of the delay difference equation $DX_t = 0$.

Linear Delay Difference Equations – Motivations

- Can be used to represent Linear First-Order Hyperbolic PDEs

$$\begin{cases} u_t + \Lambda^+(x)u_x = \Sigma^{++}(x)u + \Sigma^{+-}(x)v \\ v_t - \Lambda^-(x)v_x = \Sigma^{-+}(x)u + \Sigma^{--}(x)v \\ u(0,t) = Q_0 v(0,t), \quad v(1,t) = R_1 u(1,t) + U(t) \end{cases} \tag{3}$$

with $\Lambda^+ = \text{diag}(\lambda_1, \dots, \lambda_n)$ in which $-\mu_p < \dots < -\mu_1 < 0 < \lambda_1 < \dots < \lambda_n$.
$\Lambda^- = \text{diag}(\mu_1, \dots, \mu_p)$



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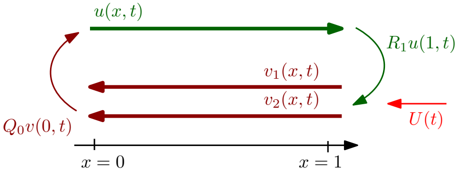
Example :

$$\begin{cases} u_t + \lambda u_x = 0 \\ v_t - \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} v_x = 0 \\ u(0,t) = q_1 v_1(0,t) + q_2 v_2(0,t) \\ v(1,t) = R_1 u(1,t) + U(t) \end{cases}$$



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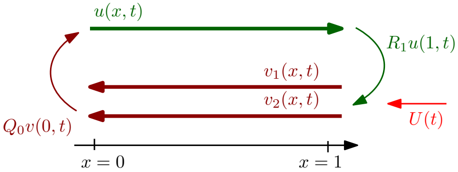

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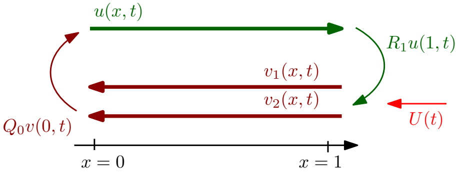

Then, it holds $v(1,t) = R_1 u(0, t - 1/\lambda) + U(t)$

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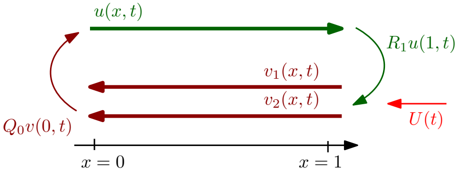

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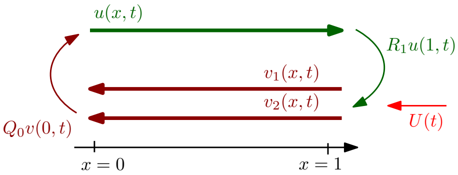

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$$v(1,t) = A_1 v \left(1, t - \frac{1}{\lambda} - \frac{1}{\mu_1} \right) + A_2 v \left(1, t - \frac{1}{\lambda} - \frac{1}{\mu_2} \right) + U(t)$$

with $A_1 = \begin{pmatrix} R_1 q_1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & R_1 q_2 \end{pmatrix}$.

Linear Delay Difference Equations – Stability results

$$\begin{cases} X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} N(-\nu) X(t+\nu) d\nu + f(t) \\ X_0 = \psi \end{cases} \tag{3}$$

with $X \in \mathbb{R}^n$,
A_k $n \times n$ constant matrices,
$N : [0, \tau_M] \mapsto \mathbb{R}^{n \times n}$ a piecewise continuous function
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Existence and uniqueness

For any $\psi \in C_{pw}([-\tau_M, 0], \mathbb{R}^n) \triangleq C_{\tau_M}^{pw}$, there exists a unique function $X(\cdot; \psi)$ defined and ***piecewise continuous*** on $[-\tau_M, +\infty)$ that satisfies (5) on $[0, +\infty)$.

Linear Delay Difference Equations – Stability results

Consider $D : \varphi \in C^{pw}_{\tau_M} \mapsto \varphi(0) - \sum_{k=1}^{M} A_k \varphi(-\tau_k) - \int_{-\tau_M}^{0} N(-\nu)\varphi(\nu)d\nu$.

Stability [Hale, Verduyn Lunel, Introduction to Functional Differential Equations]

The following statements are equivalent :

1. D is **stable**, i.e. the zero solution of $DX_{[t]} = 0$ is asymptotically stable ;
2. let $\Delta(\lambda) = I - \sum_{k=1}^{M} A_k e^{-\lambda\tau_k} - \int_{-\tau_M}^{0} N(-\nu)e^{\nu\lambda}d\nu$, then
$$\sup\{\mathrm{Re}\,\lambda \mid \det\Delta(\lambda) = 0\} < 0$$
3. there exist $\delta > 0$ such that
$$\det\Delta(\lambda) = 0 \Rightarrow \mathrm{Re}\,\lambda \leq -\delta$$
4. there exist $C, \alpha > 0$ s.t., for any $f \in C_{pw}(\mathbb{R}_+, \mathbb{R}^n)$, the solution to $DX_{[t]} = f(t)$ with $X_0 = \psi \in C^{pw}_{\tau_M}$ satisfies
$$|X(t,\psi)| \leq C\left(\|\psi\|e^{-\alpha t} + \sup_{0\leq s\leq t} |f(s)|\right)$$

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Unfortunately, the concept of stability is not sufficient to guarantee continuity with respect to the delays $\tau = (\tau_1, \ldots, \tau_M)$.

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Strong stability [Hale, Verduyn Lunel]

Let $\tau \in (\mathbb{R}_+^*)^M$ with rationally independent τ_k and $D(\tau)\varphi = \varphi(0) - \sum_{k=1}^{M} A_k \varphi(-\tau_k)$. The following statements are equivalent :

1. $D(\tau)$ is **stable** ;
2. denoting Sp the spectral radius of a matrix,
$$\sup_{\theta_k \in [0,2\pi]^M} \mathrm{Sp} \left(\sum_{k=1}^{M} A_k \exp(i\theta_k) \right) < 1$$
3. $D(\tau)$ is *stable locally in the delays* : there is an open neighborhood $I(r) \in (\mathbb{R}_+^*)^M$ of r such that $D(s)$ is stable for each $s \in I(r)$
4. D is *stable globally in the delays* : $D(s)$ is stable for every $s \in (\mathbb{R}_+^*)^M$

Outline

Problem at stake

Difference system with delayed actuation

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} N(-\nu) X(t+\nu) d\nu + U(t-\delta), \quad t \geq 0$$
$$X_0 = \psi \in C_{\tau_M}^{pw}$$

Robustness assumption

$D : \varphi \mapsto \varphi(0) - \sum_{k=1}^{M} A_k \varphi(-\tau_k)$ is (strongly) stable.

- If $X(t) = \sum_{k=1}^{M} A_K X(t-\tau_k)$ unstable $\rightarrow$ **infinite** number of unstable poles.
- Finite number of unstable poles $\rightarrow$ necessary for **(delay)-robust stabilization** (Logemann, Rebarber and Weiss, 1996)

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The **distributed term** $\int_{-\tau_M}^{0} N(-\nu)X(t+\nu)d\nu$ can destabilize the system

Remark 1

The control input dimension equals to the state dimension.

- Current limitation of our approach.

Remark 2

The delay δ is the same for all the components of U.

- Possibility to artificially delay the components of U.

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$$X_0 = \psi \in C_{\tau_M}^{pw}$$

- Use the control law $U(t)$ to stabilize the system.
- No delay $\delta = 0 \Rightarrow U(t) = -\int_{-\tau_M}^{0} N(-\nu) X(t+\nu) d\nu$
- $\delta \neq 0 \Rightarrow$ **Prediction** of the state ?

Predictor for difference systems

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} N(-\nu) X(t+\nu) d\nu + U(t-\delta), \quad t \geq 0$$
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Exponential stabilization using a predictor

The control law

$$U_{\text{pred}}(t) = -\int_{-\delta}^{0} N(-\nu) P_{[t]}(\nu) d\nu,$$

in which the **prediction** $P_{[t]}$ is implicitly defined as

$$P_{[t]}(s) = \sum_{k=1}^{M} A_k P_{[t]}(s-\tau_k) + \int_{-\delta}^{0} N(-\nu) P_{[t]}(s+\nu) d\nu + U_{[t]}(s), \quad t \geq -\delta, \ s \in [-\delta, 0]$$

with initial condition $P_{[-\delta]} = X^0$, exponentially stabilizes the system.

- Integral relation of Volterra type $\rightarrow$ Prediction well-defined.
- Possible to explicitly compute this predictor ?

Explicit realization of the predictor

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} N(-\nu) X(t+\nu) d\nu + U(t-\delta), \quad t \geq 0$$

We consider the following candidate control law

$$U(t) = \int_{-\delta}^{0} [f(-\nu) X(t+\nu) + g(-\nu) U(t+\nu)] \, d\nu,$$

with f and g **piecewise continuous matrix-valued functions**

Find f and g s.t. U stabilizes the system

.

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$$X(t) - \int_{-\delta}^{0} g(-\nu)X(t+\nu)d\nu = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_{-\delta}^{0} (N(-\nu)X(t+\nu) - g(-\nu)U(t+\nu-\delta))d\nu$$

$$+ U(t-\delta) - \sum_{k=1}^{M} \int_{-\delta}^{0} g(-\nu) A_k X(t+\nu-\tau_k) d\nu - \int_{-\delta}^{0}\int_{-\delta}^{0} g(-\nu) N(-\eta) X(t+\nu+\eta) d\eta d\nu,$$

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$$- \int_{-2\delta}^{-\delta}\int_{-\delta}^{s+\delta} g(-s+\eta)N(-\eta)d\eta X(t+s)ds$$

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$$- \int_{-2\delta}^{-\delta} \int_{-\delta}^{s+\delta} g(-s+\eta) N(-\eta) d\eta X(t+s) ds + \int_{-2\delta}^{-\delta} f(-\nu-\delta) X(t+\nu) d\nu$$

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Volterra equations and theorem

We have $X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k)$ if

$$0 = g(\nu) + N(\nu) - \int_0^\nu g(\nu-\eta)N(\eta)d\eta - \sum_{k=1}^{M} \mathbb{1}_{[\tau_k,\delta]}(\nu)g(\nu-\tau_k)A_k, \tag{3}$$

$$0 = f(\nu-\delta) - \int_{\nu-\delta}^{\delta} g(\nu-\eta)N(\eta)d\eta - \sum_{k=1}^{M} \mathbb{1}_{[\delta,\tau_k+\delta]}(\nu)g(\nu-\tau_k)A_k, \tag{4}$$

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Existence of the functions f and g

There exist two unique piecewise continuous functions (f, g) that are solutions of (3)-(4).

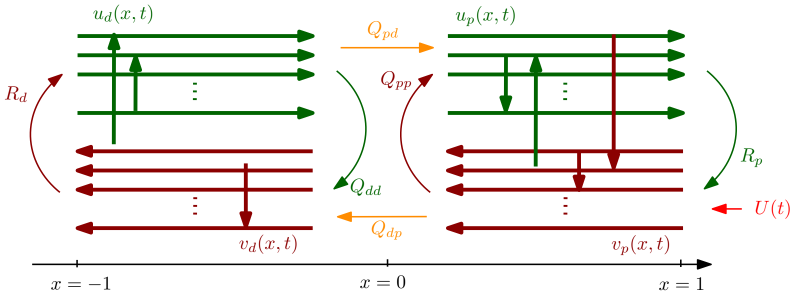
Closed-loop exponential stability

The control law $U(t) = \int_{-\delta}^{0} [f(-\nu)X(t+\nu) + g(-\nu)U(t+\nu)]\, d\nu$, where f and g are solutions of (1)-(2) **exponentially stabilizes** the original system in the sense of the $C^{pw}_{\tau_M}$-norm. Moreover, the control law is **strictly proper and exponentially converges to zero**.

This control law corresponds to an **explicit realization of the predictor**.

Outline

Control of a network of LFOH-PDEs



where $u^p = (u_1^p, \ldots, u_{n_p}^p)$, $v^p = (v_1^p, \ldots, v_{m_p}^p)$, $u^d = (u_1^d, \ldots, u_{n_d}^d)$, $v^d = (v_1^d, \ldots, v_{m_d}^d)$.

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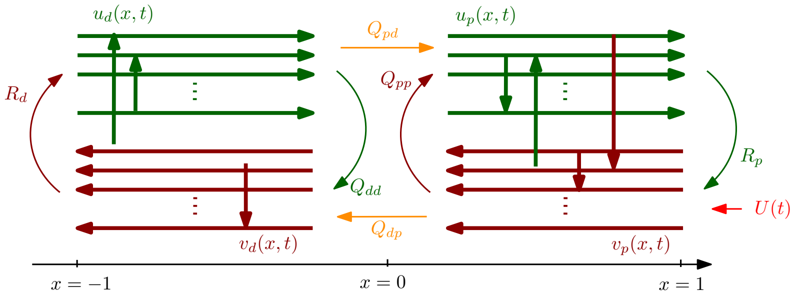

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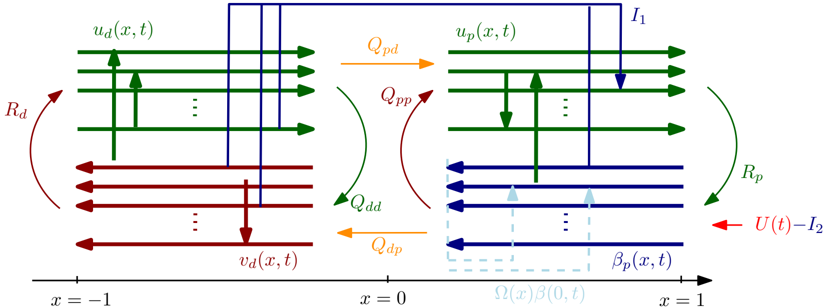
Assumption

The rank of the matrix Q_{dp} is equal to m_d (existence of a right inverse).

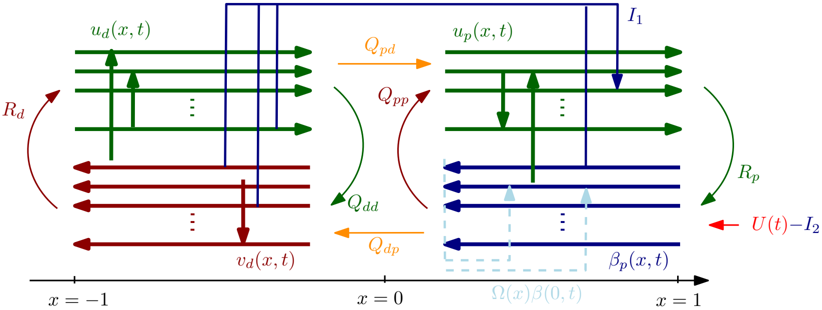
Control of a network of LFOH-PDEs. Step 1 : backstepping



with Ω **upper-triangular** and

$$\beta^p(t,x) = v^p(t,x) - \int_0^x K_p^u(x,\xi)u^p(t,\xi)d\xi - \int_0^x K_p^v(x,\xi)v^p(t,\xi)d\xi$$
$$- \int_{-1}^0 K_d^u(x,\xi)u^d(t,\xi)d\xi - \int_{-1}^0 K_d^v(x,\xi)v^d(t,\xi)d\xi$$

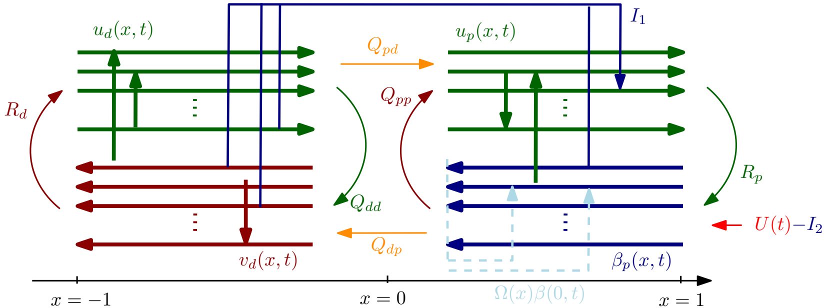
Control of a network of LFOH-PDEs. Step 1 : backstepping



with Ω **upper-triangular**

$$\beta_t^p - \Lambda_p^-(x)\beta_x^p = \begin{pmatrix} 0 & & & \\ 0 & 0 & \star & \\ \vdots & & \ddots & \\ 0 & \dots & & 0 \end{pmatrix} \beta^p(0,t)$$

Control of a network of LFOH-PDEs. Step 1 : backstepping

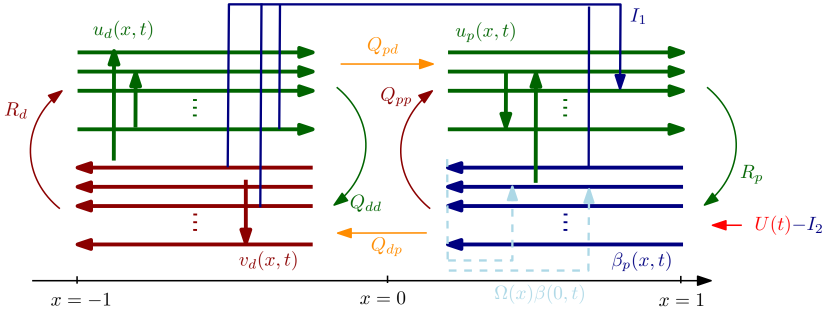


Lemma : $\beta(0,t)$ is a flat output [see Hu, Krstic, Mounier, Meurer]

Let $U_i(t) = -(R_p u^p(t,1) + I_2)_i + \zeta_i\left(t + \frac{1}{\mu_i^p}\right) - \sum_{j=i+1}^{m_p} \int_0^{\frac{1}{\mu_i^p}} \Omega_{i,j}(\mu_i^p \nu)\zeta_j\left(t + \frac{1}{\mu_i^p} - \nu\right) d\nu$

where ζ is an arbitrary known function. Then, for any $t \geq \sum_{j=1}^{m_p} \frac{1}{\mu_j^p}$, $\beta^p(t,0) \equiv \zeta(t)$.

Control of a network of LFOH-PDEs. Step 1 : backstepping



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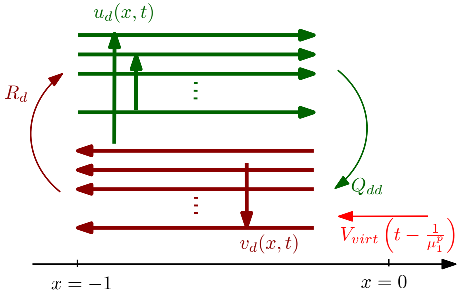
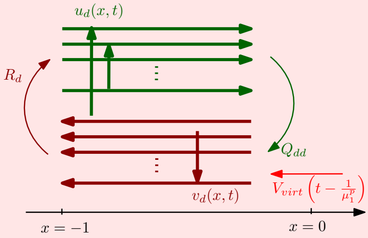
Control of a network of LFOH-PDEs. Step 2 : prediction



Control of a network of LFOH-PDEs. Step 2 : prediction

Theorem [see Auriol, Di Meglio, SCL 2019]

There exist $L^\infty([0,\tau_{m_d+n_d}],\mathbb{R})$-functions G_{ij} (with $i\in\{1,\ldots,n_d\}$ and $j\in\{1,\ldots,m_d\}$) which only depend on the system parameters such that the **stability properties of the above PDE are equivalent to those of the difference system** defined for all $1\le i\le m_d$ by



$$z_i(t)=\sum_{k=1}^{n_d}\sum_{l=1}^{m_d}(Q_{dd})_{ik}(R_d)_{kl}z_l\left(t-\frac{1}{\lambda_k^d}-\frac{1}{\mu_l^d}\right)+\sum_{l=1}^{m_d}\int_0^{\tau_{m_d+n_d}}G_{il}(\nu)z_l(t-\nu)d\nu+(V_{\text{virt}})_i\left(t-\frac{1}{\mu_1^p}\right),$$

i.e., there exist two constants $C_1>0$ and $C_2>0$ and a constant $r>0$ s.t. for all $t>\tau_{m_d+n_d}$,

$$C_1||z_{[t]}||_{L^2,r}\le||(u_d,v_d)(t)||_{L^2}\le C_2||z_{[t]}||_{L^2}.$$

Moreover, for all $t\ge 0$, the state $z(t))$ can be expressed as a function of $u^d(\cdot,t),v^d(\cdot,t)$, that is, there exists a linear operator $\mathcal{F}$ such that for all $t\ge0$, $z(t)=\mathcal{F}(u^d(\cdot,t),v^d(\cdot,t))$.

Control of a network of LFOH-PDEs. Final design

Assumption

$$D : \varphi \mapsto \varphi(0) - \sum_{k=1}^{n_d} \sum_{l=1}^{m_d} (Q_{dd})_{ik} (R_d)_{kl} \varphi \left(-\frac{1}{\lambda_k^d} - \frac{1}{\mu_l^d} \right)$$

is exponentially stable.

- Choose the **virtual control law** as $V_{virt}(t) = \int_{-\delta}^{0} [f(-\nu) z(t+\nu) + g(-\nu) V_{virt}(t+\nu)] \, d\nu$, where f and g are solutions of the previous Volterra equations.

Control of a network of LFOH-PDEs. Final design

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- An **exponentially stabilizing control law** is then

$$U_i(t) = -(R_p u^p(t,1) + I_2)_i + \zeta_i \left(t + \frac{1}{\mu_i^p} \right) - \sum_{j=i+1}^{m_p} \int_0^{\frac{1}{\mu_i^p}} \Omega_{i,j}(\mu_i^p \nu) \zeta_j(t + \frac{1}{\mu_i^p} - \nu) d\nu$$

with

$$\zeta(t) = Q_{dp}^T (Q_{dp} Q_{dp}^T)^{-1} V_{\mathrm{virt}}(t - \frac{1}{\mu_1^p})$$

1. J. Auriol and DBP, Robust state-feedback stabilization of an underactuated network of interconnected n + m hyperbolic PDE systems, Automatica, 2021

Robustness ?

- Prediction-based controller are known to be **sensitive** to uncertain input delays
- Important direction of work : evaluate the robustness of this controller to this feature.
- Analysis has been carried out for prediction-based control of ODEs with an input delay [Krstic, Bekiaris-Liberis, Léchappé]
- Based on a PDE/backstepping methodology + **Lyapunov analysis**

Outline

Questions at stake

$$\begin{cases} X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + f(t) \\ X_0 = \psi \end{cases} \tag{5}$$

with $0 < \tau_1 < \tau_2 < \ldots < \tau_M$ constant and ordered delays.

Input-to-State Stability with respect to the L^2-norm

The system (5) is called Input-to-State Stable with respect to the L^2-norm if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ s. t., for every piecewise continuous signal f and $t \geq 0$,

$$\|X_{[t]}\|_{L_2} \leq \beta(\|\psi\|_{L_2}, t) + \gamma\left(\sup_{s\in[0,t]}(|f(s)|)\right)$$

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- D stable iff L_∞-ISS [Hale, Verduyn Lunel] iff L_2-ISS [Carvalho]

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$$\begin{cases} X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + f(t) \\ X_0 = \psi \end{cases} \quad (5)$$

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- D stable iff L_∞-ISS [Hale, Verduyn Lunel] iff L_2-ISS [Carvalho]

Questions

How to choose an appropriate ISS Lyapunov function ? Does there exist one (converse Lyapunov theorem) ?

Related existing results

- For hyperbolic PDEs :
  - **existence of strict-Lyapunov functional** for the exponential stability ($f \equiv 0$) : Coron at al (2007, 2008, 2012)

$$V(t) = \int_0^1 \left[u^T P_0 u e^{-\mu x} + v^T P_1 v\, e^{\mu x} \right] dx$$

    for dissipative boundary conditions, namely, such that

$$\rho\left(\begin{pmatrix} 0 & Q_0 \\ R_1 & 0 \end{pmatrix}\right) = \inf\left\{ \left\| \Delta \begin{pmatrix} 0 & Q_0 \\ R_1 & 0 \end{pmatrix} \Delta^{-1} \right\|, \quad \Delta \in \mathcal{D}^+_{n+m} \right\} < 1$$

  - *Tanwani, Prieur and Tarbouriech* (2018) : ISS for a system of balance laws under a dissipative boundary condition
  - *Karafyllis and Krstic*, Input-to-State Stability for PDEs, 2019 : Lyapunov theorems for hyperbolic and parabolic PDEs, but only unidirectional hyperbolic PDEs
- For Difference Equations :
  - **Necessary stability conditions** in terms of a Lyapunov function (*Rocha Campos, Mondié and Di Loreto*, 2018)
  - Pepe 2014, Lyapunov characterization but **with difference operator**
  - Lyapunov characterization of the asymptotic stability of the nonlinear DDE $\eta(t) = g(\eta_{[t]}, f(t))$ (*Karafyllis, Krstic*, 2014), but only sufficient condition for ISS

Necessary stability condition : Lyapunov-Krasovskii functional [Rocha Campos et al, 2018]

Cauchy Formula

The solution to

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) \text{ with } 0 < \tau_1 < \ldots < \tau_M, \quad X_{[0]} = \varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n)$$

is

$$X(t) = \sum_{k=1}^{M} D^{+} \int_{-\tau_k}^{0} K(t-\theta-\tau_k) A_k \varphi(\theta) d\theta$$

in which the fundamental matrix K satisfies

$$\begin{cases} K(t) = \sum_{k=1}^{M} K(t-\tau_k) A_k, & t \geq 0 \\ K(\theta) = K_0 \triangleq \left(\sum_{k=1}^{M} A_k - I \right)^{-1}, & \theta \in [-\tau_M, 0) \end{cases}$$

and is piecewise constant with the set $\mathcal{T} = \left\{ t_{j+1} = \sum_{k=1}^{M} n_k \tau_k \,\middle|\, n_k \in \mathbb{N}, \quad t_{j+1} \geq t_j \right\}$ of discontinuity points .

Necessary stability condition : Lyapunov-Krasovskii functional [Rocha Campos et al, 2018]

Let us consider a quadratic functional v_0 such that

$$D^+ v_0(X_{[t]}) = -X(t)^T W X(t), \quad t \geq 0$$

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and, if the DDE is exponentially stable, taking $T \to \infty$,

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$$= \int_0^\infty \left(\sum_{i=1}^M D_\xi^+ \int_{-\tau_i}^0 \varphi(\xi)^T A_i^T K(s-\xi-\tau_i)^T d\xi \right) W \left(\sum_{j=1}^M D_\theta^+ \int_{-\tau_j}^0 K(s-\theta-\tau_j) A_j \varphi(\theta) d\theta \right)$$

$$= \sum_{i=1}^M \sum_{j=1}^M \int_{-\tau_i}^0 \int_{-\tau_j}^0 \varphi(\xi)^T A_i^T D_\xi^+ D_\theta^+ \int_0^\infty K(s-\xi-\tau_i)^T W K(s-\theta-\tau_j) ds A_j \varphi(\theta) d\xi d\theta$$

Necessary stability condition : Lyapunov-Krasovskii functional [Rocha Campos et al, 2018]

Lyapunov matrix

Suppose that the DDE is exponentially stable. Then, for every symmetric positive definite matrix W, the Lyapunov matrix

$$U(t) = \int_0^{\infty} (K(s) - K_0)^T W K(s+t) ds$$

is well-defined for $t \geq -\tau_M$ and its derivative can be expressed as

$$U'(t) = \sum_{t_k \in \mathcal{T}} (K^T(t_k - t) - K_0^T) W \Delta K(t_k) \quad \text{a.e}$$

with $\mathcal{T} = \left\{ t_{j+1} = \sum_{k=1}^{M} n_k \tau_k \,|\, n_k \in \mathbb{N}, \quad t_{j+1} \geq t_j \right\}$ and $\Delta K(t_k) = K(t_k^+) - K(t_k^-)$.

The previous functional writes in terms of the Lyapunov matrix as

$$v_0(\varphi) = \sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \varphi(\xi)^T A_i^T \underbrace{D_\xi^+ D_\theta^+ \int_0^{\infty} K(s - \xi - \tau_i)^T W K(s - \theta - \tau_j) ds}_{= U''(-\theta - \tau_j + \xi + \tau_i)} A_j \varphi(\theta) d\xi d\theta$$

Necessary stability condition : Lyapunov-Krasovskii functional [Rocha Campos et al]

Equivalence with the L_2-norm

If the DDE is exponentially stable, then there exists $\alpha > 0$ such that

$$v_1(\varphi) = v_0(\varphi) + \int_{-\tau_M}^{0} \varphi(s)^T W \varphi(s) ds \tag{6}$$

satisfies $v_1(\varphi) \geq \alpha \|\varphi\|^2_{\mathcal{L}_2}$ for any $\varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n)$.

Proof : Consider $\tilde{v}(\varphi) = v_1(\varphi) - \alpha \|\varphi\|^2_{L_2}$. Then

$$\begin{aligned} D^+ \tilde{v}(X_{[t]}) &= D^+ v_0(X_{[t]}) + X(t)^T (W - \alpha I) X(t) - X(t - \tau_M)^T (W - \alpha I) X(t - \tau_M) \\ &= -\alpha X(t)^T X(t) - X(t - \tau_M)^T (W - \alpha I) X(t - \tau_M) \end{aligned}$$

which is negative if $\alpha < \lambda_{min}(W)$. As the DDE is exponentially stable, $\tilde{v}(X_{[t]}) \to 0$ for $t \to \infty$ and thus $\tilde{v}(\varphi) \geq 0$.

How to study the effect of an additive signal ?

Objective

We are now interested in the dynamics

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + f(t) \text{ with } 0 < \tau_1 < \ldots < \tau_M , \quad X_{[0]} = \varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n) \tag{7}$$

that we wish to study with the Lyapunov functional

$$v_0(\varphi) = \sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \varphi(\xi)^T A_i^T U''(-\theta - \tau_j + \xi + \tau_i) A_j \varphi(\theta) d\xi d\theta$$

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Recall that

$$U'(t) = \sum_{t_k \in \mathcal{T}} (K^T(t_k - t) - K_0^T) W \Delta K(t_k) \quad \text{a.e}$$

with $\mathcal{T} = \left\{ t_{j+1} = \sum_{k=1}^{M} n_k \tau_k \,\middle|\, n_k \in \mathbb{N}, \quad t_{j+1} \geq t_j \right\}$.

The discontinuities of U' are countable

Lemma

The matrix U' is piecewise constant and its set $\mathcal{I}$ of discontinuity points in $(-\tau_M, \tau_M)$ is countable. Moreover, if the homogeneous system

$$X(t) = \sum_{k=1}^{M} A_k X(t - \tau_k)$$

is exponentially stable, then the quantity $\sum_{\tau_c \in \mathcal{I}} ||\Delta U'(\tau_c)||$ is finite.

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Observe that

$$\begin{aligned} v_0(\varphi) &= \sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \int_{-\tau_j}^{0} \varphi(\xi)^T A_i^T U''(-\theta - \tau_j + \xi + \tau_i) A_j \varphi(\theta) d\xi d\theta \\ &= \sum_{i,j=1}^{M} \int_{-\tau_i}^{0} \sum_{\tau_c \in \mathcal{I} \cap (-\tau_j, 0)} \varphi(\xi)^T A_i^T \Delta U'(\tau_c) A_j \varphi(\xi + \tau_i - \tau_j - \tau_c) d\xi \end{aligned}$$

The Dini derivative of v_0 can then be computed

Lemma

Consider $X_{[t]}$ the solution of

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + f(t), \quad X_{[0]} = \varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n)$$

and assume that the homogeneous equation is exponentially stable. Then, for all $t \geq 0$,

$$D^+ v_0(X_{[t]}) = -X^T(t) W X(t) - 2X^T(t)\Delta U'(0) f(t) + f^T(t)\Delta U'(0) f(t)$$
$$-2\sum_{i=1}^{M} \sum_{\tau_c \in \mathcal{I} \cap (0,\tau_i)} X^T(t+\tau_c-\tau_i) A_i^T \Delta U'(\tau_c) f(t)$$

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Proof : Uses the inhomogeneous equation and the following properties of U' :

- the Symmetry property $\Delta U'(-\tau) = [\Delta U'(\tau)]^T$,
- the Generalized algebraic property $W\Delta K(\tau) = \sum_{i,j=1}^{M} A_i^T \Delta U'(\tau+\tau_i-\tau_j) A_j - \Delta U'(\tau)$
- the Dynamic property

$$\Delta U'(\tau) = \begin{cases} \sum_{k=1}^{M} \Delta U'(\tau-\tau_k) A_k, & \tau > 0, \\ \sum_{k=1}^{M} A_k^T \Delta U'(\tau+\tau_k), & \tau < 0, \end{cases}$$

Remaining issues...

- tackle the term $-2\sum_{i=1}^{M}\sum_{\tau_c\in I\cap(0,\tau_i)} X^T(t+\tau_k-\tau_i)A_i^T\Delta U'(\tau_c)f(t)$
- obtain equivalence with the $\mathcal{L}_2$-norm
- get a strict Lyapunov function

Remaining issues...and their solutions

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  $\rightarrow$ we add integral terms of the form $\sum_{q\geq 1} b_q\int_{-\bar{\tau}_q}^{0}\varphi(\nu)^T\varphi(\nu)d\nu$ for a certain increasing sequence of delays $\bar{\tau}_q$ with suitable b_q

- obtain equivalence with the $\mathcal{L}_2$-norm

  $\rightarrow$ we add an integral term $b\int_{-\tau_M}^{0}\varphi(\nu)^T\varphi(\nu)d\nu$ with b small enough

- get a strict Lyapunov function

  $\rightarrow$ we include forgetting factors of the form $e^{\rho\nu}$ in the integrals

Remaining issues...and their solutions

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- get a strict Lyapunov function

  $\rightarrow$ we include forgetting factors of the form $e^{\rho\nu}$ in the integrals

This leads to the following Lyapunov functional, for suitable $\rho, b, b_q > 0$,

$$v_1(\varphi)=\sum_{i=1}^{M}\sum_{j=1}^{M}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0}\varphi^T(\xi)A_i^T U''(-\theta-\tau_j+\xi+\tau_i)A_j\varphi(\theta)e^{\frac{\rho}{2}(\theta+\xi)}d\theta d\xi$$

$$+\sum_{q\geq 1} b_q\int_{-\bar{\tau}_q}^{0}\varphi(\nu)^T\varphi(\nu)e^{\rho\nu}d\nu+b\int_{-\tau_M}^{0}\varphi(\nu)^T\varphi(\nu)e^{\rho\nu}d\nu$$

Main result

Theorem

Consider the inhomogenous equation

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + f(t), \quad X_{[0]} = \varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n) \tag{8}$$

The following statements are equivalent :

- (i) the solution to the inhomogeneous equation is $\mathcal{L}_2$-ISS w.r.t. f
- (ii) there exists a locally Lipschitz function $v_1 : \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n) \to \mathbb{R}_+$ such that
  1. $\exists\, \rho, \sigma > 0 \quad D^+ v_1(X_{[t]}) \leq -\rho v_1(X_{[t]}) + \sigma \|f(t)\|^2, \quad t \geq 0$
  2. $\exists\, \alpha_1, \alpha_2 > 0 \quad \forall \varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n) \quad \alpha_1 \|\varphi\|^2_{\mathcal{L}_2} \leq v_1(\varphi) \leq \alpha_2 \|\varphi\|^2_{\mathcal{L}_2},$

Main result

Theorem

Consider the inhomogenous equation

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + f(t), \quad X_{[0]} = \varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n) \tag{8}$$

The following statements are equivalent :

1. (i) the solution to the inhomogeneous equation is $\mathcal{L}_2$-ISS w.r.t. f
2. (ii) there exists a locally Lipschitz function $v_1 : \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n) \to \mathbb{R}_+$ such that
   1. $\exists\, \rho, \sigma > 0 \quad D^+ v_1(X_{[t]}) \le -\rho v_1(X_{[t]}) + \sigma \|f(t)\|^2, \quad t \ge 0$
   2. $\exists\, \alpha_1, \alpha_2 > 0 \quad \forall \varphi \in \mathcal{C}_{pw}([-\tau_M, 0], \mathbb{R}^n) \quad \alpha_1 \|\varphi\|^2_{\mathcal{L}_2} \le v_1(\varphi) \le \alpha_2 \|\varphi\|^2_{\mathcal{L}_2}$,

What does it imply for the corresponding system of hyperbolic PDEs ?

Outline

Conclusion and perspectives

- Explicit prediction for Linear Difference Equation
- Full-state boundary stabilization of a network of hyperbolic PDEs
- Lyapunov characterization of L_2-ISS of linear Difference Equations

Perspectives

- Extension to multiple input delays ?
- Underactuated difference equations ?
- Delay-robustness analysis (stochastic, time-varying, etc)
- Similar results have been obtained by [Ortiz, Mondié et al, 2019] for

$$X(t) = \int_{-\tau_M}^{0} N(\nu) X(t+\nu) d\nu$$

- Extension to

$$X(t) = \sum_{k=1}^{\tau_M} A_k X(t-\tau_k) + \int_{-\tau_M}^{0} N(\nu) X(t+\nu) d\nu$$

- **Expression of the Lyapunov functional for the PDE**, that is, in the (u, v) coordinates ?