On the regularity of abnormal minimizers for rank 2 sub-Riemannian structures

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Joint work with

This is based on a joint work with

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→ Main references:

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Outline

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Introduction

- The question of regularity of length-minimizers is one of the main open problems in sub-Riemannian geometry,
- Length-minimizers are solutions to a variational problem with constraints and satisfy a first-order necessary condition ($\to$ Pontryagin Maximum Principle).

To a length-minimizer $\gamma : [0,T] \to M$ $\to$ a lift $\lambda : [0,T] \to T^*M$ satisfying a suitable Hamiltonian equation. This splits length-minimizers into

normal & abnormal (or singular)

A length-minimizer γ can indeed admit several lifts, each of them being either normal or abnormal.

- If a length-minimizer admits a normal lift, then it is C^∞, solutions of smooth autonomous Hamiltonian systems in T^*M.

Regularity of length-minimizers

- The question of regularity in SR is reduced to length-minimizers that are strictly abnormal
- For such length-minimizers, from the first order necessary condition (and second order as well) it is not possible to deduce a priori any regularity other than Lipschitz, see e.g. [Monti, '14].

Here we investigate the following:

Question

Are all length-minimizers in a sub-Riemannian manifold of class C^1?

$\rightarrow$ We assume length-minimizers to be arclength parameterized and their regularity is meant with respect to this time parameterization.

Sub-Riemannian: notations

A sub-Riemannian manifold is a triple (M, D, g) where

- M is a smooth manifold
- D is a smooth vector distribution
- g is a smooth metric on D

We *do not assume* a priori that D has constant rank.

$\rightarrow$ one reduces to the constant rank case through a desingularisation procedure.

One can always reduce to the following global situation

- $D = \operatorname{span}\{X_1, \ldots, X_k\}$,
- $g(X_i, X_j) = \delta_{ij}$

We always assume that the structure is *bracket generating*

$$\operatorname{span}\{X_i, [X_j, X_k], \ldots\}(x) = T_x M.$$

Sub-Riemannian: length-minimizers

Fix a generating frame $X_1, \ldots, X_k$ of (D, g). For any horizontal curve γ of finite length, there exists $u \in L^\infty([0,T], \mathbb{R}^k)$ satisfying

$$\dot\gamma(t) = \sum_{i=1}^{k} u_i(t) X_i(\gamma(t)), \qquad \text{for a.e. } t \in [0,T]. \tag{1}$$

The sub-Riemannian distance is obtained by minimizing the length

$$d(x,y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(T) = y\}, \qquad \ell(\gamma) = \int_0^T \left(\sum_{i=1}^{k} u_i(t)^2 \right)^{1/2} dt.$$

A horizontal curve $\gamma : [0,T] \to M$ is

- *arclength parameterized* if $u \in L^\infty([0,T], \mathbb{S}^{k-1})$. Thus $\ell(\gamma) = T$.
- *length-minimizer* if $d(\gamma(0), \gamma(T)) = \ell(\gamma)$.

Hamiltonians: normal vs abnormal

Let $h_i : T^*M \to \mathbb{R}$ the functions $h_i(\lambda) = p \cdot X_i(x)$ for $\lambda = (p, x)$.

Theorem (First order conditions)

Let $\gamma : [0, T] \to M$ be an arclength parametrized length-minimizer. There exists a Lipschitz continuous curve $t \mapsto \lambda(t) \in T^*_{\gamma(t)}M$ such that (at least) one of the following conditions is satisfied:

(N) $\dot\lambda(t) = \vec{H}(\lambda(t))$ *for all* $t \in [0, T]$*, where* $H = \frac{1}{2}\sum_{i=1}^k h_i^2$*,*

(A) $\dot\lambda(t) = \sum_{i=1}^k u_i(t)\vec{h}_i(\lambda(t))$ *and* $h_i(\lambda(t)) \equiv 0$ *for* $i = 1, \ldots, k$.

- condition (N) implies that $\lambda(t)$ is C^∞
- condition (A) implies that $\lambda(t) \in (D_{\gamma(t)})^\perp$ for all t
- condition (A) does not imply further regularity than Lipschitz
- if (A) is satisfied and (N) is not, then $\lambda(t) \in (D^2_{\gamma(t)})^\perp$ (Goh condition)

Some previous results

- If (M, D, g) has step 2, there are no strictly abnormal length-minimizers $\rightarrow$ every length-minimizer is C^∞.
- If (M, D, g) has step 3, the situation is already more complicated. A positive answer is known only for Carnot groups ($\rightarrow$ length-minimizers are C^∞). [Leonardi-Le Donne-Monti-Vittone, '13]
- If (M, D, g) is real-analytic, every length-minimizer is real-analytic on an open dense subset of its interval of definition [Sussmann, '14].

$\rightarrow$ these results hold with no restriction on the rank of the distribution

Theorem (Chitour-Jean-Trélat, '06)

Let M be a smooth manifold and $k \geq 3$. For an open dense set of SR structures (D, g) with $\dim D = k$, there are no strictly abnormal length-minimizers.

$\rightarrow$ key study: when the rank of the distribution is 2

Corners are not minimizers

Theorem (Hakavouri-Le Donne, '16)

Let M be a sub-Riemannian manifold. Let $T > 0$ and let $\gamma : [-T, T] \to M$ be a horizontal curve such that, in local coordinates, there exist

$$\dot{\gamma}^+(0) := \lim_{t \downarrow 0} \frac{\gamma(t) - \gamma(0)}{t}, \qquad \dot{\gamma}^-(0) := \lim_{t \uparrow 0} \frac{\gamma(t) - \gamma(0)}{t}.$$

If $\dot{\gamma}^+(0) \neq \dot{\gamma}^-(0)$, then γ is not a length-minimizer.

- → based on a two main arguments "cut and adjust" + "blowup" techniques first introduced by [Leonardi-Monti, '08].
- → permits to reduce the proof of corners to Carnot groups of rank 2.
- → other recent generalizations about blowup and existence of tangent lines [Monti-Pigati-Vittone, '17], cf. also very recent [Hakavouri-Le Donne, '18]

Outline

Let (D, g) on M be a bracket generating distribution D endowed with a metric g. Hence D defines a flag of subspaces at every point $x \in M$

$$D_x = D_x^1 \subset D_x^2 \subset D_x^3 \subset \cdots \subset D_x^r = T_x M,$$

This induces a dual decreasing sequence of subspaces of $T_x^* M$

$$\{0\} = (D_x^r)^\perp \subset \cdots \subset (D_x^4)^\perp \subset (D_x^3)^\perp \subset (D_x^2)^\perp \subset (D_x^1)^\perp \subset T_x^* M,$$

- By construction, any abnormal lift satisfies $\lambda(t) \in (D^1)^\perp$.
- For strictly abnormal minimizer, then by Goh conditions $\lambda(t) \in (D^2)^\perp$.

When the distribution has rank 2

- if $\lambda(t)$ does not cross $(D^3)^\perp$, then the length-minimizer is C^∞ [Liu-Sussmann, '95, Agrachev-Sarychev, '95]

Question?

What can we say if $\lambda(t)$ enters in $(D^3)^\perp$?

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Main results: one step further!

Let (D, g) be a rank 2 sub-Riemannian structure on M. Assume that $\gamma : [0, T] \to M$ is an arclength parameterized abnormal minimizer.

Theorem

If γ admits a lift satisfying $\lambda(t) \notin (D^4)^\perp$ for every $t \in [0, T]$, then γ is C^1.

If the sub-Riemannian manifold has rank 2 and step at most 4, the assumption is trivially satisfied

Corollary

Assume that the sub-Riemannian structure has rank 2 and step at most 4. Then all length-minimizers are of class C^1.

→ The assumption in the corollary implies $\dim M \leq 8$

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Main results, II

In the case of a nilpotent algebra the argument improves to C^∞ regularity.

Theorem

Assume that D is generated by two vector fields X_1, X_2 such that

- $\mathrm{Lie}\{X_1, X_2\}$ *is nilpotent of step at most* 4

Then for every SR structure (D, g) on M, length-minimizers are C^∞.

- in particular to Carnot groups of rank 2 and step at most 4 $\to$ we recover [Leonardi-Monti, '08]
- $C^{1,\alpha}$ regularity in some Carnot with step $s > 4$ [Monti, '15]

Theorem (Belotto, Figalli, Parusinski, Rifford, '18)

For every 3D analytic manifold all length minimizers are of class C^1

$\to$ generic case [Zelenko,Zhitomirskii '95], improved [Belotto, Rifford '17]

Outline

Basic idea of the proof

- abnormal length-minimizer $\gamma : [0,T] \to M$ parameterized by arclength: there exists $u \in L^\infty([0,T],\mathbb{S}^1)$ such that

$$\dot\gamma(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)), \quad \text{a.e. } t \in [0,T].$$

- we prove that the control $u = (u_1,u_2)$ admits left/right limits at every $t \in [0,T]$.
- the two limits must coincide

$\rightarrow$ since corners are not minimizing!

- then u is continuous, and the curve γ is C^1

Desingularisation

- In general a sub-Riemannian manifold may admit non-regular points.
- However, for our purposes, we can restrict ourselves with no loss of generality to equiregular manifolds thanks to a desingularisation procedure.

We consider the following property

Statement (S)

Every arclength parameterized abnormal length-minimizer admitting a lift $\lambda(t) \notin (D^4)^\perp$ is of class C^1.

Lemma

Fix an integer $k \geq 2$. Assume that (S) holds for every rank k equiregular sub-Riemannian structure.
Then (S) holds true for every rank k sub-Riemannian structure.

Setting

- (M, D, g) be an equiregular sub-Riemannian manifold of rank $k = 2$.
- local generating frame $\{X_1, X_2\}$ of (D, g).
- abnormal length-minimizer $\gamma : [0, T] \to M$ parameterized by arclength: there exists $u \in L^\infty([0, T], \mathbb{S}^1)$ such that

$$\dot{\gamma}(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)), \quad \text{a.e. } t \in [0, T].$$

- γ admits a lift $\lambda : [0, T] \to T^*M$ which satisfies

$$\dot{\lambda}(t) = u_1\vec{h}_1(\lambda(t)) + u_2\vec{h}_2(\lambda(t)) \quad \text{and} \quad h_1(\lambda(t)) \equiv h_2(\lambda(t)) \equiv 0.$$

Set for every $i_1, \ldots, i_m \in \{1, 2\}$,

$$h_{i_1 \cdots i_m}(t) = \langle \lambda(t), [X_{i_1}, \ldots, [X_{i_m - 1}, X_{i_m}]](\gamma(t)) \rangle.$$

Such a function $h_{i_1 \cdots i_m}$ is absolutely continuous and satisfies

$$\dot{h}_{i_1 \cdots i_m}(t) = u_1(t)h_{1 i_1 \cdots i_m}(t) + u_2(t)h_{2 i_1 \cdots i_m}(t) \quad \text{for a.e. } t \in [0, T].$$

Further necessary conditions

- Differentiating the equalities $h_1 \equiv h_2 \equiv 0$ we obtain

$$0 = u_1 h_{11} + u_2 h_{12}, \qquad 0 = u_1 h_{21} + u_2 h_{22}$$

that implies $h_{12} \equiv 0$ since $u \neq 0$ ($\rightarrow$ no constant curves).

- The identities $h_1(t) = h_2(t) = h_{12}(t) = 0$ imply that $\lambda(t) \in (D^2)^\perp$
- The latter is known as *Goh condition*: a necessary condition for the associated curve to be length-minimizing.

Further conditions can be obtained by differentiating more!

- Differentiating again we get only one new relation

$$0 = \dot{h}_{12} = u_1 h_{112} + u_2 h_{212} \quad \text{a.e. on } [0, T].$$

Further necessary conditions, II

We get only one new relation

$$0 = \dot{h}_{12} = u_1 h_{112} + u_2 h_{212} \quad \text{a.e. on } [0,T].$$

- If the vector $\zeta := (-h_{212}, h_{112})$ is never zero on $[0,T]$ then we can recover $u = (u_1, u_2)$ from this equality ($\rightarrow$ up to a sign)
- Indeed in this case we have

$$u = \sigma \frac{1}{|\zeta|}\zeta, \qquad \sigma = \pm 1 \text{ constant}$$

- u is continuous! the corresponding trajectory is C^1

Problems arise when

- the vector $\zeta = (-h_{212}, h_{112})$ vanishes on $[0,T]$

$\rightarrow$ the lift $\lambda(t)$ enters in $(D^3)^\perp$

Basic observation

Differentiating once more one obtains

$$\dot{\zeta} = Au, \qquad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix}, \qquad \text{on } [0,T].$$

- the matrix $A = A(t)$ is non-autonomous, and is absolutely continuous on the whole interval $[0,T]$.
- notice that A has zero trace (since $h_{1212} = h_{2112}$ from the Jacobi identity)
- $u = \pm\frac{\zeta}{|\zeta|}$ on any interval on which $\zeta \neq 0$

The geometric condition "not enter in $(D^4)^\perp$" is translated into a property of A

Lemma

Assume that $\lambda(t) \notin (D^4_{\gamma(t)})^\perp$ for every $t \in [0,T]$. If $\zeta(t_0) = 0$ for some $t_0 \in [0,T]$, then $A(t_0) \neq 0$.

Main technical result

The key point of the whole argument is the following statement

Theorem

Let (t_0, t_1) be a maximal interval on which $\zeta \neq 0$. Assume that $t_1 < T$ (i.e. $\zeta(t_1) = 0$) and $A(t_1) \neq 0$. Then $u(t)$ has a limit as $t \uparrow t_1$.

Notice that $A(t_1) \neq 0$ and $\mathrm{trace}(A(t_1)) = 0$ hence we have the following possibilities:

1. $\det A(t_1) > 0 \rightarrow$ imaginary eigenvalues, excluded by general arguments
2. $\det A(t_1) < 0 \rightarrow$ two real eigenvalues, there exists a limit
3. $\det A(t_1) = 0 \rightarrow$ nilpotent case, maybe spirals (the difficult case!)

Recall, that $t \mapsto A(t)$ is not constant, the analysis is a priori non trivial

Outline

Case $\det A(t_1) > 0$

Lemma

Let (t_0, t_1) be a maximal open interval of $[0, T]$ on which $\zeta \neq 0$ and assume that $t_1 < T$. Then $\det A(t_1) \leq 0$.

Assume by contradiction that $\det A(t_1) > 0$. Since $\operatorname{tr} A(t_1) = 0$, up to a constant linear change of coordinates we assume

$$A(t_1) = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}.$$

Then we can write

$$A(t) = \begin{pmatrix} -\varepsilon_1(t) & \varepsilon_2(t) - \alpha(t) \\ \varepsilon_2(t) + \alpha(t) & \varepsilon_1(t) \end{pmatrix},$$

- $\varepsilon_1, \varepsilon_2, \alpha$ are absolutely continuous with bounded derivatives on (t_0, t_1),
- $\varepsilon_1(t), \varepsilon_2(t) \to 0$, and $\alpha(t) \to a$ as $t \to t_1$.

- Consider polar coordinates

$$\zeta(t) = \rho(t) e^{i\theta(t)}$$

- the "main equation" can be rewritten

$$\begin{cases} \rho' = (-\varepsilon_1 \cos 2\theta + \varepsilon_2 \sin 2\theta), \\ \theta' = \frac{1}{\rho}(\varepsilon_1 \sin 2\theta + \varepsilon_2 \cos 2\theta + \alpha). \end{cases}$$

- Set $w := \varepsilon_1 \sin 2\theta + \varepsilon_2 \cos 2\theta + \alpha$
- notice that $a/2 < w < 2a$ in a left-neighborhood of t_1. Hence

$$(\rho^2 w)' = \rho^2 w \frac{\varepsilon_1' \sin 2\theta + \varepsilon_2' \cos 2\theta + \alpha'}{w} \geq -M\rho^2 w,$$

- $t \mapsto e^{Mt}\rho^2(t) w(t)$ is increasing
- it is impossible for $\rho^2 w$ to tend to zero as $t \to t_1$

$\implies$ impossible for ρ to tend to zero

Outline

Case when A is non-autonomous

Recall that we want to prove the following result:

Theorem

Let (t_0, t_1) be a maximal interval on which $\zeta \neq 0$. Assume $\zeta(t_1) = 0$ and $A(t_1) \neq 0$. Then $u(t)$ has a limit as $t \uparrow t_1$.

Notice that $A(t_1) \neq 0$ and $\mathrm{trace}(A(t_1)) = 0$ hence we have the following possibility

1. $\det A(t_1) > 0 \rightarrow$ excluded by general arguments
2. $\det A(t_1) < 0 \rightarrow$ "two real eigenvalues", there exists a limit for u
3. $\det A(t_1) = 0 \rightarrow$ nilpotent case, maybe spirals (the difficult case!)

$\rightarrow$ we have to prove that there exists a limit in the last case!

The case $\det A(t_1) = 0$

Assume that $\det A(t_1) = 0$ and recall that $\mathrm{tr} A(t_1) = 0$. Up to a constant linear change of coordinates,

$$A(t_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Taking a suitable change of variables for $t_* \in (t_0, t_1)$,

$$t \mapsto s(t) = \int_{t_*}^{t} \frac{d\tau}{|\zeta(\tau)|}, \qquad s \to +\infty \quad \text{as} \quad t \to t_1,$$

we have

$$\frac{d\zeta}{ds} = A(s)\zeta(s), \qquad A(s) = \begin{pmatrix} -\varepsilon_1(s) & 1 + \varepsilon_2(s) \\ \varepsilon_3(s) & \varepsilon_1(s) \end{pmatrix},$$

where

- $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are absolutely continuous with bounded derivatives on $(0, +\infty)$
- $\varepsilon_1, \varepsilon_2\, \varepsilon_3 \to 0$ as $s \to +\infty$.

The case $\det A(t_1) = 0$, II

Writing in polar coordinates

$$\frac{\rho'}{\rho} = \sin\theta\cos\theta + \varepsilon_\rho, \qquad \theta' = -\sin^2\theta + \varepsilon_\theta,$$

where $\varepsilon_\rho, \varepsilon_\theta \to 0$ as $s \to +\infty$

Notice that the dynamics of θ is a perturbation via ε_θ of

$$\theta' = -\sin^2\theta$$

$\rightarrow$ Two equilibria joined by two clock-wise oriented heteroclinic trajectories.

Lemma

We have the following dichotomy:

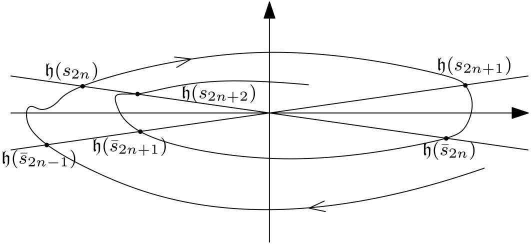
1. $\sin\theta \to 0$ *as* $s \to +\infty$ *(*$\Longrightarrow$ u *has a limit!)*
2. $\theta \to -\infty$ *as* $s \to +\infty$

Moreover, in case 2, for any $0 < \varepsilon < \pi/2$ there exist $(s_n)_{n\in\mathbb{N}}$ and $(\bar{s}_n)_{n\in\mathbb{N}}$ tending to infinity and such that

$$\bar{s}_{2n-1} < s_{2n} < s_{2n+1} < \bar{s}_{2n} < \bar{s}_{2n+1},$$

$$\theta(s_{2n}) = \pi - \varepsilon \mod 2\pi, \qquad \theta(s_{2n+1}) = \varepsilon \mod 2\pi,$$

$$\theta(\bar{s}_{2n}) = -\varepsilon \mod 2\pi, \qquad \theta(\bar{s}_{2n+1}) = \varepsilon - \pi \mod 2\pi,$$



Assume we are in case (2)

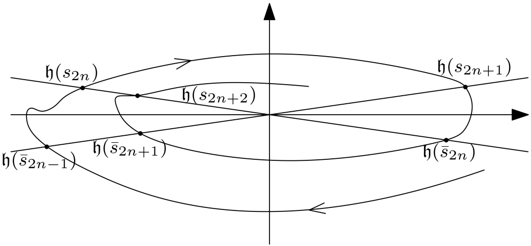
Lemma

For any ε small enough there exists N_ε such that for $n \geq N_\varepsilon$ one has

$$\frac{2}{\varepsilon}\left(1-\varepsilon^2\right) \leq s_{2n+1}-s_{2n} \leq \frac{2}{\varepsilon}\left(1+\varepsilon^2\right),$$

and for $s \in [s_{2n}, s_{2n+1}]$ one has

$$(1-\varepsilon)\varepsilon\rho(s_{2n}) \leq \rho(s)\sin\theta(s) \leq (1+\varepsilon)\varepsilon\rho(s_{2n}),$$



To the Limit!

- Fix a sequence $(\varepsilon^{(k)})_{k\in\mathbb{N}}$, strictly decreasing to 0.
- Denote $(s_n^{(k)})_{n\in\mathbb{N}}$ the sequence $(s_n)_{n\in\mathbb{N}}$ corresponding to $\varepsilon = \varepsilon^{(k)}$.
- with a diagonal procedure we build a sequence of times $(\sigma_\ell)_{\ell\in\mathbb{N}}$ that is strictly increasing and tends to infinity

- Let t_ℓ be σ_ℓ in the original time scale.
- For every $\ell \geq 0$ consider the function $u_\ell \in L^\infty([0,1],\mathbb{S}^1)$ defined by

$$u_\ell = u|_{[t_{2\ell},t_{2\ell+1}]}$$

- By the weak-$\star$ compactness of all bounded subsets of $L^\infty([0,1],\mathbb{R}^2)$, we can assume without loss of generality that $u_\ell \rightharpoonup u_\star$ in the weak-$\star$ topology.
- By a nilpotentization and minimality argument $u_\star$ corresponds to a length-minimizer

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- For every $\ell \geq 0$ consider the function $u_\ell \in L^\infty([0,1],\mathbb{S}^1)$ defined by

$$u_\ell(\tau) = u(t_{2\ell} + \tau(t_{2\ell+1} - t_{2\ell})).$$

- By the weak-$\star$ compactness of all bounded subsets of $L^\infty([0,1],\mathbb{R}^2)$, we can assume without loss of generality that $u_\ell \rightharpoonup u_\star$ in the weak-$\star$ topology.
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Lemma

Under the above assumptions, there exists a unit vector $v_\star \in \mathbb{R}^2$ such that $u_\star(t) = v_\star$ for a.e. $t \in [0,1]$. Moreover, $v_\star$ is parallel to $(1,0)$ (i.e. horizontal).

→ Contradiction with the estimates (and their consequences)

- These spirals are not length-minimizers
- there exists a left/right limit for the control u.

This solves the cases where ζ vanishes at isolated points!

One still has to treat

- Segments where ζ identically vanishes (→ not difficult)
- Accumulation of points where ζ is not zero (→ adaptation)

It gets technical. . .

Conclusions and natural questions

C^1 regularity for curves that do not enter in $(D^4)^\perp$

- Can be extended to deeper singularity in rank 2? e.g. $(D^5)^\perp$
  - higher order analysis (one step more? recursivity?)
  - generic case: $\lambda(t) \notin (D^4)^\perp$ a.e.
- Can be extended to rank > 2?
  - a priori yes but no more available tools from 2D ODEs
- Can one obtain further regularity than C^1?
  - not evident

THANKS FOR YOUR ATTENTION

Conclusions and natural questions

C^1 regularity for curves that do not enter in $(D^4)^\perp$

- Can be extended to deeper singularity in rank 2? e.g. $(D^5)^\perp$
  - higher order analysis (one step more? recursivity?)
  - generic case: $\lambda(t) \notin (D^4)^\perp$ a.e.
- Can be extended to rank > 2?
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THANKS FOR YOUR ATTENTION

The case of Carnot groups of step ≤ 4

- Without loss of generality, we assume that the step is equal to 4.
- Recall that for an abnormal minimizer on an interval I we have

$$h_1 \equiv h_2 \equiv h_{12} \equiv 0, \qquad 0 = \dot{h}_{12} = u_1 h_{112} + u_2 h_{212} \quad \text{a.e. on } I.$$

- The vector $\zeta = (-h_{212}, h_{112})$ satisfies the differential equation

$$\dot{\zeta} = Au, \qquad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix}, \quad \text{a.e. on } I.$$

- A is a constant matrix, with zero trace ($\rightarrow$ nilpotent group)
- We reduced to the case where ζ vanishes at some point $\bar{t} \in I$.

$$\dot{\zeta} = Au \quad \text{a.e.}, \qquad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix} \quad \text{constant}$$

We have the following cases:

(a) $\zeta(t) \equiv 0$ on I;

(b) there exist $\bar{t}, t_* \in I$ such that $\zeta(\bar{t}) = 0$ and $\zeta(t_*) \neq 0$.

Case (a). Easy

- $u(t)$ is in the kernel of A for a.e. t.
- A has one-dimensional kernel $\ker A = \mathrm{span}\{\bar{u}\}$, where $\bar{u}$ has norm one.
- Then $u(t) = \sigma(t)\bar{u}$ for a.e. t, with $\sigma(t) \in \{-1, 1\}$ and

$$\dot{\gamma}(t) = \sigma(t) X_{\bar{u}}(\gamma(t)) \quad \text{a.e.},$$

  with $X_{\bar{u}}$ a constant vector field.
- By length-minimality, σ is constant, and u is constant (hence smooth).

$$\dot{\zeta} = Au \quad \text{a.e.,} \qquad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix} \quad \text{constant}$$

We have the following cases:

(a) $\zeta(t) \equiv 0$ on I;

(b) there exist $\bar{t}, t_* \in I$ such that $\zeta(\bar{t}) = 0$ and $\zeta(t_*) \neq 0$.

Case (b). Less trivial.

- Consider the maximal neighborhood $J = (t_0, t_1)$ of t_* on which ζ is non-vanishing. Assume for instance $\zeta(t_1) = 0$.
- $u = \pm \frac{\zeta}{|\zeta|}$ on J
- The trajectories of ζ are time reparameterizations of $\dot{z} = Az$.
- Hence ζ stays in the stable or in the unstable subspace of A.
- Recall that $\det A \leq 0$. We deduce that $\det A < 0$. ($\rightarrow$ picture)
- No corners $\implies$ u is constant on I, and in particular it is smooth.