Stabilization of hyperbolic systems with a distributed scalar input

Christophe Zhang

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Journée jeunes contrôleurs

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Summary

Classical pole shifting

Consider the finite-dimensional **controllable** control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in \mathcal{M}_n(\mathbb{C}), B \in \mathcal{M}_{n,1}(\mathbb{C}).$$

Kalman condition: $rank\{A^n B \mid n = 0, \cdots, n-1\} = n$.

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Kalman condition: $rank\{A^n B \mid n = 0, \cdots, n-1\} = n$.

Poleshifting: $\forall P, \ \exists K \in \mathcal{M}_{1,n}(\mathbb{C}), \quad \chi(A + BK) = P$.
Idea: Brunovski normal form

Brunovski form for PDEs?

D.L. Russell, *Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems*, JMAA 62, 1978.

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} - A(x)\begin{pmatrix} u \\ v \end{pmatrix} = g(x)u(t) \qquad (1)$$

Canonical form: time-delay system

$$\zeta(t+2) = e^{2\alpha}\zeta(t) + \int_0^2 \overline{p(2-s)}\zeta(t+s)ds + u(t) \qquad (2)$$

Works for *bounded* feedback laws!

$$\sum \left| \frac{\lambda_i - \rho_i}{g_i} \right|^2 < \infty \qquad (3)$$

Summary

Integrator backstepping

$$\dot{x}_1 = ax_1 + x_2$$
$$\dot{x}_2 = u$$

Feedback for the first equation: $x_2 = -(\lambda + a)x_1$.
How do you "backstep" that through the integrator?

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Backstepping change of variable:

$$z_1 = x_1 \qquad \dot{z}_1 = -\lambda z_1 + z_2$$
$$z_2 = x_2 + (\lambda + a)x_1 \qquad \dot{z}_2 = u - \lambda(\lambda + a)z_1 + (\lambda + a)z_2$$

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Backstepping change of variable:

$$z_1 = x_1 \quad \text{lower triangular} \qquad \dot{z_1} = -\lambda z_1 + z_2$$
$$z_2 = x_2 + (\lambda + a)x_1 \qquad \dot{z_2} = u - \lambda(\lambda + a)z_1 + (\lambda + a)z_2$$

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Backstepping change of variable: $u = \lambda(\lambda + a)z_1 - (2\lambda + a)z_2$

$$z_1 = x_1 \quad \text{lower triangular}$$
$$z_2 = x_2 + (\lambda + a)x_1$$

$$\dot{z}_1 = -\lambda z_1 + z_2$$
$$\dot{z}_2 = -\lambda z_2$$

What about controllability?

Another example:

$$\dot{x}_1 = -x_1^2 x_2 \\ \dot{x}_2 = u. \tag{4}$$

Not controllable: 0 stays at 0 for any control u.

But drill with Lyapunov function works! Feedback: $x_2 = x_1$

$$V = x_1^2 + (x_2 - x_1)^2. \tag{5}$$

Summary

A “simple” idea

Boskovic, Balogh, Krstic, *Backstepping in infinite dimension for a class of parabolic distributed parameter systems*, MCSS 2003.
Unstable heat equation:

$$\begin{cases} u_t - u_{xx} = \lambda u, \\ u(0) = 0, \quad u(1) = U(t). \end{cases} \tag{6}$$

Discretization:

$$\begin{aligned} u^0 &= 0 \\ \dot{u}_i &= \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \lambda u_i \\ u_{n+1} &= \alpha_n(u_1, \cdots, u_n). \end{aligned} \tag{7}$$

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ODE backstepping on the discretized system? Careful! Triangular change of variable to an exponentially stable target system:

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Transformation (Volterra):

$$w(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy$$

Control design: $U(t) = \int_0^1 k(1,y)u(t,y)dy.$

Kernel equations

T is a kernel operator: $f \mapsto f - \int_0^x k(x,y)f(y)dy$.

Target equation $\xrightarrow{\text{Formal computations (IBP...)}}$ PDE for $k(x,y)$.

Kernel equations on $\mathcal{T} := \{0 \leq y \leq x \leq 1\}$:

$$\begin{cases} k_{xx} - k_{yy} = \lambda k, \\ k(x,0) = 0, \\ k(x,x) = -\lambda \frac{x}{2} \end{cases} \tag{9}$$

Remarks

- k is regular: formal computations actually valid.

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}$$

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- Good estimates on decay:

$$\begin{aligned} \|y(t)\|_{L^2} &\leq |||(T^\lambda)^{-1}|||\,|||T^\lambda|||e^{-\frac{t}{4}}\|y_0\|_{L^2} \\ &\leq C_1\lambda^2 e^{C_2\sqrt{\lambda}}e^{-\frac{t}{4}}\|y_0\|_{L^2}, \end{aligned} \tag{10}$$

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- **Explicit** feedback law!

Remarks

$$\begin{cases} u_t - u_{xx} = \lambda u, \\ u(0) = 0, \quad u(1) = U(t). \end{cases} \tag{11}$$

Only a finite number of unstable eigenvalues: can be dealt with using a **compact** perturbation:

$$w(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy$$

Infinite number of unstable eigenvalues?
More general transformations for a new pole placement technique?

Summary

Backstepping with controllability

We keep the spirit of a *transformation*: map

$$\dot{x} = Ax + B(Kx + v(t))$$

into the stable system

$$\dot{x} = \tilde{A}x + Bv(t).$$

The mapping T should be invertible and satisfy

$$\begin{aligned} T(A + BK) &= \tilde{A}T, \\ TB &= B. \end{aligned}$$

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“Backstepping equations”

Backtepping with controllability

Proposition

If (A, B) ***and*** $(\tilde{A}, B)$ ***are controllable****, then there exists a unique pair* (T, K) *satisfying conditions* (15)

Proof in Brunovski form.

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$$\tilde{A}T - TA = TBK,$$

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K is a parameter of T.

From finite dimension to PDEs

Suppose A is diagonalizable, with eigenvectors and eigenvalues (e_i, λ_i), $\lambda_i \notin \sigma(\tilde{A}), \forall i$.

$$(\tilde{A}T - TA) \quad = BK \quad , \quad \forall i$$

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$$Te_i = (Ke_i)(\tilde{A} - \lambda_i I)^{-1}B.$$

1. **Basis property**: $f_i := ((\tilde{A} - \lambda_i I)^{-1}B)$ is a basis.
   controllability of $(\tilde{A}, B)$

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$$B^*T^*f_i = B^*f_i \to (Ke_i) = \frac{B^*f_i}{B^*e_i}.$$

   Controllability of (A, B): $B^*e_i \neq 0$.
3. **Invertibility of** T Also controllability of $(\tilde{A}, B)$.

Summary

Linear transport equation

Linear feedbacks:

$$\langle \alpha(t), F \rangle = \sum_{n \in \mathbb{Z}} \overline{F_n} \alpha_n(t) = \int_0^L \bar{F}(s) \alpha(s) ds$$

Closed-loop system:

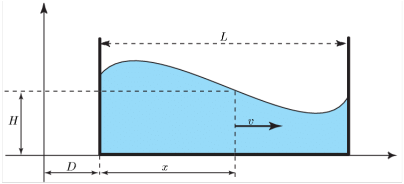
$$\begin{cases} \alpha_t + \alpha_x = \langle \alpha(t), F \rangle \varphi(x), \ x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0. \end{cases}$$

Target system:

$$\begin{cases} z_t + z_x + \lambda z = 0, \quad x \in (0, L), \\ z(t, 0) = z(t, L), \quad t \geq 0. \end{cases}$$

The water tank

$$\begin{cases} H_t + (HV)_x = 0, \\ V_t + \left(gH + \dfrac{V^2}{2}\right)_x = \underbrace{-u(t)}_{\text{acceleration}}, \\ V(t,0) = V(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$



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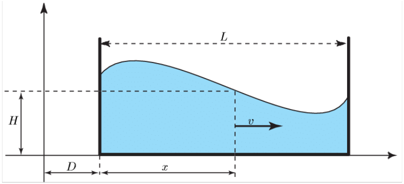

Linearised around $(H^\gamma, V^\gamma) := (H_0 - \gamma x, 0)$ (constant **nonzero** acceleration):

$$\begin{cases} h_t + h^\gamma (V)_x = 0, \\ v_t + g\,(h)_x = -u(t), \\ v(t,0) = v(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$

The water tank

After several variable changes:

$$\begin{cases} \partial_t \zeta + \Lambda \partial_x \zeta + \delta(s) J \zeta = u(t)\mathcal{I}, \\ \quad \zeta_1(t,0) = -\zeta_2(t,0), \quad \zeta_2(t,1) = -\zeta_1(t,1). \end{cases} \tag{12}$$

where $\delta(s) \sim \gamma/(1-\gamma x)$, $J = \begin{pmatrix} 0 & \frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix}$ and $\mathcal{I} = e^{\int_0^x \delta(s) ds} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Target:

$$\begin{cases} \partial_t z + \Lambda \partial_x z + \delta(x) J z = 0, \\ z_1(t,0) = -e^{-2\mu L} z_2(t,0), \\ z_2(t,L) = -z_1(t,L). \end{cases} \tag{13}$$

Exponentially stable (proof with Lyapunov function).

Result

Theorem (Coron, Hayat, Xiang, CZ)

There exists $\gamma_0 > 0$ such that, for all $\gamma \leq \gamma_0$, for all $\mu \leq -C\ln(\gamma)$ there exists a feedback law that stabilizes the linearized water tank around (H^γ, V^γ) exponentially, with decay rate μ.

We build explicit feedback laws.
Important lemma:

Lemma 3.1

The water tank system, for γ small enough, and the target system, if $\mu \leq -C\ln(\gamma)$, are controllable with the controller $\mathcal{I}$.

Trial and error

We look for a **general** T. Not Volterra!

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Water tank:

$$\mathcal{A} := \Lambda\partial_x + \delta(x)J, \quad f_n, \quad \mu_n \ (\sim n). \tag{15}$$

Target system:

$$\tilde{\mathcal{A}} := \Lambda\partial_x + \delta(x)J, \quad \tilde{f}_n, \widetilde{\phi}_n, \quad \tilde{\mu}_n \ (\sim \mu + i\pi n/L). \tag{16}$$

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$$g_n := Tf_n, \quad \begin{cases} \Lambda\partial_x g_n - \mu_n g_n + \delta(x)Jg_n = -\langle F, f_n\rangle\mathcal{I} \\ g_n^1(0) + e^{-2\mu L}g_n^2(0) = 0, \\ g_n^1(L) + g_n^2(L) = 0. \end{cases} \tag{17}$$

basis property $\Leftrightarrow$ we can build an invertible T.

Invertibility and feedback

Project on $\widetilde{\phi}_n$:

$$g_n = -\langle F, f_n \rangle \sum_{p \in \mathbb{Z}} \frac{\left\langle \mathcal{I}, \widetilde{\phi}_p \right\rangle}{\widetilde{\mu}_p - \mu_n} \tilde{f}_p \tag{18}$$

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Riesz basis of L^2 : $f_n = \left(1 - e^{-2\mu L}\right) f_n^1(0) \sum_{p \in \mathbb{Z}} \frac{\overline{\widetilde{\phi_p^1}(0)}}{\widetilde{\mu}_p - \mu_n} \widetilde{f}_p$

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Riesz basis of $L^2 \rightarrow D(\widetilde{\mathcal{A}})$: $\sum_{p \in \mathbb{Z}} \frac{\left\langle \mathcal{I}, \widetilde{\phi}_p \right\rangle}{\widetilde{\mu}_p - \mu_n} \widetilde{f}_p$

Controllability of **target system**

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Controllability of **target system**

$$T\alpha = -\sum_{n\in\mathbb{Z}} \alpha_n \langle F, f_n\rangle \sum_{p\in\mathbb{Z}} \frac{\left\langle \mathcal{I}, \widetilde{\phi}_p \right\rangle}{\widetilde{\mu}_p - \mu_n} \tilde{f}_p, \quad \alpha \in D(\mathcal{A}) \tag{19}$$

Invertible iff $|\langle F, f_n\rangle| \sim n$ $(\mu_n \alpha_n \in \ell^2)$.

Verification

Let F be such that $\langle F, f_n \rangle \sim n$.
Now we test our isomorphism: $T : D(\mathcal{A}) \to D(\tilde{A})$

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$$T(-\mathcal{A}\alpha + \langle F, \alpha \rangle \mathcal{I}) = -\widetilde{\mathcal{A}} T\alpha \quad , \quad \alpha \in D(\mathcal{A} + \langle F, \cdot \rangle \mathcal{I}). \tag{20}$$

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Truncations:

$$T(-\mathcal{A}\alpha^{(N)}+\langle F, \alpha\rangle\mathcal{I}^{(N)}) = \overbrace{-\widetilde{A}T\alpha^{(N)} - \langle F, \alpha^{(N)}\rangle\mathcal{I}}+\langle F, \alpha\rangle T\mathcal{I}^{(N)}. \tag{21}$$

The return of TB=B

We want

$$\langle -\langle F, \alpha^{(N)}\rangle \mathcal{I} + \langle F, \alpha\rangle T\mathcal{I}^{(N)}, \widetilde{\phi}_m\rangle \xrightarrow[N\to\infty]{} 0 \qquad (22)$$

The return of TB=B

We want

$$\langle (\langle F, \alpha \rangle - \langle F, \alpha^{(N)} \rangle) \mathcal{I} + \langle F, \alpha \rangle (T\mathcal{I}^{(N)} - \mathcal{I}), \widetilde{\phi}_m \rangle \xrightarrow[N \to \infty]{} 0 \quad (22)$$

The return of TB=B

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$$\langle (\langle F,\alpha\rangle - \langle F,\alpha^{(N)}\rangle)\mathcal{I} + \langle F,\alpha\rangle (T\mathcal{I}^{(N)} - \mathcal{I}), \widetilde{\phi}_m\rangle \xrightarrow[N\to\infty]{} 0 \quad (22)$$

Hence feedback F is determined by

$$\begin{aligned} \langle \alpha^{(N)}, F\rangle &\xrightarrow[N\to\infty]{} \langle \alpha, F\rangle, \quad \forall \alpha \in D_F, \\ \langle T\mathcal{I}^{(N)}, \widetilde{\phi}_m\rangle &\xrightarrow[N\to\infty]{} \langle \mathcal{I}, \widetilde{\phi}_m\rangle, \quad \forall m \in \mathbb{Z}. \end{aligned} \quad (23)$$

$$TB = B$$

Determining the feedback law

$$-\sum_{n\in\mathbb{Z}} \alpha_n \langle F, f_n\rangle f_n \tag{24}$$

Essentially, $T = - \quad \alpha \star F$

Determining the feedback law

$$-\sum_{n\in\mathbb{Z}} \alpha_n \langle F, f_n \rangle \left(1 - e^{-2\mu L}\right) f_n^1(0) \sum_{p\in\mathbb{Z}} \frac{\overline{\widetilde{\phi_p^1}(0)}}{\widetilde{\mu}_p - \mu_n} \widetilde{f}_p \qquad (24)$$

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$$-\sum_{n\in\mathbb{Z}} \alpha_n \langle F, f_n\rangle \sum_{p\in\mathbb{Z}} \frac{\overline{\widetilde{\phi_p^1}(0)}}{\widetilde{\mu}_p - \mu_n} \widetilde{f}_p \tag{24}$$

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Essentially, $T = -\, D_{\widetilde{\mathcal{A}}}^{-1} D_{\mathcal{A}}^{-1} (\alpha \star F)$

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$$T\alpha = -\sum_{n\in\mathbb{Z}} \alpha_n \langle F, f_n\rangle \sum_{p\in\mathbb{Z}} \frac{\left\langle \mathcal{I}, \widetilde{\phi}_p \right\rangle}{\widetilde{\mu}_p - \mu_n} \widetilde{f}_p \tag{24}$$

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$$= -\frac{\langle \mathcal{I}, \widetilde{\phi}_m\rangle}{\overline{\widetilde{\phi}_m^1(0)}(1 - e^{-2\mu L})} \underbrace{\sum_{n=-N}^{N} \langle \mathcal{I}, f_n\rangle \frac{\langle F, f_n\rangle}{f_n^1(0)} \left\langle f_n, \widetilde{\phi}_m\right\rangle}$$

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Almost done...

$$T(-\mathcal{A}+\mathcal{I}F) = -\widetilde{\mathcal{A}}T \tag{25}$$

$$\begin{aligned} \langle \alpha^{(N)}, F\rangle &\xrightarrow[N\to\infty]{} \langle \alpha, F\rangle, \quad \forall \alpha \in D_F, \\ \langle T\mathcal{I}^{(N)}, \widetilde{\phi}_m\rangle &\xrightarrow[N\to\infty]{} \langle \mathcal{I}, \widetilde{\phi}_m\rangle, \quad \forall m \in \mathbb{Z}. \end{aligned} \tag{26}$$

First limit: fine asymptotic study of $\langle F, f_n\rangle$.

Last step: well-posedness of closed-loop system with feedback K (Lumer-Phillips).

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- Next: nonlinear system? Finite time?

Thank you for your attention!