Boundary controls as the limit of internal controls: the parabolic case

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Joint work with M. C. Santos and J.-P. Puel

Internal controllability problem

For every $y_0 \in L^2(\Omega)$, any $T > 0$ and any non-empty subset $\omega \subset \Omega$, there exists

$$(y, f) \in C([0,T]; L^2(\Omega)) \times L^2(\omega \times (0,T))$$

which solves the distributed null controllability problem:

$$\left| \begin{array}{lll} y_t - \Delta y = f 1_\omega & \text{in} & \Omega \times (0,T), \\ y = 0 & \text{on} & \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), \; y(x,T) = 0 & \text{in} & \Omega. \end{array} \right. \tag{D-C}$$

Boundary controllability problem

For every $y_0 \in L^2(\Omega)$, any $T > 0$ and any non-empty subset $\Gamma_0 \subset \partial\Omega$, there is also

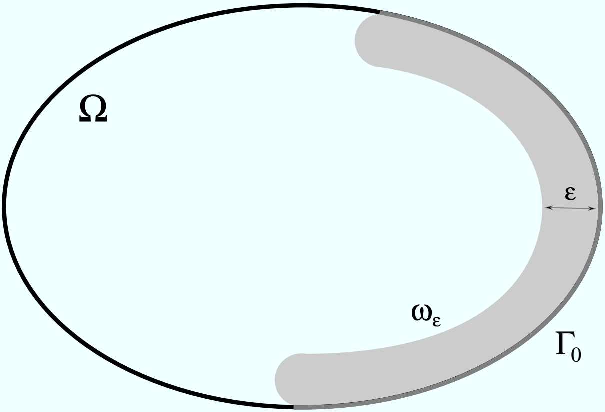
$$(y, g) \in C([0, T]; H^{-1}(\Omega)) \times L^2(\Gamma_0 \times (0, T)),$$

which solves the following boundary null controllability problem:

$$\left| \begin{array}{lll} y_t - \Delta y = 0 & \text{in} & \Omega \times (0, T), \\ y = g 1_{\Gamma_0} & \text{on} & \partial\Omega \times (0, T), \\ y(x, 0) = y_0(x), \ y(x, T) = 0 & \text{in} & \Omega. \end{array} \right. \tag{B-C}$$

Question

Let $\epsilon > 0$ and let ω_ϵ be an ϵ-neighborhood of Γ_0 which shrinks to Γ_0 as $\epsilon \to 0^+$. Can we find a sequence (y_ϵ, f_ϵ), with *supp* $f_\epsilon \subset \omega_\epsilon$, such that the distributed null control problem (D-C) converges, in some sense, to the boundary null control problem (B-C) as $\epsilon \to 0^+$?



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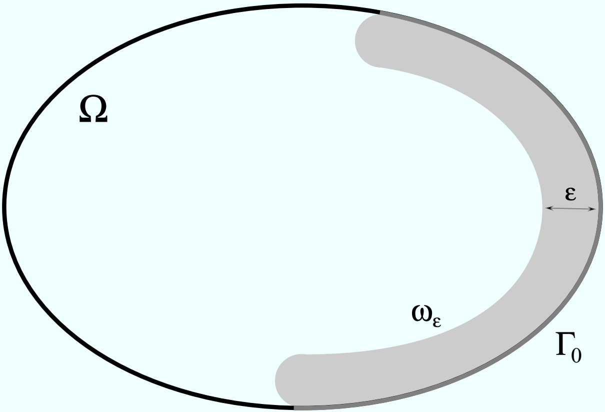

In other words...

Is it true, in some sense, that

$$\left| \begin{array}{ll} y_t - \Delta y = f 1_{\omega_\epsilon} & \Omega \times (0, T), \\ y = 0 & \partial\Omega \times (0, T), \\ y(0) = y_0,\ y(T) = 0 & \Omega \end{array} \right.$$

$$\Downarrow_{\epsilon \to 0^+}$$

$$\left| \begin{array}{ll} \widehat{y}_t - \Delta \widehat{y} = 0 & \Omega \times (0, T), \\ \widehat{y} = g 1_{\Gamma_0} & \partial\Omega \times (0, T), \\ \widehat{y}(0) = y_0,\ \widehat{y}(T) = 0 & \Omega, \end{array} \right.$$

where ω_ϵ shrinks to Γ_0?

Warning

The limiting problem has different boundary conditions!!!!!

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What is known?

Nothing for the heat equation...

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The hyperbolic case

For the wave equation, *Caroline Fabre* in 1992 gave a positive answer to the previous question (under GCC!).

The key point is Caroline's proof is the following observability inequality

Observability inequality

There exists $C > 0$, independent of ϵ, such that

$$\|\phi_0\|^2_{H^1_0(\Omega)} + \|\phi_1\|^2_{L^2(\Omega)} \leq C\epsilon^{-3}\int_0^T \int_{\omega_\epsilon} |\varphi|^2 \, dxdt,$$

for every solution of

$$\left| \begin{array}{lll} \varphi_{tt} - \Delta\varphi = 0 & \text{in} & \Omega \times (0, T), \\ \varphi = 0 & \text{on} & \partial\Omega \times (0, T), \\ \varphi(x, 0) = \phi_0(x), \ \varphi_t(x, 0) = \phi_1(x) & \text{in} & \Omega. \end{array} \right. \quad (1)$$

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We can also cite the following two papers:

- S. W. Hansen and M. Tucsnak, *Some new applications of Russells principle to infinite dimensional vibrating systems*, Annual Reviews in Control, 2017.

- R. Joly, *Convergence of the wave equation damped on the interior to the one damped on the boundary*, JDE, 2006.

A similar result for the heat equation in 1d

C. Letrouit, *From internal to pointwise control for the 1D heat equation and minimal control time*, Systems and Control Letters, 2019.

Given $T > 0$ and $x_0 \in (0,1)$:

$$\left| \begin{array}{ll} y_t - y_{xx} = f 1_{(x_0-\epsilon, x_0+\epsilon)} & (0,1)\times(0,T), \\ y(0,t) = y(1,t) = 0 & (0,1)\times(0,T), \\ y(0,x) = y_0(x),\ y(T) = 0 & (0,1) \end{array} \right.$$

$$\Downarrow_{\epsilon\to 0^+} \quad ???$$

$$\left| \begin{array}{ll} y_t - y_{xx} = f(t)\delta x_0 & (0,1)\times(0,T), \\ y(0,t) = y(1,t) = 0 & (0,T), \\ y(0,x) = y_0(x),\ y(T) = 0 & (0,1). \end{array} \right.$$

Remark

Depending on how well x_0 is approximated by rational numbers, the heat equation may or may not be observable at x_0 in time T.

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Depending on how well x_0 is approximated by rational numbers, the heat equation may or may not be observable at x_0 in time T.

Our main result

Theorem (C.-S., Santos, Puel)

Let $y_0 \in L^2(\Omega)$, $T > 0$ and, for $\epsilon > 0$, let ω_ϵ be a non-empty open neighborhood of Γ_0, which converges to Γ_0 as $\epsilon \to 0^+$.
There exists a sequence $(y_\epsilon, f_\epsilon) \in C([0,T]; L^2(\Omega)) \times L^2(\omega_\epsilon \times (0,T))$ such that the problem

$$\textbf{(DC)} \quad \left| \begin{array}{lll} y_{\epsilon\, t} - \Delta y_\epsilon = f_\epsilon 1_{\omega_\epsilon} & in & \Omega \times (0,T), \\ y_\epsilon = 0 & on & \partial\Omega \times (0,T), \\ y_\epsilon(x,0) = y_0(x),\ y_\epsilon(x,T) = 0 & in & \Omega, \end{array} \right.$$

"converges" to the boundary problem

$$\textbf{(BC)} \quad \left| \begin{array}{lll} y_t - \Delta y = 0 & in & \Omega \times (0,T), \\ y = g 1_{\Gamma_0} & on & \partial\Omega \times (0,T), \\ y(x,0) = y_0(x),\ y(x,T) = 0 & in & \Omega, \end{array} \right.$$

where $(y,g) \in C([0,T]; H^{-1}(\Omega)) \times L^2(\Gamma_0 \times (0,T))$.

Let us give some ideas on the proof.

The meaning of the convergence

If z is a solution of

$$\begin{cases} -z_t + \Delta z = h & \text{in} \quad \Omega \times (0, T), \\ z = 0 & \text{in} \quad \partial\Omega \times (0, T), \\ z(x, T) = z^T & \text{in} \quad \Omega, \end{cases} \tag{2}$$

then, a weak solution of (D-C) must satisfy

$$\iint_Q y_\epsilon h \, dxdt - \int_\Omega y_0 z(0) \, dx = \iint_{\omega_\epsilon \times (0,T)} f_\epsilon z \, dxdt. \tag{3}$$

We say that (D-C) converges to (B-C), if

$$\iint_Q y_\epsilon h \, dxdt \to \iint_Q yh \, dxdt \quad \text{and} \quad \iint_{\omega_\epsilon \times (0,T)} f_\epsilon z \, dxdt \to \int_0^T \int_{\Gamma_0} g \frac{\partial z}{\partial \nu} \, dxdt.$$

In fact, from (3), we get

$$\iint_Q yh \, dxdt - \int_\Omega y_0 z(0) \, dx = \int_0^T \int_{\Gamma_0} g \frac{\partial z}{\partial \nu} \, dxdt, \tag{4}$$

which is exactly the weak formulation for the (**BC**) problem.

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Main steps of the proof

- A sharp Carleman inequality for the adjoint system: optimal cost with respect to ϵ.
- Construction, in the sense of Fursikov-Imanuvilov, of an optimal pair (y_ϵ, f_ϵ) state-control.
- Obtainment of good estimates for (y_ϵ, f_ϵ) in the correct spaces.
- Proof of the convergence of one problem to the other one.

Step 1: Optimal Carleman inequality

Let

$$\left| \begin{array}{lll} \varphi_t - \Delta\varphi = F & \text{in} & \Omega \times (0,T), \\ \varphi = 0 & \text{on} & \partial\Omega \times (0,T), \\ \varphi(x,0) = \phi_0(x), & \text{in} & \Omega. \end{array} \right.$$

It is well-known that both problems (B-C) and (D-C) are solvable iff

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C \left(\iint_{\Gamma_0\times(0,T)} |\frac{\partial\varphi}{\partial\nu}(x,t)|^2 \, d\sigma dt + \iint_Q |F|^2 \, dxdt \right) \qquad (5)$$

and

$$\|\varphi(0)\|^2_{L^2(\Omega)} \le C \left(C(\epsilon) \iint_{\omega_\epsilon\times(0,T)} |\varphi|^2 \, dxdt + \iint_Q |F|^2 \, dxdt \right), \qquad (6)$$

respectively.

Remark

As in the hyperbolic case, we must know how $C(\epsilon)$ depends on ϵ.
This is the key point of the whole proof!

The observability constant

Lemma

The observability constant for the internal observability inequality has the form

$$C(\epsilon) = C\epsilon^{-3},$$

where C does not depend on ϵ.

Carleman estimates

On what follows, we consider a weight function $\psi \in C^2(\overline{\Omega})$ verifying

$$|\nabla \psi(x)| \neq 0, \ \forall x \in \overline{\Omega},$$

$$\frac{\partial \psi}{\partial \nu}(x) \leq 0, \quad \forall x \in \partial\Omega \setminus \Gamma_0$$

and for a positive real number λ:

$$\phi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}; \ \alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda||\psi||_\infty}}{t(T-t)}$$

Carleman estimate

$$I(s;\varphi) := s^3 \iint_Q e^{2s\alpha}\phi^3|\varphi|^2 dxdt + s\iint_Q e^{2s\alpha}\phi|\nabla\varphi|^2 dxdt$$
$$+ s^{-1}\iint_Q e^{2s\alpha}\phi^{-1}(|\varphi_t|^2 + |\Delta\varphi|^2)dxdt. \quad (7)$$

Theorem (Optimal Carleman inequality)

There exist positive constants C and λ_0 such that, for every $\lambda \geq \lambda_0$, there exists $s_0 > 0$ such that, for any $s \geq s_0$, the following estimate holds

$$I(s,\varphi) \leq C\left(\iint_Q e^{2s\alpha}|F|^2\,dxdt + \epsilon^{-3}s^6\iint_{\omega_\epsilon\times(0,T)} e^{2s\alpha}\phi^6|\varphi|^2\,dxdt\right),$$

for every φ solution of

$$\left|\begin{array}{lll} \varphi_t - \Delta\varphi = F & in & \Omega\times(0,T), \\ \varphi = 0 & on & \partial\Omega\times(0,T), \\ \varphi(x,0) = \varphi^T(x) & in & \Omega. \end{array}\right.$$

Remark 1

If we try to find $C(\epsilon)$ by only following the original proof of Fursikov-Imanouvilov, we find that

$$C(\epsilon) = O(\epsilon^{-4})$$

Remark 2

The behavior $C(\epsilon) = O(\epsilon^{-3})$ is optimal. Indeed, taking $\varphi(x,t) = e^{-\pi^2 t}\sin(\pi x)$ we have

$$\int_0^1 |\sin(\pi x)|^2 \, dx = \frac{1}{2}$$

and

$$\int_0^T \int_0^\epsilon e^{-2\pi^2 t} |\sin(\pi x)|^2 \, dx dt = O(\epsilon^3).$$

Sketch of the proof

We start the proof with a boundary Carleman inequality.

Theorem

There exist positive constants C and λ_0 such that, for every $\lambda \geq \lambda_0$, there exists $s_0 > 0$ such that, for any $s \geq s_0$, the following estimate holds

$$I(s,\varphi) \leq C \left(\iint_Q e^{2s\alpha} |F|^2 \, dxdt + s \iint_{\Gamma_0 \times (0,T)} e^{2s\alpha} \phi \left| \frac{\partial \varphi}{\partial \nu} \right|^2 dt \right),$$

for every φ solution of

$$\left| \begin{array}{lll} \varphi_t - \Delta \varphi = F & in & \Omega \times (0,T), \\ \varphi = 0 & on & \partial\Omega \times (0,T), \\ \varphi(x,0) = \varphi^T(x) & in & \Omega. \end{array} \right.$$

Obviously, the constant C does not depend on ϵ.

Sketch of the proof

The idea now is to bound the boundary term in terms of a local term, using a suitable cutoff function. In fact, we show that

Estimate for the boundary term

$$s \iint_{\Gamma_0 \times (0,T)} e^{2s\alpha} \phi \left| \frac{\partial \varphi}{\partial \nu} \right|^2 dt \leq C\epsilon^{-3} s^5 \iint_{\omega_\epsilon \times (0,T)} e^{2s\alpha} \phi^5 |\varphi|^2 \, dxdt + \delta I(s;\varphi),$$

for any $\delta > 0$.

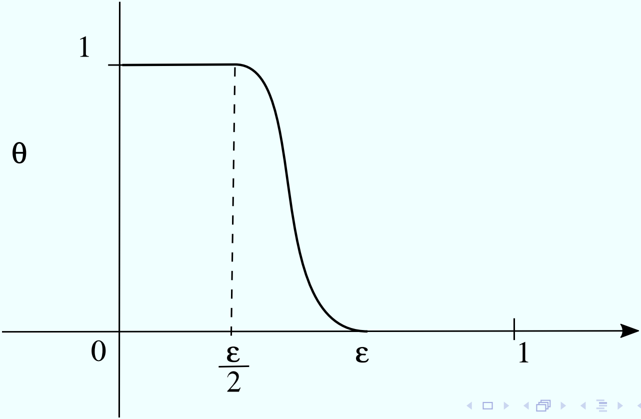
To prove this, we will assume that we are in 1D.

Sketch of the proof

We begin choosing a cut-off function

cut-off function

$\theta \in C^3(0,1)$ such that $\theta = 1$ in $(0, \frac{\epsilon}{2})$ and $\theta(\epsilon) = 0$. The function θ has the property $\theta_x = O(\epsilon^{-1})$.



Sketch of the proof:

Multiply the equation of φ by $\theta_x e^{2s\alpha}\phi^3\varphi$, integrating by parts several times, and performing a lot of estimates, we obtain

$$s^3 \int_0^T \int_0^\epsilon e^{2s\alpha}\phi^3 |\varphi_x(x,t)|^2 \theta_x \, dxdt \\ \leq C\left(\epsilon^{-3} s^6 \int_0^T \int_0^\epsilon e^{2s\alpha}\phi^6 |\varphi|^2 \, dxdt + \iint_Q e^{2s\alpha}|F|^2 \, dxdt\right). \quad (8)$$

and the proof is done.

Remark

The proof in the 1d case is simpler.
In the multi-dimensional case, one must work with a normal coordinate system and perform a lot of boring calculations....

Step 2-3: Fursikov-Imanuvilov strategy and bounds

For each $\epsilon > 0$, we construct a pair $(\widehat{y}_\epsilon, \widehat{f}_\epsilon)$ solution of (D-C).

Let $P_0 := \{w \in C^2(\overline{Q}), \quad w = 0 \text{ in } \partial\Omega \times (0,T)\}$ and the symmetric, positive definite bilinear form:

$$a_\epsilon(w_1, w_2) := \iint_Q \rho_1^2(t)\mathcal{L}^* w_1 \mathcal{L}^* w_2 dxdt + \frac{1}{\epsilon^3}\iint_{\omega_\epsilon \times (0,T)} \rho_2^2(t) w_1 w_2 dxdt,$$

where $\mathcal{L}^* := \partial_t + \Delta$, and ρ_i is an appropriate weight.

Let $P = P(\epsilon)$ the completion of P_0 with respect to the norm associated to $a_\epsilon(\cdot,\cdot)$.

By Lax-Milgram theorem, there exists a unique $\widehat{\varphi}_\epsilon \in P$ such that

$$a_\epsilon(\widehat{\varphi}_\epsilon, \varphi) = \int_\Omega y_0 \varphi(0) dx., \quad \forall \varphi \in P.$$

Next, we show that we can take $(\widehat{y}_\epsilon, \widehat{f}_\epsilon) = (\rho_1 \mathcal{L}^* \widehat{\varphi}_\epsilon, -\frac{\rho_2}{\epsilon^3} \widehat{\varphi}_\epsilon)$ as a solution to the problem (D-C).

Moreover

$$\|\rho_1 \widehat{f}_\epsilon\|^2_{L^2(\omega_\epsilon \times (0,T))} \leq C\epsilon^{-3} \|y_0\|^2_{L^2(\Omega)}, \tag{9}$$

and

$$\|\rho_2 \widehat{y}_\epsilon\|^2_{L^2(Q)} \leq C \|y_0\|^2_{L^2(\Omega)}, \tag{10}$$

where C does not depend on ϵ.

We prove

Proposition

We have that $\rho_3 \widehat{\varphi}_\epsilon \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ with

$$\|\rho_3 \widehat{\varphi}_\epsilon\|^2_{L^2((0,T);H^2(\Omega)\cap H_0^1(\Omega))} \leq \|\widehat{\varphi}_\epsilon\|^2_P \leq C\|y_0\|^2_2,$$

where C does not depend on ϵ and $\rho_3 = \rho(t)$.
Moreover, there exists a function φ such that $\rho_3 \widehat{\varphi}_\epsilon \rightharpoonup \rho_3 \varphi$ weakly in $L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ and $\rho_3 \widehat{\gamma}^{-1} \frac{\partial \varphi}{\partial \nu} \in L^2(\Sigma_0)$.

This last result allow us to prove the following:

Lemma

Let

$$L_\epsilon : L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \to \mathbb{R}$$

$$v \to \frac{1}{\epsilon^3} \iint_{\omega_{\epsilon \times (0,T)}} \rho_3(t) \widehat{\varphi}_\epsilon(x,t) v(x,t)\, dxdt.$$

Then, L_ϵ are bounded in $L^2(0,T; (H^2(\Omega) \cap H_0^1(\Omega))')$ and converge (up to a subsequence) for the weak topology of this space to

$$L : L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) \to \mathbb{R}$$

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Step 4: Convergence of one problem to the other

Homework for the weekend!!!

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Ongoing works

- Systems of equations

$$\begin{cases} y^1_t - \Delta y^1 = a y^2 & Q, \\ y^2_t - \Delta y^2 = f_\epsilon 1_{\omega_\epsilon} & Q, \\ y^1 = y^2 = 0 & \Sigma, \\ y^1(0) = y^1_0,\ y^2(0) = y^2_0 & \Omega. \end{cases} \Rightarrow \begin{cases} y^1_t - \Delta y^1 = a y^2 & Q, \\ y^2_t - \Delta y^2 = 0 & Q, \\ y^1 = 0,\ y^2 = g 1_{\Gamma_0} & \Sigma, \\ y^1(0) = y^1_0,\ y^2(0) = y^2_0 & \Omega. \end{cases}$$

Minimal time of controllability?

Ongoing works

- Stokes equation

There exists $(y^\epsilon, f_\epsilon 1_{\omega_\epsilon})$ and $(y, g1_{\Gamma_0})$

$$\begin{cases} y_t^\epsilon - \Delta y^\epsilon + \nabla \pi = f_\epsilon 1_{\omega_\epsilon} & Q, \\ y^\epsilon = 0 & \Sigma, \\ y^\epsilon(0) = y_0,\ y^\epsilon(T) = 0 & \Omega. \end{cases} \quad \Rightarrow \quad \begin{cases} y_t - \Delta y + \nabla \tilde{\pi} = 0 & Q, \\ y = g1_{\Gamma_0} & \Sigma, \\ y(0) = y_0, y(T) = 0 & \Omega. \end{cases}$$

where ω_ϵ shrinks to Γ_0?

Is it possible to eliminate controls?

Other type of equations

- KdV? Critical length?
- Kuramoto Sivashinsky? Dirichelet control and Neumann boundary controls?

Merci!!