Boundary controls as the limit of internal controls: the parabolic case

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Joint work with M. C. Santos and J.-P. Puel
Internal controllability problem

For every $y_0 \in L^2(\Omega)$, any $T > 0$ and any non-empty subset $\omega \subset \Omega$, there exists

$$(y, f) \in C([0, T]; L^2(\Omega)) \times L^2(\omega \times (0, T))$$

which solves the distributed null controllability problem:

$$\begin{align*}
\frac{\partial y}{\partial t} - \Delta y &= f 1_\omega & \text{in } \Omega \times (0, T), \\
y &= 0 & \text{on } \partial \Omega \times (0, T), \\
y(x, 0) &= y_0(x), \quad y(x, T) = 0 & \text{in } \Omega.
\end{align*}$$

(D-C)

Boundary controls as the limit of internal controls
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Boundary controllability problem

For every $y_0 \in L^2(\Omega)$, any $T > 0$ and any non-empty subset $\Gamma_0 \subset \partial \Omega$, there is also

$$(y, g) \in C([0, T]; H^{-1}(\Omega)) \times L^2(\Gamma_0 \times (0, T)),$$

which solves the following boundary null controllability problem:

$$\begin{align*}
  y_t - \Delta y &= 0 & \text{in} & \Omega \times (0, T), \\
  y &= g1_{\Gamma_0} & \text{on} & \partial \Omega \times (0, T), \\
  y(x, 0) &= y_0(x), y(x, T) &= 0 & \text{in} & \Omega.
\end{align*}$$

(B-C)
Let $\epsilon > 0$ and let $\omega_\epsilon$ be an $\epsilon$-neighborhood of $\Gamma_0$ which shrinks to $\Gamma_0$ as $\epsilon \to 0^+$. Can we find a sequence $(y_\epsilon, f_\epsilon)$, with $\text{supp } f_\epsilon \subset \omega_\epsilon$, such that the distributed null control problem (D-C) converges, in some sense, to the boundary null control problem (B-C) as $\epsilon \to 0^+$?
Question

Let $\epsilon > 0$ and let $\omega_{\epsilon}$ be an $\epsilon$-neighborhood of $\Gamma_0$ which shrinks to $\Gamma_0$ as $\epsilon \to 0^+$. Can we find a sequence $(y_{\epsilon}, f_{\epsilon})$, with $\text{supp } f_{\epsilon} \subset \omega_{\epsilon}$, such that the distributed null control problem (D-C) converges, in some sense, to the boundary null control problem (B-C) as $\epsilon \to 0^+$?
In other words...

Is it true, in some sense, that

\[ y_t - \Delta y = f 1_{\omega_\epsilon} \quad \text{\(\Omega \times (0, T),\)} \]
\[ y = 0 \quad \text{\(\partial \Omega \times (0, T),\)} \]
\[ y(0) = y_0, \ y(T) = 0 \quad \Omega \]

\[ \Downarrow_{\epsilon \to 0^+} \]

\[ \hat{y}_t - \Delta \hat{y} = 0 \quad \text{\(\Omega \times (0, T),\)} \]
\[ \hat{y} = g 1_{\Gamma_0} \quad \text{\(\partial \Omega \times (0, T),\)} \]
\[ \hat{y}(0) = y_0, \ \hat{y}(T) = 0 \quad \Omega, \]

where \(\omega_\epsilon\) shrinks to \(\Gamma_0\)?

**Warning**

*The limiting problem has different boundary conditions!!!!!*
In other words...

Is it true, in some sense, that

\[ y_t - \Delta y = f1_{\omega_\epsilon} \quad \Omega \times (0, T), \]
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\[ \downarrow_{\epsilon \to 0^+} \]

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where \( \omega_\epsilon \) shrinks to \( \Gamma_0 \)?

**Warning**

The limiting problem has different boundary conditions!!!!!
What is known?

Nothing for the heat equation...
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Nothing for the heat equation...
The hyperbolic case

For the wave equation, Caroline Fabre in 1992 gave a positive answer to the previous question (under GCC!).

The key point is Caroline’s proof is the following observability inequality

**Observability inequality**

There exists \( C > 0 \), independent of \( \epsilon \), such that

\[
\| \phi_0 \|_{H^1_0(\Omega)}^2 + \| \phi_1 \|_{L^2(\Omega)}^2 \leq C \epsilon^{-3} \int_0^T \int_{\omega_\epsilon} |\varphi|^2 \, dx \, dt,
\]

for every solution of

\[
\begin{align*}
\varphi_{tt} - \Delta \varphi &= 0 & \text{in} & \Omega \times (0, T), \\
\varphi &= 0 & \text{on} & \partial \Omega \times (0, T), \\
\varphi(x, 0) &= \phi_0(x), \quad \varphi_t(x, 0) = \phi_1(x) & \text{in} & \Omega.
\end{align*}
\]
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$$\|\phi_0\|_{H^1_0(\Omega)}^2 + \|\phi_1\|_{L^2(\Omega)}^2 \leq C\epsilon^{-3} \int_0^T \int_{\omega_{\epsilon}} |\varphi|^2 \, dx \, dt,$$

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\varphi(x, 0) &= \phi_0(x), \quad \varphi_t(x, 0) = \phi_1(x) & \text{in } \Omega.
\end{aligned}$$

(1)
We can also cite the following two papers:


A similar result for the heat equation in 1d

C. Letrouit, *From internal to pointwise control for the 1D heat equation and minimal control time*, Systems and Control Letters, 2019.

Given $T > 0$ and $x_0 \in (0, 1)$:

\[
\begin{align*}
    y_t - y_{xx} &= f_1(x_0 - \epsilon, x_0 + \epsilon) & (0, 1) \times (0, T), \\
    y(0, t) &= y(1, t) = 0 & (0, 1) \times (0, T), \\
    y(0, x) &= y_0(x), y(T) = 0 & (0, 1) \\
\end{align*}
\]

$\downarrow_{\epsilon \to 0^+}$

\[
\begin{align*}
    y_t - y_{xx} &= f(t) \delta x_0 & (0, 1) \times (0, T), \\
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    y(0, x) &= y_0(x), y(T) = 0 & (0, 1). \\
\end{align*}
\]

**Remark**

Depending on how well $x_0$ is approximated by rational numbers, the heat equation may or may not be observable at $x_0$ in time $T$. 
A similar result for the heat equation in 1d

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Given $T > 0$ and $x_0 \in (0, 1)$:

$$
\begin{align*}
\dot{y}_t - y_{xx} &= f 1_{(x_0 - \epsilon, x_0 + \epsilon)}(0, 1) \times (0, T), \\
y(0, t) &= y(1, t) = 0 (0, 1) \times (0, T), \\
y(0, x) &= y_0(x), y(T) = 0 (0, 1)
\end{align*}
$$

\[\downarrow_{\epsilon \to 0^+} \text{???}\]

$$
\begin{align*}
\dot{y}_t - y_{xx} &= f(t) \delta x_0 (0, 1) \times (0, T), \\
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**Remark**

Depending on how well $x_0$ is approximated by rational numbers, the heat equation may or may not be observable at $x_0$ in time $T$. 
Our main result

Theorem (C.-S., Santos, Puel)

Let $y_0 \in L^2(\Omega)$, $T > 0$ and, for $\epsilon > 0$, let $\omega_\epsilon$ be a non-empty open neighborhood of $\Gamma_0$, which converges to $\Gamma_0$ as $\epsilon \to 0^+$.

There exists a sequence $(y_\epsilon, f_\epsilon) \in C([0, T]; L^2(\Omega)) \times L^2(\omega_\epsilon \times (0, T))$ such that the problem

\[ y_\epsilon t - \Delta y_\epsilon = f_\epsilon 1_{\omega_\epsilon} \quad \text{in} \quad \Omega \times (0, T), \]
\[ y_\epsilon = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \]
\[ y_\epsilon(x, 0) = y_0(x), \quad y_\epsilon(x, T) = 0 \quad \text{in} \quad \Omega, \]

“converges” to the boundary problem

\[ y_t - \Delta y = 0 \quad \text{in} \quad \Omega \times (0, T), \]
\[ y = g 1_{\Gamma_0} \quad \text{on} \quad \partial\Omega \times (0, T), \]
\[ y(x, 0) = y_0(x), \quad y(x, T) = 0 \quad \text{in} \quad \Omega, \]

where $(y, g) \in C([0, T]; H^{-1}(\Omega)) \times L^2(\Gamma_0 \times (0, T))$. 
Let us give some ideas on the proof.
The meaning of the convergence

If $z$ is a solution of

\[
\begin{cases}
-z_t + \Delta z = h & \text{in } \Omega \times (0, T), \\
z = 0 & \text{in } \partial \Omega \times (0, T), \\
z(x, T) = z^T & \text{in } \Omega,
\end{cases}
\]  

(2)

then, a weak solution of (D-C) must satisfy

\[
\int \int_Q y_\epsilon h \, dx \, dt - \int \Omega y_0 z(0) \, dx = \int \int_{\omega_\epsilon \times (0, T)} f_\epsilon z \, dx \, dt.
\]  

(3)

We say that (D-C) converges to (B-C), if

\[
\int \int_Q y_\epsilon h \, dx \, dt \to \int \int_Q y h \, dx \, dt \quad \text{and} \quad \int \int_{\omega_\epsilon \times (0, T)} f_\epsilon z \, dx \, dt \to \int_0^T \int_{\Gamma_0} g \frac{\partial z}{\partial \nu} \, dx \, dt.
\]

In fact, from (3), we get

\[
\int \int_Q y h \, dx \, dt - \int \Omega y_0 z(0) \, dx = \int_0^T \int_{\Gamma_0} g \frac{\partial z}{\partial \nu} \, dx \, dt,
\]  

(4)

which is exactly the weak formulation for the (BC) problem.
The meaning of the convergence

If $z$ is a solution of

\[
\begin{cases}
-z_t + \Delta z = h & \text{in } \Omega \times (0, T), \\
z = 0 & \text{in } \partial \Omega \times (0, T), \\
z(x, T) = z^T & \text{in } \Omega,
\end{cases}
\]

then, a weak solution of (D-C) must satisfy

\[
\iint_Q y_\epsilon h \, dx \, dt - \int_\Omega y_0 z(0) \, dx = \iint_{\omega_\epsilon \times (0, T)} f_\epsilon z \, dx \, dt.
\]

We say that (D-C) converges to (B-C), if

\[
\iint_Q y_\epsilon h \, dx \, dt \to \iint_Q y h \, dx \, dt \quad \text{and} \quad \iint_{\omega_\epsilon \times (0, T)} f_\epsilon z \, dx \, dt \to \int_0^T \int_{\Gamma_0} g \frac{\partial z}{\partial \nu} \, dx \, dt.
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In fact, from (3), we get

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\]

which is exactly the weak formulation for the (BC) problem.
Main steps of the proof

- A sharp Carleman inequality for the adjoint system: optimal cost with respect to $\epsilon$.
- Construction, in the sense of Fursikov-Imanuvilov, of an optimal pair $(y_\epsilon, f_\epsilon)$ state-control.
- Obtainment of good estimates for $(y_\epsilon, f_\epsilon)$ in the correct spaces.
- Proof of the convergence of one problem to the other one.
Step 1: Optimal Carleman inequality

Let

\[
\begin{align*}
\varphi_t - \Delta \varphi &= F \quad \text{in} \quad \Omega \times (0, T), \\
\varphi &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
\varphi(x, 0) &= \phi_0(x), \quad \text{in} \quad \Omega.
\end{align*}
\]

It is well-known that both problems (B-C) and (D-C) are solvable iff

\[
\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \left( \iint_{\Gamma_0 \times (0, T)} |\frac{\partial \varphi}{\partial \nu}(x, t)|^2 \, d\sigma \, dt + \iint_{Q} |F|^2 \, dx \, dt \right)
\]

(5)

and

\[
\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \left( C(\epsilon) \iint_{\omega_\epsilon \times (0, T)} |\varphi|^2 \, dx \, dt + \iint_{Q} |F|^2 \, dx \, dt \right),
\]

(6)

respectively.
Remark

As in the hyperbolic case, we must know how $C(\epsilon)$ depends on $\epsilon$. This is the key point of the whole proof!
The observability constant

Lemma

The observability constant for the internal observability inequality has the form

$$C(\epsilon) = C\epsilon^{-3},$$

where $C$ does not depend on $\epsilon$. 
Carleman estimates

On what follows, we consider a weight function \( \psi \in C^2(\bar{\Omega}) \) verifying

\[
|\nabla \psi(x)| \neq 0, \quad \forall x \in \bar{\Omega},
\]

\[
\frac{\partial \psi}{\partial \nu}(x) \leq 0, \quad \forall x \in \partial \Omega \setminus \Gamma_0
\]

and for a positive real number \( \lambda \):

\[
\phi(x, t) = \frac{e^{\lambda \psi(x)}}{t(T - t)}; \quad \alpha(x, t) = \frac{e^{\lambda \psi(x)} - e^{2\lambda \|\psi\|_{\infty}}}{t(T - t)}
\]
Carleman estimate

\[ I(s; \varphi) := s^3 \int\int_Q e^{2s\alpha} \varphi^3 |\varphi|^2 \, dx\, dt + s \int\int_Q e^{2s\alpha} |\nabla\varphi|^2 \, dx\, dt \]

\[ + s^{-1} \int\int_Q e^{2s\alpha} \varphi^{-1}(|\varphi_t|^2 + |\Delta \varphi|^2) \, dx\, dt. \quad (7) \]

Theorem (Optimal Carleman inequality)

There exist positive constants \( C \) and \( \lambda_0 \) such that, for every \( \lambda \geq \lambda_0 \), there exists \( s_0 > 0 \) such that, for any \( s \geq s_0 \), the following estimate holds

\[ I(s, \varphi) \leq C \left( \int\int_Q e^{2s\alpha} |F|^2 \, dx\, dt + \epsilon^{-3} s^6 \int\int_{\omega_\epsilon \times (0, T)} e^{2s\alpha} \varphi^6 |\varphi|^2 \, dx\, dt \right), \]

for every \( \varphi \) solution of

\[
\begin{align*}
\varphi_t - \Delta \varphi &= F \quad \text{in} \quad \Omega \times (0, T), \\
\varphi &= 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\
\varphi(x, 0) &= \varphi^T(x) \quad \text{in} \quad \Omega.
\end{align*}
\]
Remark 1

If we try to find $C(\epsilon)$ by only following the original proof of Fursikov-Imanouvilov, we find that

$$C(\epsilon) = O(\epsilon^{-4})$$

Remark 2

The behavior $C(\epsilon) = O(\epsilon^{-3})$ is optimal. Indeed, taking $\varphi(x, t) = e^{-\pi^2 t} \sin(\pi x)$ we have

$$\int_0^1 |\sin(\pi x)|^2 \, dx = \frac{1}{2}$$

and

$$\int_0^T \int_0^\epsilon e^{-2\pi^2 t} |\sin(\pi x)|^2 \, dx \, dt = O(\epsilon^3).$$
Sketch of the proof

We start the proof with a boundary Carleman inequality.

Theorem

There exist positive constants $C$ and $\lambda_0$ such that, for every $\lambda \geq \lambda_0$, there exists $s_0 > 0$ such that, for any $s \geq s_0$, the following estimate holds

$$I(s, \varphi) \leq C \left( \iint_Q e^{2s\alpha} |F|^2 \, dx \, dt + s \iint_{\Gamma_0 \times (0, T)} e^{2s\alpha} \phi \left| \frac{\partial \varphi}{\partial \nu} \right|^2 \, dt \right),$$

for every $\varphi$ solution of

$$\begin{align*}
\varphi_t - \Delta \varphi &= F & \text{in} & \Omega \times (0, T), \\
\varphi &= 0 & \text{on} & \partial \Omega \times (0, T), \\
\varphi(x, 0) &= \varphi^T(x) & \text{in} & \Omega.
\end{align*}$$

Obviously, the constant $C$ does not depend on $\epsilon$. 
Sketch of the proof

The idea now is to bound the boundary term in terms of a local term, using a suitable cutoff function. In fact, we show that

**Estimate for the boundary term**

\[
s \int_0^T \int_{\Gamma_0} e^{2s\alpha} \phi \left| \frac{\partial \varphi}{\partial \nu} \right|^2 \, dt \leq C \epsilon^{-3} s^5 \int_0^T \int_{\omega_{\epsilon} \times (0,T)} e^{2s\alpha} \phi^5 |\varphi|^2 \, dx \, dt + \delta I(s; \varphi),
\]

for any \( \delta > 0 \).

To prove this, we will assume that we are in 1D.
Sketch of the proof

We begin choosing a cut-off function

cut-off function

$\theta \in C^3(0, 1)$ such that $\theta = 1$ in $(0, \frac{\epsilon}{2})$ and $\theta(\epsilon) = 0$. The function $\theta$ has the property $\theta_x = O(\epsilon^{-1})$. 

Figure: Cut off function
Sketch of the proof:

Multiply the equation of $\varphi$ by $\theta_x e^{2s\alpha} \phi^3 \varphi$, integrating by parts several times, and performing a lot of estimates, we obtain

$$s^3 \int_0^T \int_0^\epsilon e^{2s\alpha} \phi^3 |\varphi_x(x, t)|^2 \theta_x \, dx \, dt$$

$$\leq C \left( \epsilon^{-3} s^6 \int_0^T \int_0^\epsilon e^{2s\alpha} \phi^6 |\varphi|^2 \, dx \, dt + \iint_Q e^{2s\alpha} |F|^2 \, dx \, dt \right). \quad (8)$$

and the proof is done.

Remark

The proof in the 1d case is simpler. In the multi-dimensional case, one must work with a normal coordinate system and perform a lot of boring calculations....
Step 2-3: Fursikov-Imanuvilov strategy and bounds

For each $\epsilon > 0$, we construct a pair $(\hat{y}_\epsilon, \hat{f}_\epsilon)$ solution of (D-C).

Let $P_0 := \{ w \in C^2(\overline{Q}),\ w = 0 \text{ in } \partial \Omega \times (0, T) \}$ and the symmetric, positive definite bilinear form:

$$a_\epsilon(w_1, w_2) := \int\int_Q \rho_1^2(t) \mathcal{L}^* w_1 \mathcal{L}^* w_2 \, dx \, dt + \frac{1}{\epsilon^3} \int\int_{\omega_\epsilon \times (0, T)} \rho_2^2(t) w_1 w_2 \, dx \, dt,$$

where $\mathcal{L}^* := \partial_t + \Delta$, and $\rho_i$ is an appropriate weight.

Let $P = P(\epsilon)$ the completion of $P_0$ with respect to the norm associated to $a_\epsilon(\cdot, \cdot)$. 
By Lax-Milgram theorem, there exists a unique $\hat{\varphi}_\epsilon \in P$ such that

$$a_\epsilon(\hat{\varphi}_\epsilon, \varphi) = \int_\Omega y_0 \varphi(0) \, dx, \quad \forall \varphi \in P.$$ 

Next, we show that we can take $(\hat{y}_\epsilon, \hat{f}_\epsilon) = (\rho_1 \mathcal{L}^* \hat{\varphi}_\epsilon, -\frac{\rho_2}{\epsilon^3} \hat{\varphi}_\epsilon)$ as a solution to the problem (D-C). Moreover

$$\|\rho_1 \hat{f}_\epsilon\|^2_{L^2(\omega_\epsilon \times (0,T))} \leq C \epsilon^{-3} \|y_0\|^2_{L^2(\Omega)}, \quad (9)$$

and

$$\|\rho_2 \hat{y}_\epsilon\|^2_{L^2(Q)} \leq C \|y_0\|^2_{L^2(\Omega)}, \quad (10)$$

where $C$ does not depend on $\epsilon$. 
We prove

**Proposition**

We have that $\rho_3 \hat{\phi}_\epsilon \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ with

$$\|\rho_3 \hat{\phi}_\epsilon\|^2_{L^2((0, T); H^2(\Omega) \cap H^1_0(\Omega))} \leq \|\hat{\phi}_\epsilon\|^2_P \leq C\|y_0\|^2_2,$$

where $C$ does not depend on $\epsilon$ and $\rho_3 = \rho(t)$.

Moreover, there exists a function $\varphi$ such that $\rho_3 \hat{\phi}_\epsilon \rightharpoonup \rho_3 \varphi$ weakly in $L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ and $\rho_3 \gamma^{-1} \frac{\partial \varphi}{\partial \nu} \in L^2(\Sigma_0)$. 
This last result allow us to prove the following:

**Lemma**

Let

\[ L_\varepsilon : L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \to \mathbb{R} \]

\[ \nu \to \frac{1}{\varepsilon^3} \int \int_{\omega_\varepsilon \times (0, T)} \rho_3(t) \hat{\varphi}_\varepsilon(x, t) \nu(x, t) \, dxdt. \]

Then, \( L_\varepsilon \) are bounded in \( L^2(0, T; (H^2(\Omega) \cap H^1_0(\Omega))') \) and converge (up to a subsequence) for the weak topology of this space to

\[ L : L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \to \mathbb{R} \]

\[ \nu \to \frac{1}{3} \int \int_{\Gamma_0 \times (0, T)} \rho_3(t) \frac{\partial \varphi}{\partial \nu} (y, t) \frac{\partial \nu}{\partial \nu} (y, t) \, dydt. \]
This last result allows us to prove the following:

**Lemma**

Let

\[ L_\epsilon : L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \to \mathbb{R} \]

\[ v \to \frac{1}{\epsilon^3} \iint_{\omega \times (0, T)} \rho_3(t) \hat{\varphi}_\epsilon(x, t)v(x, t) \, dx \, dt. \]

Then, \( L_\epsilon \) are bounded in \( L^2(0, T; (H^2(\Omega) \cap H^1_0(\Omega))') \) and converge (up to a subsequence) for the weak topology of this space to

\[ L : L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \to \mathbb{R} \]

\[ v \to \frac{1}{3} \iiint_{\Gamma_0 \times (0, T)} \rho_3(t) \frac{\partial \varphi}{\partial \nu}(y, t) \frac{\partial v}{\partial \nu}(y, t) \, dy \, dt. \]
Step 4: Convergence of one problem to the other

Homework for the weekend!!!
Step 4: Convergence of one problem to the other

Homework for the weekend!!!
Ongoing works

- Systems of equations

\[
\begin{align*}
\begin{cases}
  y_1^t - \Delta y_1 &= ay^2 \\
  y_2^t - \Delta y_2 &= f_\epsilon 1_{\omega_\epsilon} \\
  y_1 &= y_2 = 0 \\
  y_1(0) &= y_1^0, \quad y_2(0) &= y_2^0
\end{cases}
&\quad\text{Q,} \\
\begin{cases}
  y_1^t - \Delta y_1 &= ay^2 \\
  y_2^t - \Delta y_2 &= 0 \\
  y_1 &= 0 \\
  y_2(0) &= g_1 \Gamma_0
\end{cases}
&\quad\text{Q,} \\
\begin{cases}
  y_1^t - \Delta y_1 &= ay^2 \\
  y_2^t - \Delta y_2 &= 0 \\
  y_1 &= 0, \quad y_2 = g_1 \Gamma_0 \\
  y_1(0) &= y_1^0, \quad y_2(0) &= y_2^0
\end{cases}
&\quad\text{Q,} \\
\begin{cases}
  y_1^t - \Delta y_1 &= ay^2 \\
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  y_1 &= 0, \quad y_2 = g_1 \Gamma_0 \\
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\end{cases}
&\quad\text{Q,} \\
\begin{cases}
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  y_2^t - \Delta y_2 &= 0 \\
  y_1 &= 0, \quad y_2 = g_1 \Gamma_0 \\
  y_1(0) &= y_1^0, \quad y_2(0) &= y_2^0
\end{cases}
&\quad\text{Q,}
\end{align*}
\]

Minimal time of controllability?
Ongoing works

- Stokes equation

There exists \((y^\varepsilon, f_\varepsilon 1_{\omega_\varepsilon})\) and \((y, g 1_{\Gamma_0})\)

\[
\begin{align*}
\begin{cases}
y_t^\varepsilon - \Delta y^\varepsilon + \nabla \pi = f_\varepsilon 1_{\omega_\varepsilon} & \text{Q,} \\
y^\varepsilon = 0 & \text{\Sigma,} \\
y^\varepsilon(0) = y_0, y^\varepsilon(T) = 0 & \text{\Omega.}
\end{cases} \quad \Rightarrow \quad \begin{cases}
y_t - \Delta y + \nabla \tilde{\pi} = 0 & \text{Q,} \\
y = g 1_{\Gamma_0} & \text{\Sigma,} \\
y(0) = y_0, y(T) = 0 & \text{\Omega.}
\end{cases}
\end{align*}
\]

where \(\omega_\varepsilon\) shrinks to \(\Gamma_0\)?

Is it possible to eliminate controls?
Other type of equations

- KdV? Critical length?
- Kuramoto Sivashinsky? Dirichelet control and Neumann boundary controls?
Merci!!