# On the intersection of control and machine learning

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GdT Contrôle, LJLL

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# Supervised learning

**Goal:** Approximate an unknown function $f : \mathbb{R}^d \to \mathbb{R}^m$ given data

$$\mathscr{D} := \left\{ x^{(i)}, f(x^{(i)}) \right\}_{i \in [n]} \subset \mathbb{R}^d \times \mathbb{R}^m.$$

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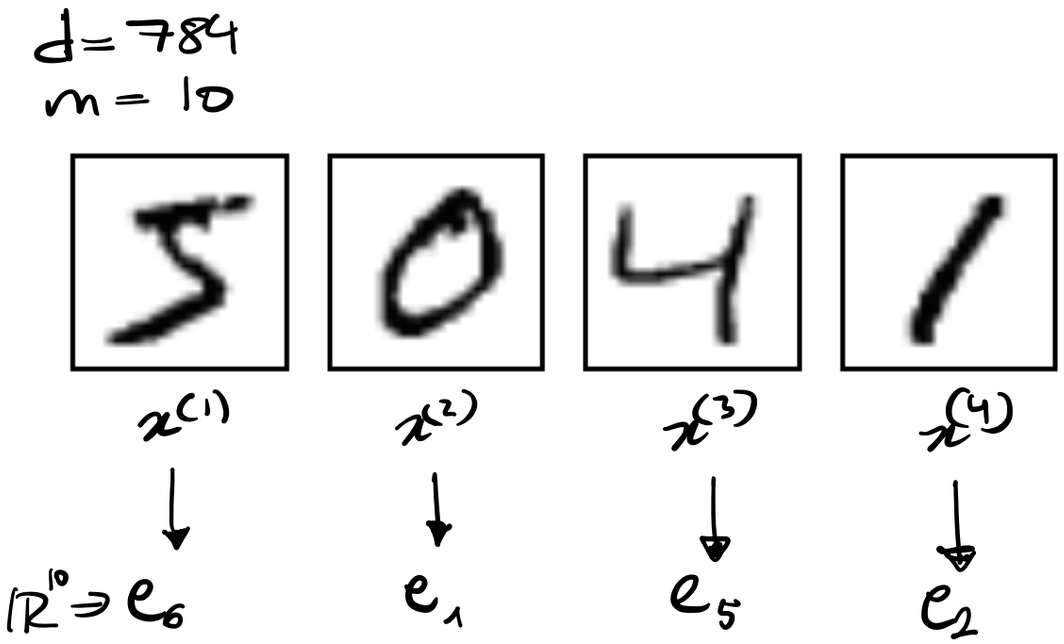
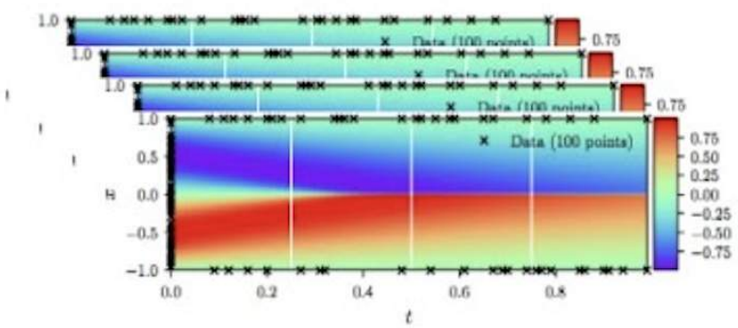
We distinguish:

- Classification: $\mathrm{ran}(f)$ is $\{e_j\}_{j \in [m]} \subset \mathbb{R}^m$.
  Image, audio: $d \geqslant 10^3$, $d \gg m$.




- Regression: $\mathrm{ran}(f)$ is $\mathbb{R}^m$.
  PDE: fixed initial/boundary data, then learn $(t, \mathrm{x}) \mapsto u(t, \mathrm{x})$



# (Feed-forward) neural networks

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where

- $P \in \mathbb{R}^{m \times d_{n_T}}$ (suppose given)
- $\mathrm{x}^{[n_T]}(x) \in \mathbb{R}^{d_{n_T}}$ is output of neural net with $n_t \geqslant 1$ *layers*:

$$\begin{cases} \mathrm{x}^{[k+1]} = c^{[k]}\sigma(a^{[k]} \cdot \mathrm{x}^{[k]} + b^{[k]}) & k \in \{0, \dots, n_T - 1\} \\ \mathrm{x}^{[0]} = x, \end{cases}$$

  **state** $\mathrm{x}^{[k+1]} \in \mathbb{R}^{d_{k+1}}$ and **weights** $c^{[k]} \in \mathbb{R}^{d_{k+1}}, a^{[k]} \in \mathbb{R}^{d_k}, b^{[k]} \in \mathbb{R}$
- $\sigma \in C^{0,1}(\mathbb{R})$, typically $\sigma(x) = (x)_+$ or $\sigma(x) = \tanh(x)$
- *widths* $d_k$ given

# Residual neural networks (ResNets)

Let $d_k = d$ for all $k$.

Consider

$$\begin{cases} \mathrm{x}^{[k+1]} = \mathrm{x}^{[k]} + \triangle t c^{[k]}(a^{[k]} \cdot \mathrm{x}^{[k]} + b^{[k]})_+ & k \in \{0, \dots, n_T - 1\} \\ \mathrm{x}^{[0]} = x, \end{cases}$$

with $c^{[k]}, a^{[k]} \in \mathbb{R}^d$ and $b^{[k]} \in \mathbb{R}$.


Deep **residual** learning for image recognition

K He, X Zhang, S Ren, J Sun - Proceedings of the IEEE …, 2016 - openaccess.thecvf.com

… as learning **residual** functions with … **residual** networks are easier to optimize, and can gain accuracy from considerably increased depth. On the ImageNet dataset we evaluate **residual** …

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# Neural ODEs

Natural to consider [E '17]:

$$\boxed{\begin{cases} \dot{\mathrm{x}}(t) = c(t)(a(t) \cdot \mathrm{x}(t) + b(t))_+ & t \in (0, T), \\ \mathrm{x}(0) = x \end{cases}}$$

And now $f_{\text{approx}}(x) := P\mathrm{x}(T)$.

Useful in practice:

- Beyond Euler schemes [Chen et al. '18]
- Structure preserving schemes [Schönlieb et al. '20, '22]
- Beyond supervised learning (wait until the end of the talk)
- More compact form for analysis

# Approximation theory

Before using the data, a "well-posedness" question can be asked.

## Problem (Universal approximation)

Given $f \in \mathscr{H}$ and $\epsilon > 0$, find $(a_\epsilon, b_\epsilon, c_\epsilon) \in L^\infty((0,T); \mathbb{R}^{2d+1})$ such that

$$\|f_{\text{approx},\epsilon} - f\|_{\mathscr{H}} \leqslant \epsilon$$

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- Feed-forward nets: [Cybenko '89], [Barron '93] ($n_t = 1$ and $\mathscr{H} = C^0([0,1]^d)$), Pinkus '99 ($n_t \geqslant 1$)
- Neural ODEs: [Li, Lin, Shen '22], [Ruiz-Balet, Zuazua '22] ($\mathscr{H} = L^2((0,1)^d;\mathbb{R}^m)$). ResNets are corollary as controls are piecewise constant
- Strategies generally **non-algorithmic** and suffer from **curse of dimensionality**

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where

$$\begin{cases} \dot{\mathrm{x}}_i(t) = c(t)(a(t)\cdot \mathrm{x}_i(t) + b(t))_+ & t \in (0,T), \\ \mathrm{x}_i(0) = x^{(i)}, \end{cases} \tag{1}$$

and hope predictions of $f(x)$ are good if we take initial data points $x$ outside $\mathscr{D}$ (*generalization*).

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- It's a simultaneous/ensemble control(lability) problem!
- Nonlinear control-state interaction is necessary

# What is done in practice

Least squares, with penalty $\lambda > 0$:

$$\min_{\substack{\theta=(a,b,c)\\(a,b)\in H^1((0,T);\mathbb{R}^{d+1})\\c\in L^2((0,T);\mathbb{R}^d)\\ \mathrm{x}_i \text{ solves (1)}}} \underbrace{\frac{1}{n}\sum_{i\in[n]} \left|P\mathrm{x}_i(T) - f(x^{(i)})\right|^2}_{:=\mathbf{E}(\boldsymbol{X}(t))} + \lambda\|\theta\|^2_{H^1\times L^2} \tag{2}$$

- Empirical risk minimization: $\mathbf{E}(\cdot)$ is the empirical risk.
- $H^1$ suffices for compactness – if $p \in [1,\infty)$, $\varphi : \mathbb{R} \to \mathbb{R}$ is s.t. $\varphi \circ u_n \rightharpoonup \varphi \circ u^*$ in $L^p(0,1)$ for any $u_n \rightharpoonup u^*$ in $L^p(0,1)$, then $\varphi$ is affine!
- Can go way beyond squared Euclidean distance (even non-distances such as cross-entropy for classification)

github.com/borjanG/dynamical.systems

$$T = 5, n_T = 16, n = 3000, \lambda = 0.01$$

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## Question

For global minimizer $\theta_T$ for (2) and $\boldsymbol{X}_T \in C^0([0,T];\mathbb{R}^{d\times n})$ matrix with unique solutions $\mathrm{x}_i$ to (1) as columns, **what happens when $T \to \infty$**?

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## Definition (Interpolation)

(1) interpolates $\mathscr{D}$ if $\exists \theta \in H^1((0,1);\mathbb{R}^{d+1}) \times L^2((0,1);\mathbb{S}^{d-1})$ such that

$$P\mathrm{x}_i(1) = f(x^{(i)}) \qquad \forall i \in [n]$$

where $\mathrm{x}_i \in C^0([0,1];\mathbb{R}^d)$ solves (1) with control $\theta$. (I.e., $\mathbf{E}(\boldsymbol{X}(1)) = 0$.)

## Theorem [Esteve-Yagüe, G., Pighin, Zuazua, '22b]

Fix $\lambda > 0$; $c$ in (2) minimized over $L^2((0,T);\mathbb{S}^{d-1})$; (1) interpolates $\mathscr{D}$. For $T \geqslant 1$, any global minimizer $\theta_T$ for (2) and $\boldsymbol{X}_T \in C^0([0,T];\mathbb{R}^{d\times n})$ matrix with columns solutions to (1) satisfy:

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1. $\exists C(\mathscr{D},\lambda) > 0$,

$$\mathbf{E}(\boldsymbol{X}_T(T)) = \frac{1}{n}\sum_{i\in[n]} \left| P\mathrm{x}_i(T) - f(x^{(i)}) \right|^2 \leqslant \frac{C}{T}$$

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2. $\boldsymbol{X}_{T_k}(T_k) \to \boldsymbol{X}^*$ for some subsequence $T_k > 0$, $T_k \to \infty$ $(k \to \infty)$ and $\boldsymbol{X}^* \in \mathbb{R}^{d\times n}$ with $\mathbf{E}(\boldsymbol{X}^*) = 0$

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3. Set $\theta_k(t) := (T_k a_{T_k}(tT_k), T_k b_{T_k}(tT_k), c_{T_k}(tT_k))$ for $t \in [0,1]$. Then $\theta_k \to \theta^*$ strongly in $H^1 \times L^2$ where $\theta^*$ is some solution to

$$\inf_{\substack{\theta=(a,b,c)\\(a,b)\in H^1((0,1);\mathbb{R}^{d+1})\\ c\in L^2((0,1);\mathbb{S}^{d-1})\\ \mathbf{E}(\boldsymbol{X}(1))=0}} \|\theta\|^2_{H^1\times L^2}$$

1. $\theta_1 = (a_1, b_1, c_1) \in H^1((0,1); \mathbb{R}^{d+1}) \times L^2((0,T); \mathbb{S}^{d-1})$ yields $\mathrm{x}_i^1$ solution to (1) on $[0,1]$. Then

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2. In turn,

$$\begin{aligned}\mathbf{E}(\boldsymbol{X}_T(T)) + \lambda \int_0^T |\theta_T(t)|^2 dt \\ = \mathbf{E}(\boldsymbol{X}_1(1)) + \frac{\lambda}{T}\int_0^1 |(a_1(s), b_1(s))|^2 ds + \lambda.\end{aligned}$$

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**Corollary**

In this setting, $T \to \infty$ is equivalent to $\lambda \to 0$.

# Interpolation, Controllability

1. *Combinatorics.* For (1): [Li, Lin, Shen '22], [Ruiz-Balet, Zuazua '22]. Distinct targets if $d = m$.
2. *Lie algebra.* For

$$\dot{\mathrm{x}}_i(t) = \theta(t)\sigma(\mathrm{x}_i(t)), \tag{3}$$

with $\sigma \in C^{0,1} \cap C^1(\mathbb{R})$ element-wise: [Agrachev, Sarychev '22]

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A digression - little homotopy method inspired by [Coron-Trélat '04]:

## Proposition [Esteve-Yagüe, G., Pighin, Zuazua '22b]

Suppose $d \geqslant n$. Fix $\boldsymbol{X}^1 \in \mathbb{R}^{d\times n}$ with

$$\mathrm{span}\{\sigma(\mathrm{x}_1^1), \dots, \sigma(\mathrm{x}_n^1)\} = \mathbb{R}^d$$

Then $\exists r, C > 0$ such that $\forall \boldsymbol{X}^0 \in B_r(\boldsymbol{X}^1)$, $\exists \theta \in L^\infty((0,1); \mathbb{R}^{d\times d})$ for which the solutions $\mathrm{x}_i$ to (3) with $\mathrm{x}_i(0) = \mathrm{x}_i^0$ satisfy $\mathrm{x}_i(1) = \mathrm{x}_i^1$ $\forall i \in [n]$. Moroever

$$\|\theta\|_{L^\infty} \leqslant \frac{C}{T}|\boldsymbol{X}^1 - \boldsymbol{X}^0|.$$

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- Look at $\{x^{(i)}, y^{(i)}\}_{i\in[n]} \subset \mathbb{R}^d \times \mathbb{R}^m$ as i.i.d. samples from unknown joint law $\mu \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^m)$. Then $f(x) := \mathbb{E}(y|x)$ which minimizes $\mathbb{E}_{(x,y)\sim\mu}|f(x) - y|^2$ over all functions $f$.

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- Associated to (2): population risk minimization

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- Generalization: $\theta_n$ minimizer of $J_n$ in (2), and $\theta^*$ of $J$ in (4), then $\exists \alpha > 0$:

$$\mathbb{E}_{(x,y)\sim\mu}\ |P\mathrm{x}^{\theta_n}(T) - y|^2 - \mathbb{E}_{(x,y)\sim\mu}|P\mathrm{x}^{\theta^*}(T) - y|^2 = \mathcal{O}\left(\frac{1}{n^\alpha}\right)$$

# What is known

1. [E, Han, Li '19]:
   - Pontryagin Maximum Principle for both (2) and (4);
   - Hamiltonian $\theta \mapsto H(x,p,\theta) = p \cdot \theta\sigma(x) + \lambda|\theta|^2$ strongly concave for any $(x,p)$
   - given $\theta^*$, with high probability $\exists \theta_n$ critical point of Hamiltonian for (2) such that

$$\mathbb{E}_{(x,y)\sim\mu} \left|P\mathrm{x}^{\theta_n}(T) - y\right|^2 - \mathbb{E}_{(x,y)\sim\mu}\left|P\mathrm{x}^{\theta^*}(T) - y\right|^2 \leqslant \frac{C(d)}{n^{\frac{1}{2}-\epsilon}}$$

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   - Ensuring that $\theta_n$ is global minimizer: true when $T \ll 1$, so $\lambda \gg 1$!
2. [Bonnet et al. '22]:
   - For $\lambda \gg 1$, $J_n$ strongly convex on any $L^2$ ball
   - Mean-field PMP ... rate $\mathcal{O}\left(\frac{1}{n^{\frac{1}{d}}}\right)$

## An observation with C. Letrouit and P. Rigollet

1. Strong convexity on any $B \subset L^2((0,T);\mathbb{R}^{d\times d})$

$$\|\theta_1 - \theta_2\|_{L^2}^2 \lesssim |\nabla J_n(\theta_1) - \nabla J_n(\theta_2)|\,, \qquad \forall \theta_1, \theta_2 \in B.$$

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2. Then

$$\begin{aligned}\|\theta_n - \theta^*\|_{L^2}^2 &\lesssim |\nabla J_n(\theta_n) - \nabla J_n(\theta^*)| \\ &= |\nabla J(\theta^*) - \nabla J_n(\theta^*)| \\ &= \left| \mathbb{E}_{(x,y)\sim\mu} \nabla_\theta \ell(\mathrm{x}^{\theta^*}(T), y) - \frac{1}{n}\sum_{i\in[n]} \nabla_\theta \ell(\mathrm{x}_i^{\theta^*}(T), y^{(i)}) \right|\end{aligned}$$

$|P_{x_i}^{\theta^*}(T) - y^{(i)}|^2$

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3. $\theta^*$ is fixed, not random. Sum of bounded, independent random variables, compared with expectation $\Rightarrow$ **concentration of measure** (Hoeffding inequality): $\forall \delta > 0$, $\exists \kappa > 0$ such that $\forall n \geqslant 1$,

$$\mathbb{P}(\|\theta_n - \theta^*\|_{L^2}^2 \leqslant \kappa/\sqrt{n}) \geqslant 1 - \delta.$$

Bound $J(\theta_n) - J(\theta^*)$ from above and get $\mathcal{O}(\frac{1}{\sqrt{n}})$ rate.

# Improving $1/T$

We consider

$$\inf_{\substack{c\in L^2((0,T);\mathbb{R}^d)\\ \mathrm{x}_i(\cdot)\text{ solves (1)}}} \int_0^T \frac{1}{n}\sum_{i\in[n]} \left|P\mathrm{x}_i(t) - f(x^{(i)})\right|^2 dt + \int_0^T |c(t)|^2 dt \tag{5}$$

Controls $a \in L^\infty(\mathbb{R}_+;\mathbb{R}^d)$ and $b \in L^\infty(\mathbb{R}_+)$ assumed fixed in (1) (need $L^2$-penalties and compactness simultaneously)

Can also consider dynamics as (3), or $(a, b)$ can be optimized over $\mathbb{S}^d$.

# Two assumptions

Recall, for $\boldsymbol{X} = [\mathrm{x}_1 \cdots \mathrm{x}_n] \in \mathbb{R}^{d\times n}$:

$$\mathbf{E}(\boldsymbol{X}) := \frac{1}{n}\sum_{i\in[n]} \left| P\mathrm{x}_i - f(x^{(i)}) \right|^2 .$$

**Assumption 1.**

Set $\mathcal{Z} := \{\boldsymbol{Z} \in \mathbb{R}^{d\times n} \colon \mathbf{E}(\boldsymbol{Z}) = 0\}$. Then $P \in \mathbb{R}^{m\times d}$ is such that

$$\kappa_1 \mathrm{dist}(\boldsymbol{X}, \mathcal{Z})^2 \leqslant \mathbf{E}(\boldsymbol{X}) \leqslant \kappa_2 \mathrm{dist}(\boldsymbol{X}, \mathcal{Z})^2$$

for some $\kappa_2 \geqslant \kappa_1 > 0$, and $\forall \boldsymbol{X} \in \mathbb{R}^{d\times n}$.

- Lower bound: global Lojasiewicz inequality for analytic functions

## Assumption 2.

Fix $\boldsymbol{X}^0 = [\mathrm{x}_1^0 \cdots \mathrm{x}_n^0] \in \mathbb{R}^{d\times n}$. We assume $\exists c \in L^2((0,1);\mathbb{R}^d)$ such that the matrix $\boldsymbol{X} \in C^0([0,1];\mathbb{R}^{d\times n})$ with columns $\mathrm{x}_i(\cdot)$ solutions to (1) with $\mathrm{x}_i(0) = \mathrm{x}_i^0$, satisfies $\boldsymbol{X}(1) \in \mathcal{Z}$.

Moreover, $\exists C(n) > 0$,

$$\int_0^1 |c(t)|^2 dt \leqslant C(n)\,\mathrm{dist}(\boldsymbol{X}^0, \mathcal{Z})^2.$$

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Moreover, $\exists C(n) > 0$,

$$\int_0^1 |c(t)|^2 dt \leqslant C(n)\,\mathrm{dist}(\boldsymbol{X}^0, \mathcal{Z})^2.$$

- When $d > m$, then $P \in \mathbb{R}^{d\times n}$ is generically surjective and

$$\mathcal{Z} = \left\{ [\mathrm{z}_1 \cdots \mathrm{z}_n] \in \mathbb{R}^{d\times n} : \mathrm{z}_i \in P^{-1}\left\{ f(x^{(i)}) \right\} \right\}$$

So $\mathrm{x}_i(1) \in P^{-1}\left\{ f(x^{(i)}) \right\}$ for all $i \in [n]$ and

$$\int_0^1 |c(t)|^2 dt \leqslant C(n) \inf_{\substack{[\mathrm{z}_1 \cdots \mathrm{z}_n] \in \mathbb{R}^{d\times n} \\ \mathrm{z}_i \in P^{-1}\{f(x^{(i)})\}}} \sum_{i\in[n]} \left| \mathrm{x}_i^0 - \mathrm{z}_i \right|^2.$$

## Theorem [Esteve-Yagüe, G., Pighin, Zuazua, '22a]

Suppose $m = d$ and $P = \mathrm{Id}$. Then $\exists T_*, \omega > 0$, $C \geqslant 1$ such that for $T \geqslant T_*$, any global minimizer $c_T \in L^2((0,T);\mathbb{R}^d)$ to (5) and $\mathrm{x}_i^T$ solution to (1) satisfy

$$\sum_{i\in[n]} \left|\mathrm{x}_i^T(t) - f(x^{(i)})\right|^2 + |c_T(t)|^2 \leqslant \left(C \sum_{i\in[n]} \left|x^{(i)} - f(x^{(i)})\right|^2\right) e^{-\omega t}$$

$\forall t \in [0,T]$.

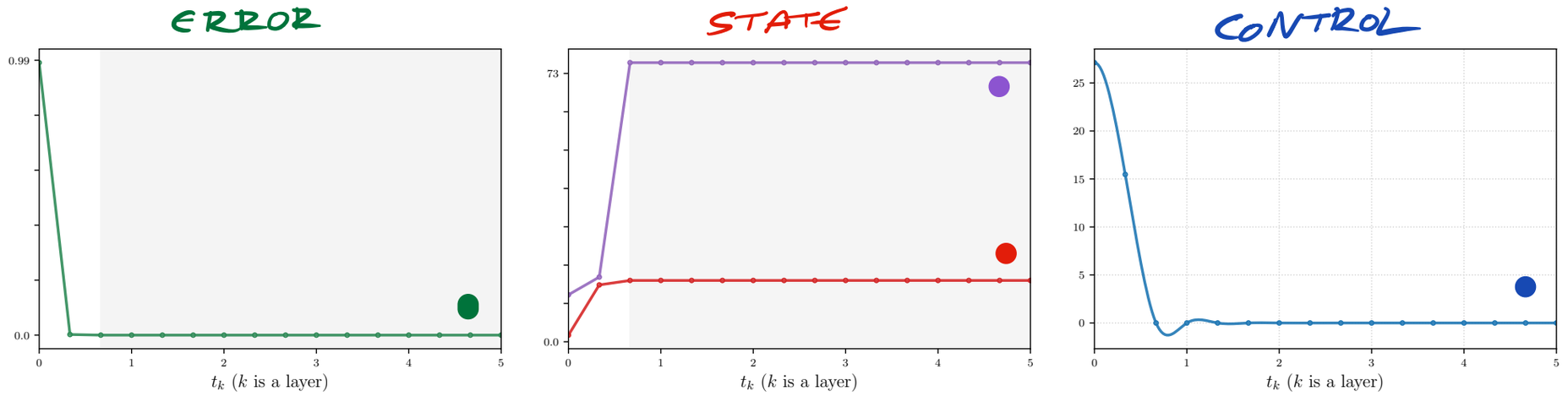
## Theorem [Esteve-Yagüe, G., Pighin, Zuazua, '22b]

Replace $(\cdot)_+$ by $\sigma \in L^\infty(\mathbb{R})$ in (1) and let $d > m$. Then $\exists T_*, \omega > 0$, $C \geqslant 1$ such that for $T \geqslant T_*$, any global minimizer $c_T \in L^2((0,T);\mathbb{R}^d)$ to (5) and $\mathrm{x}_i^T$ solution to (1) satisfy

$$\sum_{i\in[n]} \left| P\mathrm{x}_i^T(t) - f(x^{(i)}) \right|^2 + \inf_{\substack{[\mathrm{z}_1 \cdots \mathrm{z}_n]\in\mathbb{R}^{d\times n} \\ \mathrm{z}_i \in P^{-1}\{f(x^{(i)})\}}} \sum_{i\in[n]} \left|\mathrm{x}_i^T(t) - \mathrm{z}_i\right|^2 + |c_T(t)|^2$$

$$\leqslant \left( C \inf_{\substack{[\mathrm{z}_1 \cdots \mathrm{z}_n]\in\mathbb{R}^{d\times n} \\ \mathrm{z}_i \in P^{-1}\{f(x^{(i)})\}}} \sum_{i\in[n]} \left| x^{(i)} - \mathrm{z}_i \right|^2 \right) e^{-\omega t}$$

$\forall t \in [0,T]$.



# It's faster

Takeaway: when possible, proceed in *model predictive control manner:* start with small $T$, evaluate error, and proceed by increasing $T$ adaptively.

# Important tool

Focus on $d = m$, $P = \mathrm{Id}$.

## Lemma

$\exists C_1 > 0$ independent of $T$, $\forall c \in L^2$ and $\mathrm{x}_i$ solution to (1):

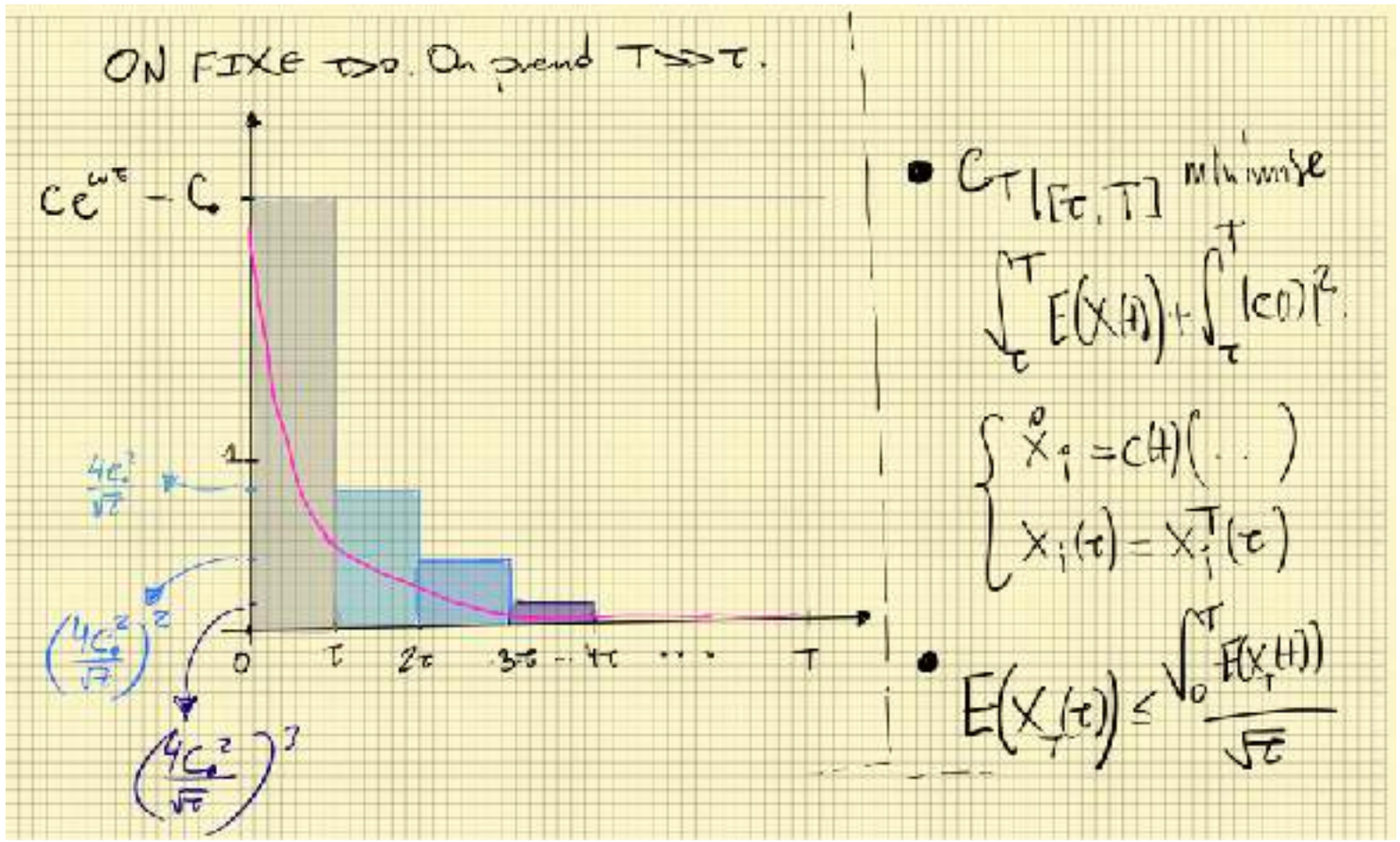
$$\sup_{t\in[0,T]} \sum_{i\in[n]} \left|\mathrm{x}_i(t) - f(x^{(i)})\right|^2 \leqslant C_1 \Bigg( \sum_{i\in[n]} \left|x^{(i)} - f(x^{(i)})\right|^2 + \int_0^T \sum_{i\in[n]} \left|\mathrm{x}_i(t) - f(x^{(i)})\right|^2 dt + \int_0^T |c(t)|^2 dt \Bigg)$$

# Ingredients

1. $c_{\mathrm{aux}}(t) := c_1(t)1_{[0,1]}(t)$, where $c_1$ ensures controllability. As $c_T$ is optimal and $c_{\mathrm{aux}}$ is not, $\exists C_2 > 0$ independent of $T$:

$$\int_0^T \sum_{i\in[n]} \left|\mathrm{x}_i^T(t) - f(x^{(i)})\right|^2 dt + \int_0^T |c_T(t)|^2 dt \leqslant C_2 \sum_{i\in[n]} \left|x^{(i)} - f(x^{(i)})\right|^2$$

   Lemma yields pointwise bound uniform in $T$.

2. 

# Extensions

- Using this method, we can't have exponential decay with $BV$-penalty for $(a, b)$ as norm tracks singularities unlike $L^2$. We can at most get decay of time-averages of the error and controls.
- **Results are more general.** Per [Esteve-Yagüe, G., Pighin, Zuazua '22a; G., Zuazua '22]: controllable PDE

$$y_t(t,x) - Ay(t,x) + Bu(t,x) = f(y(t,x))$$

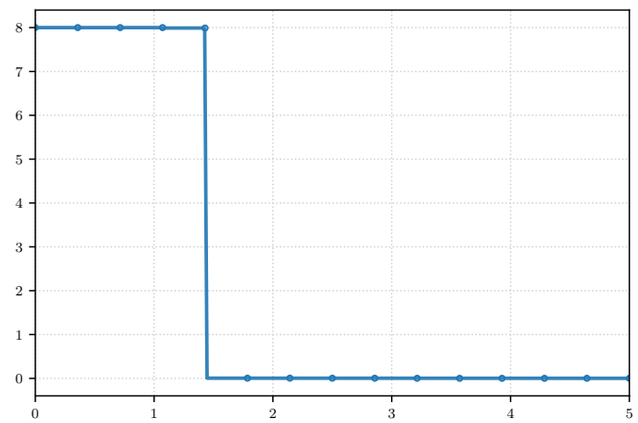
with Lipschitz (possibly non-smooth) nonlinearity $f$ and cost

$$\phi(y(T)) + \int_0^T |y(t) - \bar{y}|^2_{\mathscr{H}} dt + \int_0^T |u(t)|^2_{\mathscr{U}} dt$$

where $\bar{y}$ is any steady state $\Rightarrow$ exponential turnpike without smoothness or smallness assumptions!

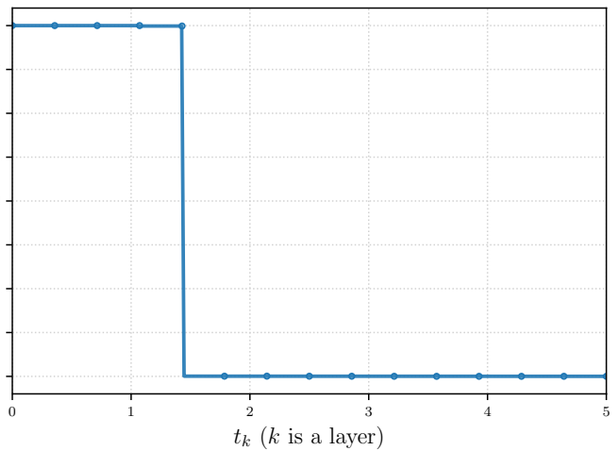
# Further comments

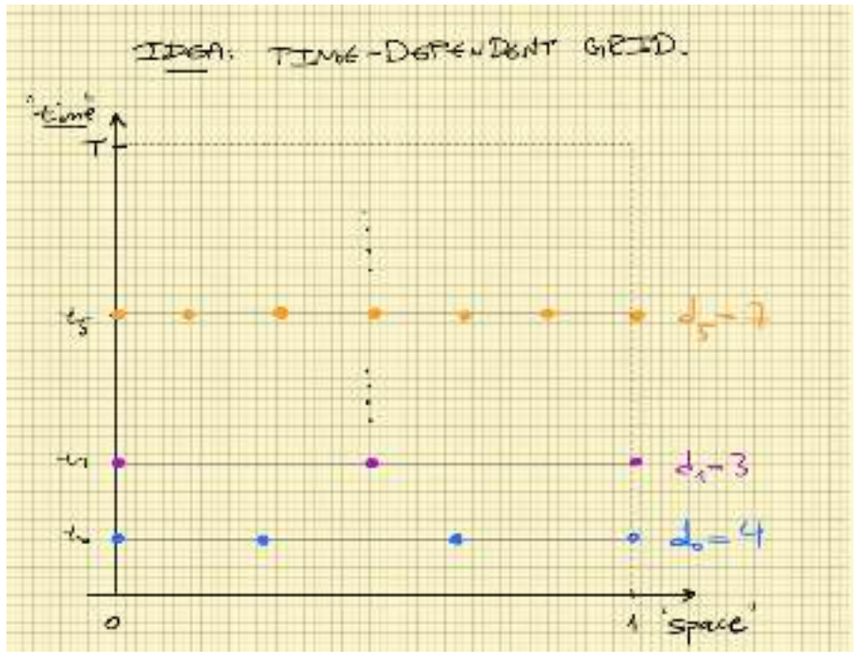
1. $L^1(0,T)$-penalties for (3) [Esteve-Yagüe, G. '22]:



# Further comments

1. $L^1(0,T)$-penalties for (3) [Esteve-Yagüe, G. '22]:




2. Variable-width ResNets:

$$\mathrm{x}^{[k+1]} = \Pi^{[k]}\mathrm{x}^{[k]} + c^{[k]}\sigma(a^{[k]}\mathrm{x}^{[k]})$$

where $\Pi^{[k]} : \mathbb{R}^{d_k} \to \mathbb{R}^{d_{k+1}}$, $a^{[k]} \in \mathbb{R}^{d_{k+1}\times d_k}$, $c^{[k]} \in \mathbb{R}^{d_{k+1}\times d_{k+1}}$. So,

$$\partial_t \mathrm{x}(t,z) = \int_0^1 c(t,z,\zeta)\sigma(a(t,z,\zeta)\mathrm{x}(t,\zeta))d\zeta \qquad (0,T)\times(0,1)$$

Helpful for structured controls (convolutional neural networks).

# Outlook

1. Control
   - Exponential decay/turnpike with $BV$-penalty for $(a, b)$?
   - Using control: are feedback controls for $n \gg 1$ trajectories possible, useful?
   - Extrapolating to control: can we get robustness using the lens of many data and statistics?
2. Unsupervised learning/**generative modeling** with normalizing flows (E. Vanden-Eijnden et al.).
   - NF: diffeomorphism $\mathfrak{T} : \mathbb{R}^d \to \mathbb{R}^d$ optimized to transport $\{z^{(i)}\}_{i\in[n]} \subset \mathbb{R}^d$ samples from a known law $\rho_0$ (Gaussian with unit variance) to unknown target law $\rho_1$ of which we know samples $\{x^{(i)}\}_{i\in[n]} \subset \mathbb{R}^d$.
   - Parametrizing $\mathfrak{T}$ by the flow of a neural ODE, and then solving an optimal control problem (KL divergence) is quite practical.

Merci pour votre attention!

*Bibliographie:*

1. [Esteve-Yagüe, G., Pighin, Zuazua '22a]: https://arxiv.org/abs/2011.11091, Nonlinearity
2. [Esteve-Yagüe, G., Pighin, Zuazua '22b]: https://arxiv.org/abs/2008.02491
3. [G., Zuazua '22]: https://arxiv.org/abs/2202.04097, Acta Numerica
4. [Esteve-Yagüe, G. '22]: https://arxiv.org/abs/2102.13566, submitted