Stabilité du Processus de Contrôle HUM

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Mai 2017

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Stability of HUM Process

Setting

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta_g u + V.\nabla u + qu = 0 \qquad \text{in} \qquad]0,+\infty[\times M \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H^1 \times L^2 \end{cases}$$

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- M Riemannian manifold, connected, compact, without boundary, with dimension d.
- $M = \Omega$ open subset of $\mathbb{R}^d$, connected, bounded, with smooth boundary (homogeneous Dirichlet condition).

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$$H = C([0,+\infty[, H^1) \cap C^1([0,+\infty[, L^2)$$

$$Eu(t) = ||u(t,.)||^2_{H^1(M)} + ||\partial_t u(t,.)||^2_{L^2(M)}$$

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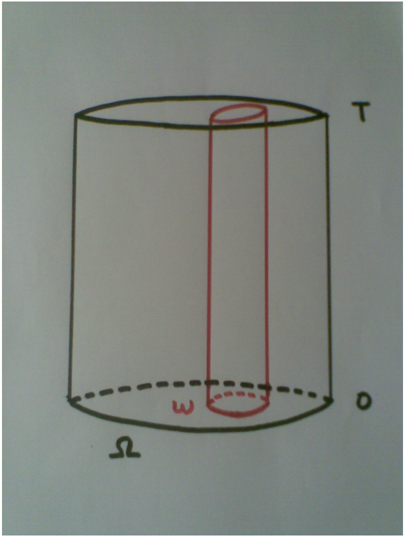

Assumption : (ω, T) satisfies the (GCC) condition of Bardos-Lebeau-Rauch, or any sufficient condition for observability/control.

The Goal

Study the problem of observation and control for the wave equation (W)

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Problem 2 : If the metric g is not smooth.

$\longrightarrow$ (P1) : Study the behavior of the HUM control operator in each case.

$\longrightarrow$ (P2) : Eventually allow obsevability estimate with loss.....

About the HUM control operator

Spectral localization operators

$(e_j, \omega_j^2)_j$ the spectral elements of M, $u = \sum_j a_j e_j$, $k \in \mathbb{N}$.

$$\psi_k(D)u = \sum_j \psi(2^{-k}\omega_j^2) a_j e_j$$

$$S_k(D) = \sum_{j=0}^{k-1} \psi_j(D) \qquad k \geq 1$$

$$\eta_k(D) = \sum_{j=k}^{\infty} \psi_j(D) = Id - S_k(D)$$

And we get the Littlewood-Paley dcomposition of u :

$$u = \sum_k \psi_k(D)u$$

$$\begin{cases} \partial_t^2 u - \Delta_x u = \chi_\omega^2(x) f & \text{in} \quad]0, T[\times\Omega \\ u = 0 & \text{on} \quad]0, T[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

We look for $f \in L^2(]0, T[\times\Omega)$, s.t

$$(u(T), \partial_t u(T)) = (0, 0)$$

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By HUM and **under** (**G**.**C**.**C**), we can take f solution of

$$\begin{cases} \partial_t^2 f - \Delta_x f = 0 & \text{in} \quad]0, T[\times\Omega \\ f = 0 & \text{on} \quad]0, T[\times\partial\Omega \\ (f(0), \partial_t f(0)) = (f_0, f_1) \in L^2 \times H^{-1} \end{cases}$$

The map

$$\begin{cases} \Lambda : H_0^1 \times L^2 \to L^2 \times H^{-1} \\ \\ (u_0, u_1) \to (f_0, f_1) \end{cases}$$

is an isomorphism; this is HUM optimal control operator.

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Theorem (D-Lebeau, 2009)

In the setting above and under (G.C.C),
a) For all $s \geq 0$,

$$\Lambda : H^{s+1} \times H^s \to H^s \times H^{s-1}$$

is an isomorphism.
b)

$$\|\Lambda \psi_k(D) - \psi_k(D)\Lambda\| \leq C2^{-k/2}$$

c) If M is a Riemannian manifold without boundary, Λ is a pseudo differential operator.

$\longrightarrow$ Remind $\quad u = \sum_j u_j e_j \in H^s \quad$ iff $\quad \sum_j (1 + \omega_j^2)^s |u_j|^2 < \infty$.

Case of unknown potential

Assume that (GCC) is satisfied.

Denote by f_q the control vector that drives the solution of $\Box u + q(x)u = \chi_\omega^2(x) f_q$ from the initial data (u_0, u_1) to $(0,0)$.

Take $q_1(x), q_2(x) \in W^{s,p}(\Omega)$, $s \in]0,1]$ and $p > max(2, d)$.

Theorem

$$f_{q_1} = f_{q_2} + K$$

where K is a smoothing compact operator :

$$K : H_0^1 \times L^2 \longrightarrow L^2(0, T; H^s)$$

Moreover, if $||q_{1,2}||_{W^{s,p}} \leq M$, one has

$$||K|| \leq C(M) ||q_1 - q_2||_{W^{s,p}}$$

Corollary (High frequency estimate)

In this setting, $0 < s \leq 1$ *and* $p > max(2, d)$*, we have for any integer* k*,*

$$||\eta_k(D)K(u_0, u_1)||_{L^2(L^2)} \leq C2^{-ks/2}||q_1 - q_2||_{W^{s,p}}||(u_0, u_1)||_{H_0^1 \times L^2}$$

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Theorem (B.D - S.Ervedoza, SIAM J. Contr. 2014)

In the same setting,

$$||\eta_k(D)(\Lambda_{q_1} - \Lambda_{q_2})(u_0, u_1)||_{L^2 \times H^{-1}} \leq C2^{-ks/2}||(u_0, u_1)||_{H_0^1 \times L^2}||q_1 - q_2||_{W^{s,p}}$$

In other words,

$$\Lambda_{q_1} - \Lambda_{q_2} : H_0^1 \times L^2 \longrightarrow H^s \times H^{s-1}$$

is bounded.

Sketch of the proof

$\longrightarrow$ Denote by S_q the semi-group generating the solution of

$$\Box f + q(x) f = 0 \qquad (f(0), \partial_t f(0)) = (f_0, f_1)$$

$\longrightarrow$ $A_q = S_q \circ \Lambda_q$

$\longrightarrow$ $f_q = A_q(u_0, u_1)$.

$$A_{q_1} - A_{q_2} = (S_{q_1} - S_{q_2}) \circ \Lambda_{q_1} + S_{q_2} \circ (\Lambda_{q_1} - \Lambda_{q_2})$$

Application

In other words, to drive the initial data $(u_0, u_1) \in H_0^1 \times L^2$ to equilibrium $(0, 0)$, one can use one of the following control processes:

$$\begin{cases} \Box u + q_1(x)u = \chi_\omega^2(x) f_{q_1} \\ \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

or

$$\begin{cases} \Box v + q_2(x)v = \chi_\omega^2(x) f_{q_1} \\ \\ (v0), \partial_t v(0)) = (u_0, u_1) \end{cases}$$

The error is then bounded as

$$||\eta_k(D)(u(T) - v(T))||_{H_0^1 \times L^2} \leq C2^{-ks/2}||q_1 - q_2||_{W^{s,p}}||(u_0, u_1)||_{H_0^1 \times L^2}$$

Comments

- A key point of the proof: a uniform observability estimate

Theorem (Precised observability estimate)

Under (G.C.C), for every $r > 0$, there exists a constant $C_r > 0$ s.t. for every $q \in L^p(\Omega)$ satisfying $\|q\|_{L^p} \leq r$, the following estimate

$$\|(\varphi(0), \partial_t \varphi(0))\|^2_{L^2 \times H^{-1}} \leq C_r \int_0^T \int_\Omega \chi_\omega^2 |\varphi|^2 \, dxdt$$

holds true for every solution of the system $\Box\varphi + q(x)\varphi = 0$.

- See also the work by M.Leautaud and C. Laurent : $q \in L^\infty$, manifold without boundary.
- Well known under classical Γ-condition. Here, micro local (GCC) condition...
- Other (interesting) point : propagation of the H^1-wave front from ω.

Case of the 1st order part

Assume here that $V_{1,2}(t,x) = V_{1,2}(x) \in C^\sigma$, $\sigma > 0$ and $||V_{1,2}||_{C^\sigma} \leq M$.

Theorem

Under (GCC), to control the data $(u_0, u_1) \in H_0^1 \times L^2$ *to equilibrium* $(0,0)$ *one can use one of the two following control processes:*

$$\begin{cases} \Box u + V_1(x).\nabla_{t,x} u = \chi_\omega^2(x) f_{V_1} \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

or

$$\begin{cases} \Box v + V_2(x).\nabla_{t,x} u = \chi_\omega^2(x) f_{V_1} + \chi_\omega^2(x) L(u_0, u_1) \\ (v(0), \partial_t v(0)) = (u_0, u_1) \end{cases}$$

where

$$L : H_0^1 \times L^2 \longrightarrow L^2((0,T) \times \Omega)$$

is a bounded operator satisfying

$$||L|| \leq C(M) ||V_1 - V_2||_{C^\sigma}$$

Case of the metric

Denote $\Box_g = \partial_t^2 - \Delta_g$ and assume (ω, T) satisfies (GCC) for the metric g.

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Theorem

For any C^2- neighborhood $\mathcal{W}$ of g, there exist a metric $g' \in \mathcal{W}$ and an initial data (u_0, u_1) , $||(\nabla_g u_0, u_1)||_{L^2 \times L^2} = 1$, s.t the respective solutions u and v of

$$\begin{cases} \Box_g u = \chi_\omega^2(x) f_g \\ \Box_{g'} v = \chi_\omega^2(x) f_g \\ (u(0), \partial_t u(0)) = (v(0), \partial_t v(0)) = (u_0, u_1) \end{cases}$$

satisfy

$$E_g(u - v)(T) = E_g(v)(T) \geq 1/2$$

Moreover,

$$||f_g - f_{g'}||_{L^2((0,T)\times\Omega)} \geq 1/8\sqrt{T}$$

Control of smooth data

Assume (ω, T) satisfies (GCC) for the metric g.

Theorem

For any metric g' and any $\alpha \in]0,1]$, the respective solutions u and v of

$$\begin{cases} \Box_g u = \chi_\omega^2(x) f_g \\ \Box_{g'} v = \chi_\omega^2(x) f_g \\ (u(0), \partial_t u(0)) = (v(0), \partial_t v(0)) = (u_0, u_1) \in H^{1+\alpha} \times H^\alpha \end{cases}$$

satisfy

$$E_g^{1/2}(u - v)(T) \leq c_\alpha ||g - g'||_{C^1}^\alpha \times ||(u_0, u_1)||_{H^{1+\alpha} \times H^\alpha}$$

Observability of waves with rough coefficients

The Goal

Under suitable conditions on (ω, T), provide an observability estimate for the wave equation $\quad \Box u = 0$:

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt \qquad (O)$$

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Or at least

Observation with loss :

$$Eu(0) \leq c||u||^2_{H^m((0,T)\times\omega)}, \qquad m > 1 \qquad (OL)$$

State of the art

- 80' : Observability estimates under the Γ-condition of J.L. Lions.
  $\rightarrow$ C^1 coefficients.
  $\rightarrow$ Multiplier techniques.

- 90' : Microlocal conditions and microlocal tools (Bardos, Lebeau and Rauch).
  The couple (ω, T) satisfies the geometric control condition (G.C.C), i.e every geodesic of Ω issued at $t = 0$ and travelling with speed 1, enters in ω before the time T.

  $\rightarrow$ Microlocal and pseudo-differential techniques : propagation of wave front sets and supports of microlocal defect measures.
  $\rightarrow$ The condition is optimal but...... smooth enough coefficients.

- N. Burq 97' C^2 metric (Boundary observability).

The OL theorem of Fanelli-Zuazua (2014)

$$\Box u = a(x)\partial_t^2 u - \Delta_x u$$

1-D setting

$\Omega =]0,1[$ and $a(x)$ is resp. Zygmund, Log-Lipschitz or Log-Zygmund continuous function.

$$\begin{cases} |a(x+h) + a(x-h) - 2a(x)| \leq K|h| \\ \\ |a(x+h) - a(x)| \leq K|hLog(1+1/h)| \\ \\ |a(x+h) + a(x-h) - 2a(x)| \leq K|hLog(1+1/h)| \end{cases}$$

For $0 < \alpha < 1$,

$$Lip \subset Z \subset LL \subset LZ \subset C^\alpha$$

Theorem (Fanelli - Zuazua 1)

Assume $a(x) \in Z$, then for $T > T_a$, there exists $C > 0$ s.t

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Theorem (Fanelli - Zuazua 2)

Assume $a(x) \in LZ$ and denote $D_a f = a(x)^{-1}\partial_x^2 f$. Then for $T > T_a$, there exist $C > 0$ and $m \in \mathbb{N}$ s.t

$$Eu(0) \leq C \int_0^T |\partial_t^m \partial_x u(t,0)|^2 dt$$

for all initial data $(u_0, u_1) \in (H^{2m+1} \cap H_0^1) \times H^{2m}$ satisfying

$$D_a^m u_0 \in H^1 \qquad D_a^m u_1 \in L^2$$

Comments

- Classical boundary observation in Th. 1 and boundary observation with **loss** in Th. 2.
- For $a(x)$ worse that LZ, infinite loss of derivatives :
  No Observability !
  See also the counter-example of Castro - Zuazua (03').
- Proof: 1-dimensional technique: the sidewise energy estimates, i.e hyperbolic energy estimates by interchanging time $\longleftrightarrow$ space. (Colombini, Spagnolo, Lerner, Métivier, Fanelli....)
- 1-D geometry: all characteristic rays reach the boundary in uniform time.

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- 1-D geometry: all characteristic rays reach the boundary in uniform time.

- **Question**: What about dimensions higher than 1, where geometry is more evolved ???

The density $a(x)$ is continuous in $\mathbb{R}^d$

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$$

Assumption H: There exists $\alpha \in (0,2]$, s.t

$$
x\cdot\nabla a(x) + (2-\alpha)a(x) \geq 0 \quad \text{in the sense of } D'(\mathbb{R}^d).
$$

$$
\forall \varphi \in D(\mathbb{R}^d), \text{ with } \varphi \geq 0, \quad \int_{\mathbb{R}^d} a(x)\left(-div(x\varphi(x)) + (2-\alpha)\varphi(x)\right)\, dx \geq 0
$$

Observation region

ω an open neighborhood (in $\overline{\Omega}$) of an open subset Γ of the boundary satisfying the multiplier condition, i.e

$$\{x \in \partial\Omega, \quad \text{such that} \quad x \cdot n_x > 0\} \subset \Gamma,$$

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$$\{x \in \partial\Omega, \quad \text{such that} \quad x \cdot n_x > 0\} \subset \Gamma,$$

Let $R = sup\{|x|, x \in \Omega\}$ and $0 < a_0 \leq a(x) \leq a_1, \forall x \in \overline{\Omega}$.

Theorem (D-Ervedoza 2016)

For $\alpha T > 4R\sqrt{a_1}$, there exists a constant $C > 0$ s.t the observability estimate

$$Eu(0) \leq C \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt$$

holds true for every solution of (W).

Some insights on Assumption H

Assume here that

- $a(x)$ is smooth ($C^2(\overline{\Omega})$ is sufficient),
- Assumption H is satisfied pointwise.

Away from the boundary, a bicharacteristic ray $\gamma(s)$ associated to the wave operator of symbol $p(t,x;\tau,\xi) = a(x)\tau^2 - |\xi|^2$, and issued from $(t_0, x_0; \tau_0, \xi_0)$ satisfying $p(t_0, x_0; \tau_0, \xi_0) = 0$ and $x_0 \in \Omega$, is given by the ODE $\frac{d}{ds}\gamma(s) = H_p\gamma(s)$:

$$\begin{cases} \frac{dt}{ds} = 2\tau a(x), & \frac{d\tau}{ds} = 0, \\ \frac{dx}{ds} = -2\xi, & \frac{d\xi}{ds} = -\tau^2 \nabla a(x). \end{cases}$$

$$\frac{d^2}{ds^2}\left(|x(s)|^2\right) = 4\tau^2\left(x\cdot\nabla a(x) + 2a(x)\right) \geq 4\tau^2\alpha a_0,$$

$\rightarrow$ $s \mapsto |x(s)|^2$ is strictly convex.

$\rightarrow$ Non captive geometry !

$\rightarrow$ The (GCC) condition of Bardos-Lebeau -Rauch is satisfied, for instance, if ω is a neighborhood of the whole boundary $\partial\Omega$.

$\rightarrow$ $\gamma(s)$ enters in ω in finite time.

Examples

- One can take $a(x) = a(r,\theta) = f(r)g(\theta)$ with f, g positive continuous, and $f'(r) \geq 0$ in the sense of distributions.

$\rightarrow$ Allows highly oscillating function $g(\theta)$.

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- One can take $a(x) = a(r, \theta) = f(r)g(\theta)$ with f, g positive continuous, and $f'(r) \geq 0$ in the sense of distributions.

  $\rightarrow$ Allows highly oscillating function $g(\theta)$.

- $\Omega = B(0, R) \setminus B(0, R_1)$, $0 < R_1 < R$, and $a(x) = 1/r^2$.

  Take $x_0 \in \Omega$, $\xi_0 \neq 0$, $x_0.\xi_0 = 0$, and $\tau_0 = |x_0||\xi_0|$

  The ray $\gamma(s)$ issued from $(0, x_0, \tau_0, \xi_0)$ satisfies

$$\frac{d^2}{ds^2}\left(|x(s)|^2\right) = 0 \quad \text{and} \quad \frac{d}{ds}\left(|x(s)|^2\right)\Big|_{s=0} = 0$$

  $\rightarrow$ $|x(s)| = |x_0|$ Captive ray !

Sketch of the proof of Th. 1

For simplicity, we assume here that $0 \notin \overline{\Omega}$.

Consider the solution u_ε of the "regularized" system

$$(W_\varepsilon) \qquad \begin{cases} a_\varepsilon(x)\,\partial_t^2 u_\varepsilon - \Delta u_\varepsilon = 0 & \text{in} \quad]0,+\infty[\times\Omega \\ u_\varepsilon(t,.) = 0 & \text{on} \quad]0,+\infty[\times\partial\Omega \\ (u_\varepsilon(0), \partial_t u_\varepsilon(0)) = (u_0, u_1) \in H_0^1 \times L^2 & \end{cases}$$

where the smooth function a_ε satisfies Assumption H.

$\eta(x) \in C_0^\infty(\mathbb{R}^d)$, even, $supp(\eta) \subset B(0,1)$, $\int \eta(x)dx = 1$.

$$\forall x \in \overline{\Omega}, \quad a_\varepsilon(x) = (\varepsilon|x|)^{-d} \int_{\mathbb{R}^d} a(y)\eta\left(\frac{x-y}{\varepsilon|x|}\right) dy = \int_{\mathbb{R}^d} a(x - \varepsilon|x|z)\eta(z)\, dz,$$

- $lim_{\varepsilon \to 0} ||a_\varepsilon(x) - a(x)||_\infty = 0$
- $x \cdot \nabla a_\varepsilon(x) + (2-\alpha)a_\varepsilon(x) \geq 0, \qquad \forall x \in \overline{\Omega}$

Lemma

We have the following strong convergence :

$$\lim_{\varepsilon \to 0} ||u_\varepsilon - u||_{L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))} = 0.$$

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- $E[u_\varepsilon](0) \leq C \int_0^T \int_\omega \left(|\partial_t u_\varepsilon(t,x)|^2 + |\nabla_x u_\varepsilon(t,x)|^2 + |u_\varepsilon(t,x)|^2 \right) dxdt.$

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We have the following strong convergence :

$$\lim_{\varepsilon\to 0} ||u_\varepsilon - u||_{L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))} = 0.$$

→ Multiplier: $x.\nabla_x u_\varepsilon + \lambda u_\varepsilon$

- $E[u_\varepsilon](0) \leq C \int_0^T \int_\omega \left(|\partial_t u_\varepsilon(t,x)|^2 + |\nabla_x u_\varepsilon(t,x)|^2 + |u_\varepsilon(t,x)|^2\right) dxdt.$
- $E[u_\varepsilon](0) \leq C \int_0^T \int_\omega \left(|\partial_t u_\varepsilon(t,x)|^2 + |u_\varepsilon(t,x)|^2\right) dxdt.$
- $E[u](0) \leq C \int_0^T \int_\omega \left(|\partial_t u(t,x)|^2 + |u(t,x)|^2\right) dxdt.$
- $E[u](0) \leq C \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt$

Case of a non smooth metric

$$Pu = \partial_t^2 u - \nabla_x C(x) \nabla_x u$$

with $C(x)$ of class C^1 and $\nabla C(x)$ Log-Lipschitz.

The hamiltonian field H_p associated to the wave symbol

$$p(t, x; \tau, \xi) = \tau^2 - \xi^T C(x) \xi$$

in $T^*((0, T) \times M)$ still defines a bicharacteristic flow.

But, this time, there is a loss of regularity during evolution:

$$\phi_s - Id \in C^{exp(-cs)}$$

$$|m - m'| \leq 1 \Longrightarrow |\phi_s(m) - \phi_s(m')| \leq C|m - m'|^{exp(-cs)}$$

Theorem

Assume that $C(x) \in C^1$, $\nabla_x C \in LL$, and that $C(x)$ is smooth near the boundary. Then under (GCC), the following observability estimate holds true

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt$$

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Comments

- Microlocal assumption with a low regularity flow.
- Tools : propagation result for micro local defect measures.
- Similar result for observation from the boundary.

Sketch of the proof of Th.2

Strategy

$\rightarrow$ In the smooth case

Contradiction argument and propagation of micro local defect measures.

To be achieved Prove a propagation result for μ in a low regularity setting.

$\rightarrow$ Smooth approximation of the metric $C(x)$.