A sub-Riemannian modular approach for
diffeomorphic deformations.

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   - Definition
   - Modular large deformations
   - Sub-Riemannian structure on O
   - Study shape variability

3 Example: rigid and non-linear deformations

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Shape space

Definition (S. Arguillère)

\[ O \] is a \( C^k \)-shape space of order \( \ell \) on \( \mathbb{R}^d \) \(( d, \ell, k \in \mathbb{N}^* )\) if:
1. it is a manifold of finite dimension
2. \( \text{Diff}^\ell_0(\mathbb{R}^d) \subset \text{Id} + C^\ell_0(\mathbb{R}^d) \) continuously acts on \( O \)
3. For all \( o \in O \), \( \phi \in \text{Diff}^\ell_0(\mathbb{R}^d) \), \( \phi \cdot o \) is differentiable at \( \text{Id}_{\mathbb{R}^d} \), giving the infinitesimal action \( \xi : O \times C^\ell_0(\mathbb{R}^d) \to T_o O \).

Example of landmarks:
\[ O = \{ (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N \mid i \neq j \Rightarrow x_i \neq x_j \} \]
Definition (S. Arguillère)

$O$ is a $C^k$-shape space of order $\ell$ on $\mathbb{R}^d$ ($d, \ell, k \in \mathbb{N}^*$) if:

1. it is a manifold of finite dimension
2. $\text{Diff}^{\ell,0}(\mathbb{R}^d) \subset Id + C^{\ell,0}(\mathbb{R}^d)$ continuously acts on $O$
3. $\forall o \in O, \phi \in \text{Diff}^{\ell,0}(\mathbb{R}^d) \mapsto \phi \cdot o$ is differentiable at $Id_{\mathbb{R}^d}$, giving the infinitesimal action $\xi$:

$O \times C^{\ell,0}(\mathbb{R}^d) \mapsto T_o O$.

Example of landmarks: $O = \{(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N | i \neq j = \Rightarrow x_i \neq x_j\}$

$\xi_o(v) = (v(x_1), \ldots, v(x_N))$
Definition (S. Arguillère)

$\mathcal{O}$ is a $C^k$-shape space of order $\ell$ on $\mathbb{R}^d$ ($d, \ell, k \in \mathbb{N}^*$) if:

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\( \mathcal{O} \) is a \( \mathcal{C}^k \)-shape space of order \( \ell \) on \( \mathbb{R}^d \) (\( d, \ell, k \in \mathbb{N}^* \)) if:

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2. \( \text{Diff}^\ell_0(\mathbb{R}^d) \subset Id + \mathcal{C}^{\ell+k}_0(\mathbb{R}^d) \) continuously acts on \( \mathcal{O} \)
### Definition (S. Arguillère)

$\mathcal{O}$ is a $C^k$-shape space of order $\ell$ on $\mathbb{R}^d$ ($d, \ell, k \in \mathbb{N}^*$) if:

1. it is a manifold of finite dimension
2. $\text{Diff}^\ell_0(\mathbb{R}^d) \subset \text{Id} + C_0^{\ell+k}(\mathbb{R}^d)$ continuously acts on $\mathcal{O}$
3. $\forall o \in \mathcal{O}, \phi \in \text{Diff}^\ell_0(\mathbb{R}^d) \mapsto \phi \cdot o$ is differentiable at $\text{Id}_{\mathbb{R}^d}$, giving the infinitesimal action $\xi: \mathcal{O} \times C_0^\ell(\mathbb{R}^d) \mapsto T_o \mathcal{O}$.
Shape space

**Definition (S. Arguillère)**

$\mathcal{O}$ is a $C^k$-shape space of order $\ell$ on $\mathbb{R}^d$ ($d, \ell, k \in \mathbb{N}^*$) if:

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**Example of landmarks**:

$\mathcal{O} = \{(x_1, \cdots, x_N) \in (\mathbb{R}^d)^N \mid i \neq j \implies x_i \neq x_j\}$
**Introduction**

**Shape space**

**Definition (S. Arguillère)**

\( \mathcal{O} \) is a **\( C^k \)-shape space of order \( \ell \)** on \( \mathbb{R}^d \) \((d, \ell, k \in \mathbb{N}^*)\) if :

1. it is a manifold of finite dimension
2. \( \text{Diff}_0^\ell(\mathbb{R}^d) \subset \text{Id} + \mathcal{C}_0^{\ell+k}(\mathbb{R}^d) \) continuously acts on \( \mathcal{O} \)
3. \( \forall o \in \mathcal{O}, \ \phi \in \text{Diff}_0^\ell(\mathbb{R}^d) \mapsto \phi \cdot o \) is differentiable at \( \text{Id}_{\mathbb{R}^d} \), giving the **infinitesimal action** \( \xi : \mathcal{O} \times \mathcal{C}_0^\ell(\mathbb{R}^d) \mapsto T_o \mathcal{O} \).

**Example of landmarks** :

\[ \mathcal{O} = \{(x_1, \cdots, x_N) \in (\mathbb{R}^d)^N \mid i \neq j \implies x_i \neq x_j\} \]

\[ \text{if } o = (x_1, \cdots, x_N) \in \mathcal{O}, \ \xi_o(v) = \left(v(x_1), \cdots, v(x_N)\right) \]
Introduction
Diffeomorphic differences

D’Arcy Thompson (On Growth and Form, 1917)
For $v \in L^1([0,1], C^1_0(\mathbb{R}^d))$, we set $\varphi^v$ the flow of $v$:

\[
\begin{align*}
\dot{\varphi}^v(t) &= v(t) \circ \varphi^v(t) \\
\varphi^v(0) &= \text{Id}
\end{align*}
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**Proposition (S. Arguillère)**

Let $v \in L^1([0, 1], C^0_0(\mathbb{R}^d))$, $a \in \mathcal{O}$. Then $o : t \in [0, 1] \mapsto o(t) = \varphi^v(t) \cdot a$
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**Proposition (S. Arguillère)**

Let $v \in L^1([0, 1], C^1_0(\mathbb{R}^d))$, $a \in \mathcal{O}$. Then $o : t \in [0, 1] \mapsto o(t) = \varphi^v(t) \cdot a$ satisfies $\dot{o} = \xi_o(v)$ a.e..
Figure: Source: *Diffeomorphometry and geodesic positioning systems for human anatomy*, Miller et al, Technology 2014.

- **LDDMM** [M. I. Miller, L. Younes, and A. Trouvé. Diffeomorphometry and geodesic positioning systems for human anatomy, 2014]

- **Higher-order momentum** [S. Sommer M. Nielsen, F. Lauze, and X. Pennec. Higher-order momentum distributions and locally affine lddmm registration. SIAM Journal on Imaging Sciences, 2013]

Parametric models to model non linear patterns:

- **Poly-affine** [C. Seiler, X. Pennec, and M. Reyes. Capturing the multiscale anatomical shape variability with polyaffine transformation trees. Medical image analysis, 2012]
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2 Deformation modules
   - Definition
   - Modular large deformations
   - Sub-Riemannian structure on $\mathcal{O}$
   - Study shape variability

3 Example : rigid and non-linear deformations

4 Conclusion
A deformation module can generate vector fields:
A deformation module can generate vector fields:
- Of a particular type
A deformation module can generate vector fields:

- Of a particular type
- Parametrized by a geometrical component and a control variable
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Deformation modules
Definition and first examples

\[ M = (O, H, \zeta, \xi, c) \]
Deformation modules
Definition and first examples

\[ M = (\mathcal{O}, H, \zeta, \xi, c) \]
Deformation modules
Definition and first examples

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Deformation modules
Definition and first examples

\[ M = (\mathcal{O}, H, \zeta, \xi, c) \]

Diagram:
- Controls: \( H \)
- Geometrical descriptors: \( \mathcal{O} \)
- Infinitesimal action: \( \xi \)
- Cost: \( c \)
- Field generator: \( \zeta \)
- Space of smooth functions: \( C^\ell_0(\mathbb{R}^d) \)
Deformation modules
Definition and first examples

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Definition and first examples

\[ M = (\mathcal{O}, H, \zeta, \xi, c) \]
Deformation modules
Definition and first examples: local scaling of scale $\sigma$

Example of generated vector field
Deformation modules
Definition and first examples: local scaling of scale $\sigma$

Diagram:
- $\mathbb{R}$ (Controls)
- $\mathbb{R}^2$ (Geometrical descriptors)
- $\nu(\partial)$ (Infinitesimal action)
- $\mathbb{R}$ (Field generator)
- $C^\ell_0(\mathbb{R}^2)$
- $h^2$ (Cost)

Arrows indicate the flow of information or actions between these elements.
Deformation modules
Definition and first examples: local translation of scale $\sigma$

Example of generated vector field
Deformation modules
Definition and first examples: local translation of scale $\sigma$
Deformation modules

- Constraints on the deformation model
Deformation modules

- Constraints on the deformation model
- More complicated constraints?
Deformation modules

- Constraints on the deformation model
- More complicated constraints?
  → Combine deformation modules
Deformation modules

Combination

\[ \mathbb{R}^+ \]

\[ H_1 \]

\[ O_1 \times H_1 \]

\[ \xi^1 \text{ infinitesimal action} \]

\[ T O_1 \]

\[ \zeta^1 \text{ cost} \]

\[ c_1 \]

\[ C_0^\ell(\mathbb{R}^d) \]

\[ \mathbb{R}^+ \]

\[ H_2 \]

\[ O_2 \times H_2 \]

\[ \xi^2 \text{ infinitesimal action} \]

\[ T O_2 \]

\[ \zeta^2 \text{ cost} \]

\[ c_2 \]

\[ C_0^\ell(\mathbb{R}^d) \]

\[ \mathbb{R}^+ \]

\[ H_3 \]

\[ O_3 \times H_3 \]

\[ \xi^3 \text{ infinitesimal action} \]

\[ T O_3 \]

\[ \zeta^3 \text{ cost} \]

\[ c_3 \]

\[ C_0^\ell(\mathbb{R}^d) \]
Deformation modules

Combination

\[ C(M^l, l = 1 \cdots L) \]

\[ \mathbb{R}^+ \]

\[ H_1 \times H_2 \times H_3 \]

\[ O_1 \times O_2 \times O_3 \times H_1 \times H_2 \times H_3 \]

\[ \xi_o(v) = (\xi_{o1}^1(v), \xi_{o2}^2(v), \xi_{o3}^3(v)) \]

\[ T\mathcal{O}_1 \times T\mathcal{O}_2 \times T\mathcal{O}_3 \]

\[ \zeta_o(h) = \sum_i \zeta_{oi}^i(h_i) \]
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Definition

Let $M = (\mathcal{O}, H, \zeta, \xi, c)$ be a $C^k$-deformation module of order $\ell$. We say that $M$ satisfies the **Uniform Embedding Condition (UEC)** if there exists a Hilbert space of vector fields $V$ continuously embedded in $C^{\ell+k}_0(\mathbb{R}^d)$ and a constant $C > 0$ such that for all $o \in \mathcal{O}$ and for all $h \in H$, $\zeta_o(h) \in V$ and

$$|\zeta_o(h)|_V^2 \leq Cc_o(h)$$
Modular large deformations
Uniform Embedding Condition

**Definition**

Let $M = (\mathcal{O}, H, \zeta, \xi, c)$ be a $C^k$-deformation module of order $\ell$. We say that $M$ satisfies the **Uniform Embedding Condition (UEC)** if there exists a Hilbert space of vector fields $V$ continuously embedded in $C^{\ell+k}_0(\mathbb{R}^d)$ and a constant $C > 0$ such that for all $o \in \mathcal{O}$ and for all $h \in H$, $\zeta_o(h) \in V$ and

$$|\zeta_o(h)|_V^2 \leq Cc_o(h)$$

**Proposition**

If $M^l$, $l = 1 \cdots L$, are $C^k$-deformation modules of order $\ell$ that satisfy UEC, then $\mathcal{C}(M^l, l = 1 \cdots L)$ satisfies UEC.
Modular large deformations
A deformation module

\[ M = (\mathcal{O}, H, \zeta, \xi, c) \]
Modular large deformations
From a deformation module to a deformation model

Definition (Finite energy controled paths on $\mathcal{O}$)

We denote $\Omega$ the set of measurable curves $t \mapsto (o_t, h_t) \in \mathcal{O} \times H$ such that:

$$\text{Energy } E(o_t, h_t) = \int_0^1 c_{o_t}(h_t) \, dt < \infty$$

where $v_t = \gamma_{o_t}(h_t) \in \gamma_{o_t}(H)$.
Modular large deformations
From a deformation module to a deformation model

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- Energy $E(o, h) = \int_0^1 c_{o_t}(h_t) dt < \infty$
- $\dot{o}_t = \xi_{o_t}(v_t)$ where $v_t = \zeta_{o_t}(h_t) \in \zeta_{o_t}(H)$
Modular large deformations
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Definition (Finite energy controlled paths on $\mathcal{O}$)

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Proposition (Modular large deformations)

We suppose $M$ satisfies UEC. If $(o, h) \in \Omega$ and $v = \zeta_o(h)$, then flow $\varphi^v$ exists and $o_{t=1} = \varphi^v_{t=1} \cdot o_{t=0}$. 
Modular large deformations
An Example
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Diffeomorphic differences

D’Arcy Thompson (On Growth and Form, 1917)
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Sub-Riemannian structure on $\mathcal{O}$

**Proposition**

Wet set $\rho : (o, h) \in \mathcal{O} \times H \mapsto (o, \xi \circ \zeta_o(h)) \in T\mathcal{O}$. Then $(\mathcal{O} \times H, c, \rho)$ defines a sub-Riemannian structure on $\mathcal{O}$.
Sub-Riemannian structure on $\mathcal{O}$

**Proposition**

Wet set $\rho : (o, h) \in \mathcal{O} \times H \mapsto (o, \xi \circ \zeta_o(h)) \in T\mathcal{O}$. Then $(\mathcal{O} \times H, c, \rho)$ defines a sub-Riemannian structure on $\mathcal{O}$ and

$$\text{Dist}(a, b)^2 = \inf \{ \int_0^1 c_o(h) \mid h \in L^2([0, 1], H), \dot{o} = \rho_o(h), \quad o_{t=0} = a, o_{t=1} = b \}$$
Sub-Riemannian structure on $\mathcal{O}$

**Proposition**

Wet set $\rho : (o, h) \in \mathcal{O} \times H \mapsto (o, \xi \circ \zeta_o(h)) \in T\mathcal{O}$. Then $(\mathcal{O} \times H, c, \rho)$ defines a sub-Riemannian structure on $\mathcal{O}$ and

$$\text{Dist}(a, b)^2 = \inf \left\{ \int_0^1 c_o(h) \mid h \in L^2([0, 1], H), \dot{o} = \rho_o(h), o_{t=0} = a, o_{t=1} = b \right\}$$

**Theorem**

If $\text{Dist}(a, b) < \infty$ the energy $E$, there exists $(o, h) \in \Omega$ such that $o_{t=0} = a, o_{t=1} = b$ and $\text{Dist}(a, b) = \sqrt{\int_0^1 c_o(h)}$. 
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Study shape variability

Atlas problem

\[ E(S, (v_k), (T_k)) = \sum_{k} \text{Dist}(S, \phi_{v_k t=1} \cdot S) + \frac{1}{\sigma^2} \mu_{(\phi_{v_k t=1} \cdot S, T_k)} \]
Study shape variability

Atlas problem
Study shape variability

Atlas problem

\[ E(S, (v^k)_k, (T_k)_k) = \sum_k \text{Dist}(S, \varphi_{t=1}^v \cdot S)^2 + \frac{1}{\sigma^2} \mu(\varphi_{t=1}^v \cdot S, T_k) \]
Goal: Study $T_1, \cdots T_N \in \mathcal{O}$
Goal: Study $T_1, \ldots T_N \in O$

$$\text{Dist}(a, b)^2 = \inf \left\{ \int_0^1 c_o(h) \mid h \in L^2([0, 1], H), \ 
\dot{o} = \rho_o(h), o_{t=0} = a, o_{t=1} = b \right\}$$
Goal: Study $T_1, \cdots, T_N \in \mathcal{O}$

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\text{Dist}(a, b)^2 = \inf \left\{ \int_0^1 c_0(h) \mid h \in L^2([0, 1], H), \dot{o} = \rho_0(h), o_{t=0} = a, o_{t=1} = b \right\}
\]

Minimize:

\[
E(S, (h^k)_k, (T_k)_k) = \sum_k \int_0^1 c_{o^k}(h^k) + \frac{1}{\sigma^2} \mu(\varphi_{t=1}^{\zeta_{o^k}(h^k)} \cdot S, T_k)
\]
Study shape variability
Atlas problem

Goal: Study $T_1, \cdots, T_N \in \mathcal{O}$

$$\text{Dist}(a, b)^2 = \inf \left\{ \int_0^1 c_\circ(h) \mid h \in L^2([0, 1], H), \dot{o} = \rho_o(h), o_{t=0} = a, o_{t=1} = b \right\}$$

Minimize:

$$E(S, (h^k)_k, (T_k)_k) = \sum_k \int_0^1 c_{\circ k}(h^k) + \frac{1}{\sigma^2} \mu(\varphi_{t=1}^{\zeta_{\circ k}(h^k)} \cdot S, T_k)$$

with $o_{t=0}^k = S$, $\dot{o}_t^k = \xi_{\circ k} \circ \zeta_{\circ k}(h^k)$. 
Goal:
Goal:

- Study $T_1, \ldots, T_N \in \mathcal{F}$
Study shape variability
Atlas problem in practice

Goal:

- Study $T_1, \cdots, T_N \in \mathcal{F}$
- Thanks to a user-defined deformation module $M^1 = (\mathcal{O}^1, H^1, \zeta^1, \xi^1, c^1)$
Goal:

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We build:
- $M^2 = (\mathcal{F}, \{0\}, \zeta^2, \xi^2, c^2) = $ Silent deformation module induced by $\mathcal{F}$
Goal:

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- Thanks to a user-defined deformation module
  
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We build:

- $M^2 = (\mathcal{F}, \{0\}, \zeta^2, \xi^2, c^2) = \text{Silent deformation module induced by } \mathcal{F}: \zeta^2 = 0$
Study shape variability
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- $M^2 = (\mathcal{F}, \{0\}, \zeta^2, \xi^2, c^2) = \text{Silent deformation module induced by } \mathcal{F}: \zeta^2 = 0, \xi^2 = \xi_\mathcal{F}, c^2 = 0$
- $M = C(M^1, M^2) = (O^1 \times \mathcal{F}, H^1 \times \{0\}, \zeta, \xi, c) :$
Goal:

- Study \( T_1, \cdots, T_N \in \mathcal{F} \)
- Thanks to a user-defined deformation module
  \[ M^1 = (\mathcal{O}^1, H^1, \zeta^1, \xi^1, c^1) \]

We build :

- \[ M^2 = (\mathcal{F}, \{0\}, \zeta^2, \xi^2, c^2) = \text{Silent deformation module induced by} \]
  \( \mathcal{F} : \zeta^2 = 0, \xi^2 = \xi_\mathcal{F}, c^2 = 0 \)
- \[ M = C(M^1, M^2) = (\mathcal{O}^1 \times \mathcal{F}, H^1 \times \{0\}, \zeta, \xi, c) : \]
  \[ \zeta_o(h) = \zeta^{1}_{h^1}(o^1) + \zeta^{2}_{h^2}(o^2) \]
Study shape variability
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- $M = C(M^1, M^2) = (\mathcal{O}^1 \times \mathcal{F}, H^1 \times \{0\}, \zeta, \xi, c)$:
  - $\zeta_o(h) = \zeta_{h^1}(o^1) + \zeta_{h^2}(o^2) = \zeta_{h^1}(o^1), \xi_o(v) = (\xi_{h^1}o_1(v), \xi_{h^2}f(v)),$
Goal:

- Study $T_1, \cdots, T_N \in \mathcal{F}$
- Thanks to a user-defined deformation module $M^1 = (O^1, H^1, \zeta^1, \xi^1, c^1)$

We build:

- $M^2 = (\mathcal{F}, \{0\}, \zeta^2, \xi^2, c^2) = $ Silent deformation module induced by $\mathcal{F}$: $\zeta^2 = 0$, $\xi^2 = \xi_F$, $c^2 = 0$
- $M = C(M^1, M^2) = (O^1 \times \mathcal{F}, H^1 \times \{0\}, \zeta, \xi, c)$:
  - $\zeta_o(h) = \zeta_{h^1}(o^1) + \zeta_{h^2}(o^2) = \zeta_{h^1}(o^1)$,
  - $\xi_o(v) = (\xi_{o^1}(v), \xi_{f^2}(v))$, 
Study shape variability
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Goal:
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We build:
- $M^2 = (\mathcal{F}, \{0\}, \zeta^2, \xi^2, c^2)$ = Silent deformation module induced by $\mathcal{F}$: $\zeta^2 = 0, \xi^2 = \xi_\mathcal{F}, c^2 = 0$
- $M = C(M^1, M^2) = (\mathcal{O}^1 \times \mathcal{F}, H^1 \times \{0\}, \zeta, \xi, c)$:
  - $\zeta_\circ(h) = \zeta_{h^1}^1(\circ^1) + \zeta_{h^2}^2(\circ^2) = \zeta_{h^1}^1(\circ^1)$,
  - $\xi_\circ(v) = (\xi_{o1}^1(v), \xi_f^2(v))$,  
  - $\rho_\circ(h) = \xi_\circ \circ \zeta_\circ(h) = (\xi_{o1}^1 \circ \zeta_{o1}^1(h^1), \xi_f^2 \circ \zeta_{o1}^1(h^1))$,  

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Study shape variability
Atlas problem in practice

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Distance on $\mathcal{O} = \mathcal{O}^1 \times \mathcal{F}$:

$$D(a, b)^2 = \inf \{ \int_0^1 c_0(h) \mid \dot{o} = \xi_o \circ \zeta_0(h), o_{t=0} = a, o_{t=1} = b \}$$
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the Hamiltonian of the system, \(\eta^k_{t=1} = -\partial_o \mu(o^k_{t=1}, T_k)\) and

\[
\begin{cases}
\frac{d o^k}{dt} = \xi_{o^k} \circ \zeta_{o^k}(h^k) \\
\frac{d \eta^k}{dt} = -\frac{\partial \mathcal{H}}{\partial o}(o^k, \eta^k, h^k) \\
0 = \frac{\partial \mathcal{H}}{\partial h}(o^k, \eta^k, h^k)
\end{cases}
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- Gradient descent on $(o_{\text{temp}}, (\eta^k_{t=0})_k)$
1. Introduction

2. Deformation modules
   - Definition
   - Modular large deformations
   - Sub-Riemannian structure on \( \mathcal{O} \)
   - Study shape variability

3. Example: rigid and non-linear deformations

4. Conclusion
Example: rigid and non-linear deformations

Targets
Example: rigid and non-linear deformations

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With previous rigid registration
Example: rigid and non-linear deformations
Before optimisation
Example: rigid and non-linear deformations

After optimisation
Example: rigid and non-linear deformations

Example of trajectory
Example: rigid and non-linear deformations

Example of trajectory
Sommaire

1 Introduction

2 Deformation modules
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4 Conclusion
We have presented a coherent mathematical framework to build modular large deformations. We showed how easily incorporating constraints in a deformation model and merging different constraints in a global one.
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[B. Gris, S. Durrleman, A. Trouvé. A sub-Riemannian modular framework for diffeomorphism based analysis of shape ensembles, 2016]
Sub-space of "meaningful" $\eta$ ?
Issues

- Sub-space of "meaningful" \( \eta \) ?
- Cancelling one module ?
Thank you for your attention!