On Feedback Solutions to Noncooperative Differential Games

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Outline

- **Nash equilibria for differential games**
  - feedback strategies in finite time horizon (with Wen Shen)
  - feedback strategies in infinite time horizon, with exponential discount (with Khai Nguyen)
- **Stackelberg equilibria for stochastic games**
  - feedback strategies in infinite time horizon (with Yilun Jiang)

Differential game

Dynamics: $$\dot{x}(t) = f(x(t), u_1(t), u_2(t)), \qquad x(t_0) = x_0$$

$$x(t) = \text{state of the system}$$

$$u_1(\cdot),\ u_2(\cdot) = \text{controls implemented by the two players}$$

Payoffs: $$J_i \doteq \psi_i\big(x(T)\big) - \int_{t_0}^{T} L_i\big(x(t),\ u_1(t),\ u_2(t)\big)\,dt, \qquad i = 1, 2$$

$$= \text{[terminal payoff] - [running cost]}$$

Nash equilibria

Seek: **feedback strategies** $u_1^*(t,x)$, $u_2^*(t,x)$ with the following properties

- Given the feedback control $u_2 = u_2^*(t,x)$ adopted by the second player, for every initial data $x(t_0) = x_0$ the assignment $u_1 = u_1^*(t,x)$ provides a solution to the **optimal control problem for the first player:**

Dynamics: $$\dot{x} = f(x, u_1, u_2^*(t,x)), \qquad x(t_0) = x_0$$

Payoff: $$J_1 = \psi_1(x(T)) - \int_{t_0}^{T} L_1\big(x(t),\ u_1(t),\ u_2^*(t, x(t))\big)\, dt$$

- Similarly, given the strategy $u_1 = u_1^*(t,x)$ adopted by the first player, the feedback control $u_2 = u_2^*(t,x)$ provides a solution to the optimal control problem for the second player.

A system of H-J equations for the value functions

$V_1, V_2 =$ value functions for the two players

$$\begin{cases} V_{1,t} + \max_\omega \Big\{ \nabla_x V_1(t,x) \cdot f(x, \omega, u_2^*) - L_1(x, \omega, u_2^*) \Big\} & = & 0 \\ V_{2,t} + \max_\omega \Big\{ \nabla_x V_2(t,x) \cdot f(x, u_1^*, \omega) - L_2(x, u_1^*, \omega) \Big\} & = & 0 \end{cases} \qquad (HJ)$$

Terminal conditions:

$$\begin{cases} V_1(T,x) & = & \psi_1(x) \\ V_2(T,x) & = & \psi_2(x) \end{cases}$$

$$u_1^*(t,x) = \operatorname*{argmax}_\omega \Big\{ \nabla_x V_1(t,x) \cdot f(x, \omega, u_2^*) - L_1(x, \omega, u_2^*) \Big\}$$

$$u_2^*(t,x) = \operatorname*{argmax}_\omega \Big\{ \nabla_x V_2(t,x) \cdot f(x, u_1^*, \omega) - L_2(x, u_1^*, \omega) \Big\}$$

system of first order PDEs, highly nonlinear !

Explicit solutions: for Linear-Quadratic games

Dynamics: $\dot{x}(t) = Ax(t) + \mathbf{b}_1 u_1(t) + \mathbf{b}_2 u_2(t) \qquad x(t_0) = x_0$

Payoffs: $J_i \doteq \mathbf{p}_i \cdot x(T) + x(T)^T Q_i x(T) - \int_{t_0}^{T} \left[x(t)^T R_i x(t) + |u_i(t)|^2 \right] dt$

Value functions:

$$V_i(t,x) = x^T \Gamma_i(t) x + \beta_i(t) \cdot x + k_i(t) \qquad i = 1,2 \qquad (*)$$

To find this solution, it suffices to determine the coefficients $\Gamma_i(t)$, $\beta_i(t)$, $k_i(t)$, by solving a system of ODEs

Optimal controls: $u_i^*(t,x) = -\left(\beta_i(t) + 2x^T \Gamma_i(t)\right) \cdot \mathbf{b}_i$

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A. J. Weeren, J. M. Schumacher, and J. Engwerda, Asymptotic analysis of linear feedback Nash equilibria in nonzero-sum linear-quadratic differential games. *J. Optim. Theory Appl.* **101** (1999), 693–723.

J. Engwerda, Feedback Nash equilibria in the scalar infinite horizon LQ-game *Automatica* **36** (2000), 135–139.

Is the Linear-Quadratic approximation justified? Is the system (HJ) of nonlinear PDEs well posed?

Let $V = (V_1, V_2)$ be a smooth solution of the system (HJ), and let

$$V^\varepsilon(t,x) \;=\; V(t,x) + \varepsilon\, Z(t,x) + o(\varepsilon) \qquad\qquad x \in \mathbb{R}^n$$

describe a small perturbation. How does Z behave?

$$\begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} + \sum_{\alpha=1}^{n} A^\alpha \begin{pmatrix} Z_{1,x_\alpha} \\ Z_{2,x_\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A^\alpha \;=\; \begin{pmatrix} f_\alpha + \left(\nabla V_1 \cdot \dfrac{\partial f}{\partial u_2} - \dfrac{\partial L_1}{\partial u_2}\right) \dfrac{\partial u_2^*}{\partial V_{1,x_\alpha}} & \left(\nabla V_1 \cdot \dfrac{\partial f}{\partial u_2} - \dfrac{\partial L_1}{\partial u_2}\right) \dfrac{\partial u_2^*}{\partial V_{2,x_\alpha}} \\ \left(\nabla V_2 \cdot \dfrac{\partial f}{\partial u_1} - \dfrac{\partial L_2}{\partial u_1}\right) \dfrac{\partial u_1^*}{\partial V_{1,x_\alpha}} & f_\alpha + \left(\nabla V_2 \cdot \dfrac{\partial f}{\partial u_1} - \dfrac{\partial L_2}{\partial u_1}\right) \dfrac{\partial u_1^*}{\partial V_{2,x_\alpha}} \end{pmatrix}$$

Hyperbolic systems

The Cauchy problem for the system with constant coefficients

$$\begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} + \sum_{\alpha=1}^{n} A^\alpha \begin{pmatrix} Z_{1,x_\alpha} \\ Z_{2,x_\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is well posed only if the system is **hyperbolic**. That means:

For every $\xi = (\xi_1, \ldots, \xi_n)$, *the matrix* $A(\xi) = \sum_{\alpha=1}^n A^\alpha \xi_\alpha$ *has real eigenvalues.*

- Compute the solution by Fourier transform
- Stability in $\mathbf{L}^2(\mathbb{R}^n)$ $\Longrightarrow$ hyperbolicity

An example

$$\text{dynamics:} \qquad \dot{x} \;=\; f(x, u_1, u_2) \;=\; f_0(x) + f_1(x)u_1 + f_2(x)u_2$$

$$\text{payoffs:} \quad J_i \;=\; \psi_i(x(T)) - \int_{t_0}^{T} \frac{1}{2} u_i^2(t)\,dt$$

$$\text{optimal feedback controls:} \quad u_i^*(t,x) \;=\; \nabla V_i(t,x) \cdot f_i(x)$$

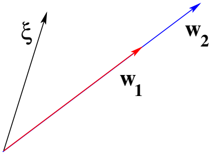
$$A(\xi) \;=\; \begin{pmatrix} f \cdot \xi & (\nabla V_1 \cdot f_2)\,(f_2 \cdot \xi) \\ (\nabla V_2 \cdot f_1)(f_1 \cdot \xi) & f \cdot \xi \end{pmatrix}$$

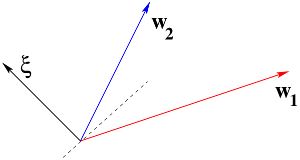
$$\text{eigenvalues are real} \qquad \Longleftrightarrow \qquad (\nabla V_1 \cdot f_2)(f_2 \cdot \xi)\,(\nabla V_2 \cdot f_1)(f_1 \cdot \xi) \;\geq\; 0$$

$$\text{eigenvalues are real} \quad \Longleftrightarrow \quad (\mathbf{w}_1 \cdot \xi)(\mathbf{w}_2 \cdot \xi) \geq 0$$

$$\mathbf{w}_1 = (\nabla V_2 \cdot f_1) f_1 \qquad \mathbf{w}_2 = (\nabla V_1 \cdot f_2) f_2$$

Cannot hold for all ξ, unless $\mathbf{w}_1, \mathbf{w}_2$ are parallel, same orientation





Possible in dimension 1, when $\nabla V_1 \cdot \nabla V_2 \geq 0$

- In one space dimension, the Cauchy Problem can be well posed for a large set of data.
- In several space dimensions, generically the system is not hyperbolic, and the Cauchy Problem is ill posed

$$\text{similar to:} \quad u_t = -u_{xx}$$

$$u(0,x) = ax^2 + bx + c \quad \Longrightarrow \quad u(t,x) = ax^2 + bx + c - 2at$$

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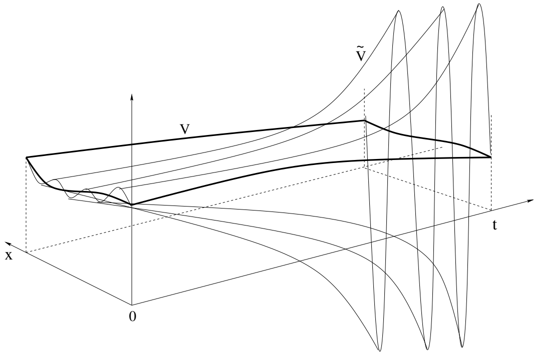


For non-cooperative differential games in finite time horizon

linear - quadratic approximations are hardly justified !

What happens for games in infinite time horizon?

Differential games in infinite time horizon

Dynamics: $$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2\,, \qquad x(0) = x_0$$

cost functionals: $$J_i \doteq \int_0^{+\infty} e^{-\gamma t}\Big(\phi_i(x) + \frac{u_i^2}{2}\Big)\,dt$$

(running cost, exponentially discounted in time)

A system of PDEs for the value functions

The value functions V_1, V_2 for the two players satisfy the system of H-J equations

$$\begin{cases} \gamma V_1 = (f \cdot \nabla V_1) - \frac{1}{2}(g_1 \cdot \nabla V_1)^2 - (g_2 \cdot \nabla V_1)(g_2 \cdot \nabla V_2) + \phi_1 \\ \gamma V_2 = (f \cdot \nabla V_2) - \frac{1}{2}(g_2 \cdot \nabla V_2)^2 - (g_1 \cdot \nabla V_1)(g_1 \cdot \nabla V_2) + \phi_2 \end{cases}$$

Optimal feedback controls: $u_i^*(x) = -\nabla V_i(x) \cdot g_i(x) \qquad i = 1, 2$

nonlinear, implicit !

Linear - Quadratic games

Assume that the dynamics is linear:

$$\dot{x} \;=\; (Ax + \mathbf{b}_0) + \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2\,, \qquad\qquad x(0) = y$$

and the cost functions are quadratic:

$$J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\Big(\mathbf{a}_i \cdot x + x^T P_i x + \frac{u_i^2}{2}\Big)\,dt$$

Then the system of PDEs has a special solution of the form

$$V_i(x) \;=\; x^T \Gamma_i x + \beta_i \cdot x + k_i \qquad\qquad i = 1,2 \qquad\qquad (*)$$

$$\text{optimal controls:}\quad u_i^*(x) \;=\; -\,(\beta_i + 2x^T\Gamma_i)\cdot \mathbf{b}_i$$

To find this solution, it suffices to determine the coefficients k_i, β_i, Γ_i by solving a system of algebraic equations

Validity of linear-quadratic approximations ?

Assume the dynamics is almost linear

$$\dot{x} \;=\; f_0(x)+g_1(x)u_1+g_2(x)u_2 \;\approx\; (Ax+\mathbf{b}_0)+\mathbf{b}_1 u_1+\mathbf{b}_2 u_2\,, \qquad\qquad x(0)=y$$

and the cost functions are almost quadratic

$$J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\Big(\phi_i(x)+\frac{u_i^2}{2}\Big)\,dt \;\approx\; \int_0^{+\infty} e^{-\gamma t}\Big(\mathbf{a}_i\cdot x+x^T P_i x+\frac{u_i^2}{2}\Big)\,dt$$

Is it true that the nonlinear game has a feedback solution close to the linear-quadratic game?

$$\dot{x} \;=\; (a_0 x + b_0) + b_1 u_1 + b_2 u_2$$

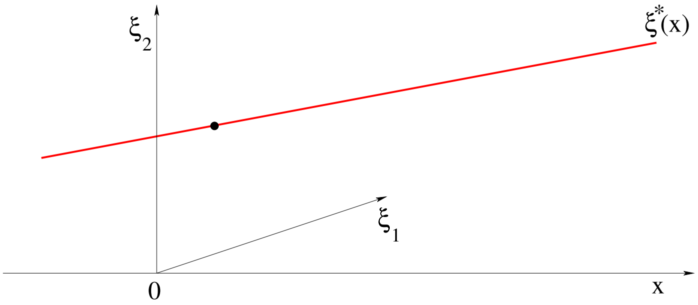
The ODE for the derivatives of the value functions $\xi_i = V_i'$ takes the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} \;=\; \begin{pmatrix} \psi_1(x,\xi_1,\xi_2) \\ \psi_2(x,\xi_1,\xi_2) \end{pmatrix}, \qquad\qquad A_{ij} = A_{ij}(x,\xi_1,\xi_2)$$

- The map $(x,\xi_1,\xi_2) \;\mapsto\; \det A(x,\xi_1,\xi_2)$ is a homogeneous quadratic polynomial
- An affine solution exists: $\xi_1^*(x) \;=\; k_1 x + \beta_1\,, \qquad \xi_2^*(x) \;=\; k_2 x + \beta_2$

Is this solution stable w.r.t. nonlinear perturbations?

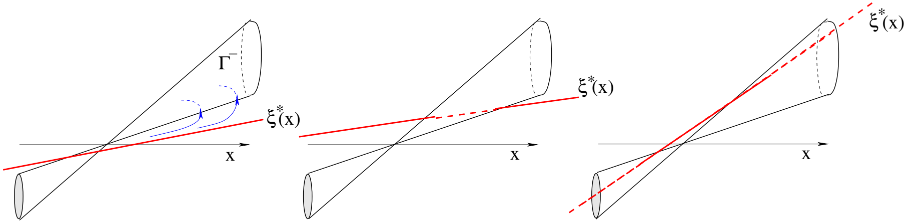
Where are all the other solutions of the ODE ?



Stability w.r.t. perturbations (A.B. - Khai Nguyen, 2016)

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}$$

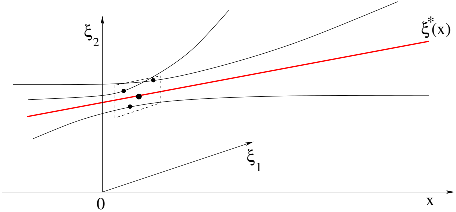
Three main cases:



$$\Gamma^- = \Big\{ (x, \xi_1, \xi_2) \,; \quad \det A(x, \xi_1, \xi_2) \le 0 \Big\}$$

$$\begin{pmatrix} \xi_1' \\ \\ \xi_2' \end{pmatrix} = A^{-1}(x, \xi_1, \xi_2) \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}$$

- The linear-quadratic game has a 2-parameter family of Nash equilibrium solutions in feedback form. One is affine, the other are nonlinear.
- All of the above solutions are stable w.r.t. small nonlinear perturbations of the dynamics and the cost functions.



Case 2: det $A(x, \xi_1^*(x), \xi_2^*(x))$ vanishes at two points $\bar{x}_1 < \bar{x}_2$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}$$

equivalent Pfaffian system: $$\begin{cases} -\psi_1\, dx + A_{11}\, d\xi_1 + A_{12}\, d\xi_2 = 0 \\ -\psi_2\, dx + A_{21}\, d\xi_1 + A_{22}\, d\xi_2 = 0 \end{cases}$$

Setting: $$\mathbf{v} \doteq \begin{pmatrix} -\psi_1 \\ A_{11} \\ A_{12} \end{pmatrix}, \qquad \mathbf{w} \doteq \begin{pmatrix} -\psi_2 \\ A_{21} \\ A_{22} \end{pmatrix},$$

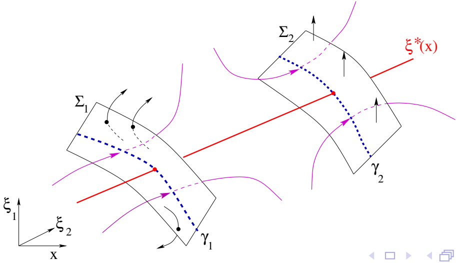
we seek continuously differentiable functions $x \mapsto (\xi_1(x),\ \xi_2(x))$ whose graph is obtained by concatenating trajectories of the system

$$\begin{pmatrix} \dot{x} \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \mathbf{v} \times \mathbf{w} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} \\ A_{22}\psi_1 - A_{12}\psi_2 \\ A_{11}\psi_2 - A_{21}\psi_1 \end{pmatrix}.$$

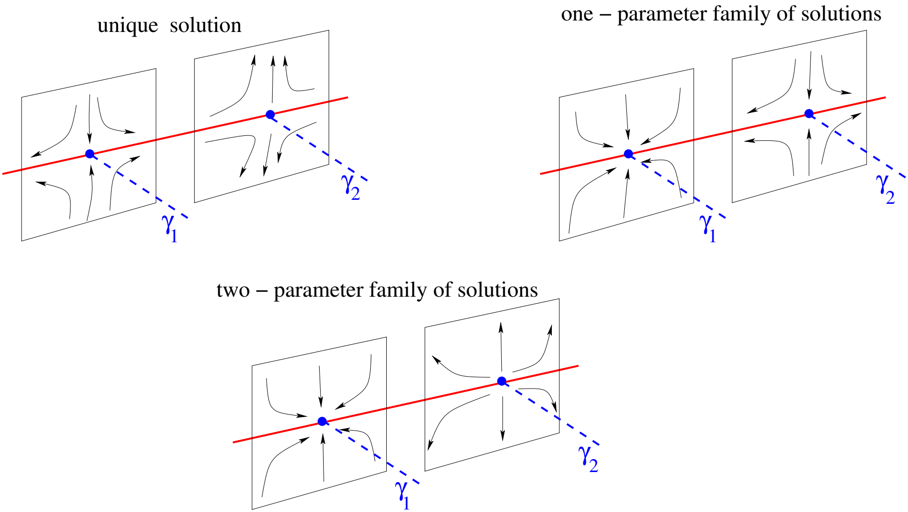
$$\begin{pmatrix} \dot{x} \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \mathbf{v} \times \mathbf{w} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} \\ A_{22}\psi_1 - A_{12}\psi_2 \\ A_{11}\psi_2 - A_{21}\psi_1 \end{pmatrix}.$$

Under generic conditions on the coefficients of the linear-quadratic problem, there exists two surfaces Σ_1, Σ_2 and two curves $\gamma_1 \subset \Sigma_1$, $\gamma_2 \subset \Sigma_2$ such that

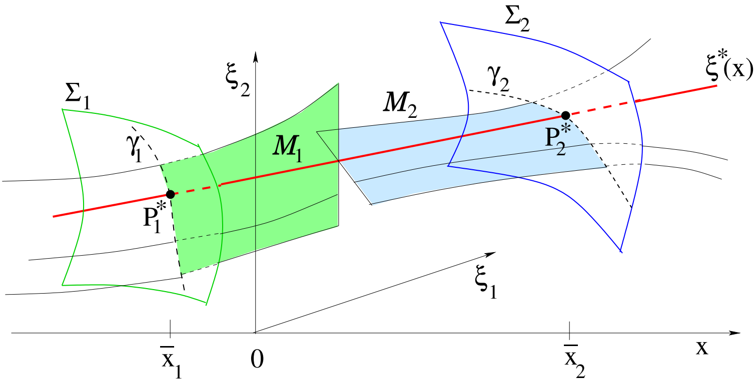
- $\det\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = 0$ on $\Sigma_1 \cup \Sigma_2$
- $\mathbf{v} \times \mathbf{w}$ is vertical on $\Sigma_1 \setminus \gamma_1$ and on $\Sigma_2 \setminus \gamma_2$
- $\mathbf{v} \times \mathbf{w} = 0$ on $\gamma_1 \cup \gamma_2$



Three generic cases



The saddle-saddle case



Under generic assumptions, a unique solution exists, also for the perturbed equation

Stability under perturbations, on the entire real line

(A.B., K.Nguyen, *Dyn. Games Appl.* **8**, 2018)

$$\text{dynamics:} \qquad \dot x \;=\; a_0 x + f_0(x) + (b_1 + h_1(x))u_1 + (b_2 + h_2(x))u_2$$

$$\text{payoffs:} \qquad J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\left(\mathbf{a}_i x + P_i x^2 + \eta_i(x) + \frac{u_i^2}{2}\right) dt \qquad i = 1,2$$

Under generic assumptions on the coefficients $a_0, b_1, b_2, \ldots$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. If the perturbations satisfy

$$\|f_0\|_{\mathcal{C}^2} + \|\eta_1\|_{\mathcal{C}^2} + \|\eta_2\|_{\mathcal{C}^2} + \|h_1\|_{\mathcal{C}^1} + \|h_2\|_{\mathcal{C}^1} \;\leq\; \delta,$$

then the perturbed equations for $\xi_1 = V_1'$, $\xi_2 = V_2'$ admit a solution such that

$$|\xi_1(x) - \xi_1^*(x)| + |\xi_2(x) - \xi_2^*(x)| \;\leq\; \varepsilon(1 + |x|) \qquad \text{for all } x \in \mathbb{R}$$

Does a similar result hold in dimension $d \geq 2$?

Expected costs: $$J_i = E^{x_0}\left[\int_0^{+\infty} e^{-\gamma_i t} L_i\Big(x(t), u_1(t), u_2(t)\Big)\, dt\right]$$

$x(t) =$ state, $u_1, u_2 =$ controls

$\gamma_1, \gamma_2 =$ discount factors $x_0 =$ initial state

- **Continuum state space:** $x \in \mathbb{R}^d$

Dynamics: $$dx = f(x, u_1, u_2)\, dt + \sigma\, dW$$

- **Discrete state space:** $x \in \{1, \ldots, N\}$

Transition probabilities: $$\text{Prob.}\Big\{x(t+\varepsilon) = j \,\Big|\, x(t) = i\Big\} = a_{ij}\,\varepsilon + o(\varepsilon),$$

where $a_{ij} = a_{ij}(u_1, u_2)$

Stackelberg equilibria in feedback form

- Given a strategy $u_1 = u_1^*(x)$ for the leader, a feedback $u_2 = u_2^*(x)$ is a **best reply** for the follower if it minimizes the expected cost

$$J_2 \;=\; E^y \left[\int_0^{+\infty} e^{-\gamma_2 t} L_2\Big(x(t), u_1^*(x(t)), u_2(x(t))\Big)\, dt \right]$$

  for every initial data $x(0) = y$

- A couple of feedback strategies (u_1^*, u_2^*) is a **Stackelberg equilibrium** if

  (i) $u_2^* \in \mathcal{R}_2(u_1^*)$ is a best reply for the follower, and

  (ii) The expected cost to the leader

$$J_1 \;=\; E^{x_0} \left[\int_0^{+\infty} e^{-\gamma_1 t} L_1\Big(x(t), u_1^*(x(t)), u_2^*(x(t))\Big)\, dt \right]$$

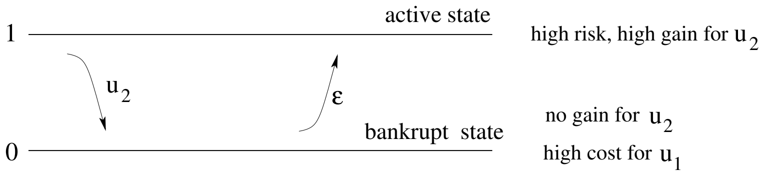
  is minimum among all couples (u_1, u_2) with $u_2 \in \mathcal{R}_2(u_1)$

$$\min_{u_1,\, u_2 \in \mathcal{R}_2(u_1)} \; E^{x_0} \left[\int_0^{+\infty} e^{-\gamma_1 t} L_1\Big(x(t), u_1^*(x(t)), u_2^*(x(t))\Big)\, dt \right]$$

Note: In general, the feedback strategies in a Stackelberg equilibrium depend on the initial state x_0, or the probability distribution μ assigned on the initial state

Can one single out a feedback equilibrium which does NOT make reference to the initial state?

An example



(u_2: an investment banker, u_1: a regulating government)

Transition probabilities: $a_{10} = u_2\,,\quad a_{01} = \varepsilon, \qquad u_1, u_2 \in [0,1]$

Expected costs: $$J_i \doteq E^{x_0}\left[\int_0^{+\infty} e^{-\gamma t} L_i\big(x(t), u_1(t), u_2(t)\big)\,dt\right]$$

$$\begin{cases} L_1(1, u_1, u_2) = u_1 \\ L_2(1, u_1, u_2) = -K_2 u_2 \end{cases} \qquad \begin{cases} L_1(0, u_1, u_2) = K_0 + u_1 \\ L_2(0, u_1, u_2) = K_1 u_1 \end{cases}$$

Self-consistent Stackelberg equilibria

$$dx \;=\; f(x, u_1(x), u_2(x))\, dt + \sigma\, dW$$

Assume: for any couple of feedback controls $u_1(\cdot), u_2(\cdot)$, as $t \to +\infty$ the state of the system will converge to a stationary probability distribution $\mu^\infty(u_1, u_2)$.

Definition. A couple of feedback controls (u_1^*, u_2^*) is a **self-consistent Stackelberg equilibrium** if

- the strategy $u_2^* \in \mathcal{R}_2(u_1^*)$ is a best reply for the follower
- the strategy u_1^* for the leader is optimal for the problem

$$\min_{u_1,\; u_2 \in \mathcal{R}_2(u_1)} \; E^{\mu^\infty} \left[\int_0^{+\infty} e^{-\gamma_1 t} L_1\Big(x(t), u_1^*(x(t)), u_2^*(x(t))\Big)\, dt \right]$$

- the probability distribution μ^∞ is invariant w.r.t. the dynamics generated by the feedbacks u_1^*, u_2^*

Construction of a SCSE

Goal: find a fixed point of the composed map

$$\mu \;\mapsto\; (u_1^*, u_2^*) \;\mapsto\; \mu^\infty$$

- (u_1^*, u_2^*) is a Stackelberg equilibrium, given the probability distribution μ on the initial state
- μ^∞ is an invariant probability distribution for the dynamical system generated by the feedback controls $u_1^*(\cdot)$, $u_2^*(\cdot)$

An existence result

- discrete state space: $X = \{1, \ldots, N\}$
- controls: $u_1, u_2 \in [0,1]$
- transition rates $a_{ij}(u_1, u_2)$ (linear w.r.t. u_1, u_2)
- Expected costs: $J_i = E^{x_0}\left[\int_0^{+\infty} e^{-\gamma_i t} L_i\Big(x(t), u_1(t), u_2(t)\Big)\, dt\right]$

Theorem. (A.B., Yilun Jiang, 2018)

Assume L_2 strictly convex w.r.t. u_2. Under generic assumptions on L_1, L_2, a_{ij}, one can find $\gamma_1^\sharp,\ \gamma_2^\sharp > 0$ such that a Self-consistent Stackelberg equilibrium exists provided that discount rates satisfy

- either $\gamma_1 \leq \gamma_1^\sharp$ (a far-sighted leader)
- or $\gamma_2 \geq \gamma_2^\sharp$ (a narrow-sighted follower)

Idea of the proof: show that the map

$$\mu \;\mapsto\; (u_1^*, u_2^*) \;\mapsto\; \mu^\infty$$

is (locally) a strict contraction if

- either γ_1 is sufficiently small (a far-sighted leader)

- or γ_2 is sufficiently large (a narrow-sighted follower)

Similar results are expected to hold also for games in continuum state space