Brouwer Degree and Applications

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January 17, 2009
Chapter 1

Introduction

Degree theory for continuous mappings from $\mathbb{R}$ to $\mathbb{R}$ can of course be traced to Bolzano’s intermediate value theorem for continuous functions [32]. Implicit in some of Gauss’ proofs of the fundamental theorem of algebra [144, 145], the concept of index or winding number around a closed curve, a forerunner of the notion of degree for continuous mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$, was first explicitly defined and developed by Cauchy in a series of memoirs published between 1831 and 1837 [52, 53, 54, 55], first devoted to the special case of holomorphic functions from $\mathbb{C}$ to $\mathbb{C}$ and then, in 1833 (published 1837), to $C^1$ mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. See [185], p. 134-136 for an interesting discussion and references. As shown in [272], Cauchy’s index was widely used by Poincaré in his early work on the qualitative theory of nonlinear differential equations [319].

Cauchy’s index was generalized in 1869 by Kronecker [222] to $C^1$ mappings $f = (f_1, f_2, \ldots, f_n)$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, on the compact $(n - 1)$-dimensional manifold $F^{-1}(0)$ associated to a smooth mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and the regular value 0 of $F$. He defined the ‘characteristic of $(F, f)$’, nowadays called Kronecker’s index of $f$ on $F^{-1}(0)$, by the integral (written here in the modern terminology of differential forms)

$$i_K[f, F^{-1}(0)] = \frac{1}{\mu_{n-1}} \int_{F^{-1}(0)} ||f||^{-n} \sum_{j=1}^{n} (-1)^{j-1} f_j \, df_1 \wedge \ldots \wedge \hat{df}_j \wedge \ldots df_n \tag{1.1}$$

where $\mu_{n-1}$ is the $(n - 1)$-dimensional measure of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, and $\hat{df}_j$ means that the factor $df_j$ is missing. One of Kronecker’s motivation was some extensions to systems of equations of the results of Sturm, Liouville and Hermite on the number of real roots of polynomial equations or systems [377, 378, 379, 380, 172]. See [185] and [366] for more details, as well as [272] for Poincaré’s use of Kronecker’s index in his qualitative study of nonlinear differential equations.

Influenced by Brouwer’s construction (using simplicial methods) of a degree theory for continuous mappings between two-oriented compact manifolds of the same finite dimension [39], Hadamard [162] extended in 1910 Kronecker’s integral to continuous mappings and more general $(n - 1)$-dimensional compact manifolds. See
for the role of Hadamard in the genesis of Brouwer’s ideas. After Brouwer’s pioneering paper, degree theory was essentially developed using algebraic topological tools (see e.g. [6, 152]), with the exception of two papers of Nagumo [292, 293], who based his analytical construction of the (localized) Brouwer degree $d_B[f, D, z]$ of a continuous mapping $f : \overline{D} \to \mathbb{R}^n$, with $D$ open bounded and $z \notin f(\partial D)$, on approximating $f$ by smooth mappings $g$ and $z$ by regular values $u$ of $g$, for which one can define

$$d_B[g, D, u] = \sum_{x \in g^{-1}(u)} \text{sign } J_g(x),$$

with $J_g$ the Jacobian of $g$. Nagumo’s approach, which also uses the structure of compact 1-dimensional manifolds, has been adopted in the monographs [10, 29, 30, 94, 174, 221, 282, 368, 400].

Inspired by de Rham cohomology theory based on differential forms [87], Heinz [171] proposed in 1959 another analytical approach to Brouwer degree. He first defined $d_B[f, D, z]$ for $f$ smooth, $z \notin f(\partial D)$, by the integral

$$\int_D c(||f(x) - z||)J_f(x) \, dx,$$

where the continuous function $c$ has support in $]0, \min_{\partial D} ||f(\cdot) - z||[$, and then approximated a continuous $f$ through smooth mappings. Heinz’ approach, which avoids the explicit use of differential forms, and does not discuss the connection of integral (1.2) with Kronecker’s one (1.1) when both are defined, has been adopted in many monographs [1, 22, 89, 90, 130, 138, 198, 254, 315, 328, 333, 353].

A variant of it, due to Lax (see [297]), explicitly uses the language of differential forms, and can be found in the monographs [23, 29, 30, 95, 130, 297, 370, 407].

In this book, we present Heinz’ approach in a somewhat simpler, shorter and better motivated way, by linking it to the Kronecker index (Proposition 2.6.1), and using the language of differential forms, at the level of an advanced calculus treatise (for example [373]). The needed versions of Stokes formula are

$$\int_D d\lambda = \int_{\partial D} \lambda,$$

when the open bounded set $D \subset \mathbb{R}^n$ has a regular oriented boundary $\partial D$ and $\lambda$ is a differential $(n-1)$-form of class $C^1$ (to relate Kronecker and Heinz definitions), and

$$\int_D d\lambda = 0.$$

when the $C^1$ differential $(n-1)$-form $\lambda$ has compact support in an arbitrary open set $D \subset \mathbb{R}^n$ (to justify Heinz definition and prove the homotopy invariance of degree). The homotopy invariance of the degree is proved using an unpublished result of Tartar [382], quoted in the Introduction of [130]. When written in terms of differential forms, this result (our Lemma 3.2.2) immediately follows from an elementary computation of exterior calculus (Lemma 3.2.1). In the setting of Kronecker index,
a similar approach was used recently by Hatziafratis and Tsarpalias [170], at the expense of much longer and tedious computations. An extension of Lemma 3.2.2 from differential $n$-forms in $\mathbb{R}^n$ to differential $k$-forms in $\mathbb{R}^n$ is given in [275].
Chapter 2

Kronecker index

2.1 Definition

Let $n \geq 2$ be an integer,

$$\omega := dy_1 \wedge ... \wedge dy_n$$  \hspace{1cm} (2.1)

be the volume $n$-form in $\mathbb{R}^n$, and let $\sigma$ be the solid angle $(n-1)$-form

$$\sigma := \sum_{j=1}^{n} (-1)^{j-1} y_j dy_1 \wedge ... \wedge \hat{dy}_j \wedge ... \wedge dy_n,$$  \hspace{1cm} (2.2)

where $\hat{dy}_j$ means that $dy_j$ is missing. One has

$$\omega = d\left(\frac{1}{n}\sigma\right).$$  \hspace{1cm} (2.3)

Let $D \subset \mathbb{R}^n$ be a bounded open set with oriented smooth boundary $\partial D$ and $f \in C^1(\overline{D}, \mathbb{R}^n)$ be such that $0 \not\in f(\partial D)$. Then

$$\mu := \min_{\partial D} \|f\| > 0.$$  \hspace{1cm} (2.4)

Definition 2.1.1 The Kronecker index $i_K[f, \partial D]$ is defined by

$$i_K[f, \partial D] = \frac{1}{\mu n^{-1}} \int_{\partial D} f^* \left[\frac{1}{\|y\|^{n}} \sigma\right]$$

$$= \frac{1}{\mu n^{-1}} \int_{\partial D} \left[\sum_{j=1}^{n} (-1)^{j-1} \frac{\tilde{f}_j}{\|f\|^{n}} df_1 \wedge ... \wedge \hat{df}_j \wedge ... \wedge df_n\right],$$

where $df_k$ is the $1$-differential form

$$df_k = \sum_{i=1}^{n} \partial_i f_k \, dx_k \quad (1 \leq k \leq n)$$
and \( \mu_{n-1} \) denotes the \((n-1)\)-dimensional measure of the unit sphere
\[
S^{n-1} = \{ y \in \mathbb{R}^n : \|y\| = 1 \}
\]
for the Euclidian norm \( \| \cdot \| \) in \( \mathbb{R}^n \).

Explicitely
\[
\mu_{n-1} = \begin{cases} 
\frac{2 \pi^{n/2}}{(n-1)!} & \text{if } n \geq 2 \text{ is even} \\
\frac{\pi^{(n-1)/2}}{(n-1)!} & \text{if } n \geq 3 \text{ is odd}.
\end{cases}
\]

Remark 2.1.1 If \( n = 1 \) and \( D = [a, b] \), the Kronecker index of a continuous function \( f : [a, b] \to \mathbb{R} \) such that \( f(a)f(b) \neq 0 \) can be accordingly defined by
\[
i_K[f, \partial]a, b[] = \frac{1}{2} \left[ \frac{f(b)}{|f(b)|} - \frac{f(a)}{|f(a)|} \right] = \frac{1}{2} \text{sign } f(b) - \text{sign } f(a). 
\] (2.6)

Hence,
\[
i_K[f, \partial]a, b[] = 0 \quad \text{if } f(a)f(b) > 0, \\
i_K[f, \partial]a, b[] = 1 \quad \text{if } f(a) < 0 < f(b), \\
i_K[f, \partial]a, b[] = -1 \quad \text{if } f(b) < 0 < f(a).
\]

This definition is compatible with (2.1.1) if one uses the usual conventions of calling 0-form on \([a, b]\) any real continuous function \( g \) on \([a, b]\), defining the (oriented) integral of \( g \) on the boundary of \([a, b]\) by \( g(b) - g(a) \), and taking the counting measure as 0-measure of \( \partial]a, b[ \).

### 2.2 The planar case: winding number

For \( n = 2 \), formula (2.5) reduces to a line integral, already defined by Cauchy, and also named the winding number or the Poincaré index of \( f \) around \( \partial D \)
\[
i_K[f, \partial D] = \frac{1}{2\pi} \int_{\partial D} f_1 df_2 - f_2 df_1.
\]

If \( \partial D \) has the parametric representation \( \varphi : [0, 2\pi] \to \mathbb{R}^2 \) (with \( \varphi(0) = \varphi(2\pi) \)), then
\[
i_K[f, \partial D] = \frac{1}{2\pi} \int_0^{2\pi} f_1[\varphi(s)] \frac{df_2[\varphi(s)]}{df_1[\varphi(s)]} - f_2[\varphi(s)] \frac{df_1[\varphi(s)]}{df_2[\varphi(s)]} ds. 
\] (2.7)

Example 2.2.1 If \( D = B(1) \subset \mathbb{R}^2 \), and \( \partial D = S^1 \) is parametrized by \( \varphi(s) = (\cos s, \sin s) \) \((s \in [0, 2\pi])\), then for \( f = I \),
\[
i_K[I, S^1] = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2 s + \sin^2 s}{\cos^2 s + \sin^2 s} ds = 1
\]
and the vector field \( f(\cos s, \sin s) = (\cos s, \sin s) \) rotates once anticlockwise around the origin when \( s \) varies from 0 to \( 2\pi \).
2.2. THE PLANAR CASE : WINDING NUMBER

Example 2.2.2 If \( f(x_1, x_2) = (x_1, -x_2) \), then
\[
\text{\( i_K[f, S^1] = \frac{1}{2\pi} \int_0^{2\pi} \left( -\cos^2 s - \sin^2 s \right) ds = -1 \)}
\]
and the vector field \( f(\cos s, \sin s) = (\cos s, -\sin s) \) rotates once clockwise around the origin when \( s \) varies from 0 to \( 2\pi \).

Example 2.2.3 If \( f(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2) \), then
\[
\text{\( f(\cos s, \sin s) = (\cos 2s, \sin 2s) \)}
\]
\[
\text{\( i_K[f, S^1] = \frac{1}{2\pi} \int_0^{2\pi} \frac{2\cos^2 2s + 2\sin^2 2s}{\cos^2 2s + \sin^2 2s} ds = 2 \)}
\]
and the vector field \( f(\cos s, \sin s) = (\cos 2s, \sin 2s) \) rotates twice anticlockwise around the origin when \( s \) varies from 0 to \( 2\pi \).

One can also give a complex formulation of Kronecker index. Set \( f = f_1 + i f_2 \), \( df = df_1 + idf_2 \). Then
\[
\frac{1}{2\pi i} \int_{\partial D} \frac{df}{f} = \frac{1}{2\pi i} \int_{\partial D} \frac{df_1 + idf_2}{f_1 + i f_2} = \frac{1}{2\pi i} \int_{\partial D} \frac{(f_1 - if_2)(df_1 + idf_2)}{\|f\|^2} = \frac{1}{2\pi i} \int_{\partial D} f_1 df_1 + f_2 df_2 \frac{1}{\|f\|^2} + \frac{1}{2\pi} \int_{\partial D} f_1 df_1 - f_2 df_2 \frac{1}{\|f\|^2} = i_K[f, \partial D], \quad (2.8)
\]
because
\[
\frac{1}{2\pi i} \int_{\partial D} \frac{f_1 df_1 + f_2 df_2}{\|f\|^2} = \frac{1}{2\pi i} \int_{\partial D} \frac{d\|f\|}{\|f\|^2} = \frac{1}{2\pi i} \int_{\partial D} d\log \|f\| = 0.
\]
Consequently
\[
\text{\( i_K[f, \partial D] = \frac{1}{2\pi i} \int_0^{2\pi} \partial_\tau f(\varphi(s)) \varphi_1'(s) + \partial_\tau f(\varphi(s)) \varphi_2'(s) ds \)}
\]
\[
= \frac{1}{2\pi i} \int_0^{2\pi} \partial_\tau f(\varphi(s)) \varphi_1'(s) ds, \quad (2.9)
\]
where \( \partial_\tau f \) denotes the tangential derivative of \( f \) along \( \partial D \).

The complex formulation allows a simple proof, due to Ahlfors [3], of the fact that \( i_K[f, \partial D] \) is an integer.

**Lemma 2.2.1** If \( \partial D \) has the parametric representation \( \varphi : [0, 2\pi] \rightarrow \mathbb{R}^2 \), then \( i_K[f, \partial D] \) is an integer.
Proof. Let $\Phi := \varphi_1 + i\varphi_2$, and let, for $t \in [0, 2\pi],$

$$h(t) := \int_0^t \frac{f(\Phi(s))}{f(\Phi(t))} ds,$$

$$g(t) := e^{-h(t)} f(\Phi(t)).$$

One has

$$g'(t) = e^{-h(t)} \frac{d}{dt} f(\Phi(t)) - \frac{d}{dt} f(\Phi(t)) e^{-h(t)} f(\Phi(t)) = 0,$$

for all $t \in [0, 2\pi]$, so that $g(0) = g(2\pi)$, which easily implies $e^{h(2\pi)} = 1$, and

$$\int_0^{2\pi} \frac{d}{dt} f(\Phi(s)) ds = h(2\pi) = 2\pi m i$$

for some $m \in \mathbb{Z}$. 

\[ \] 

2.3 The case where $f : \partial D \subset \mathbb{R}^n \to S^{n-1}$

Let again $n \geq 2$ be an integer and $D \subset \mathbb{R}^n$ be a bounded open set with oriented smooth boundary $\partial D$. Now let $g \in C^2(D, \mathbb{R}^n)$ be such that

$$\|g(x)\| = 1 \quad \text{for all} \quad x \in \partial D. \quad (2.10)$$

Let us show that, in this case, it is easy to express $i_K[g, \partial D]$ through an integral over $D$. We denote by $J_g$ the Jacobian of $g$.

Lemma 2.3.1 If condition (2.10) holds, then

$$i_K[g, \partial D] = \frac{n}{\mu_{n-1}} \int_D J_g(x) dx. \quad (2.11)$$

Proof. Using formulas (2.5), (2.3) and Stokes formula, we get

$$i_K[g, \partial D] = \frac{1}{\mu_{n-1}} \int_{\partial D} g^* \sigma = \frac{1}{\mu_{n-1}} \int_D d[g^* \sigma] = \frac{1}{\mu_{n-1}} \int_D g^* [d\sigma]$$

$$= \frac{n}{\mu_{n-1}} \int_D g^* \omega = \frac{n}{\mu_{n-1}} \int_D dg_1 \wedge \ldots \wedge dg_n$$

$$= \frac{n}{\mu_{n-1}} \int_D J_g(x) dx. \quad \square$$

In the special case where $n = 2$, we can express $i_K[g, \partial D]$ as follows.
Lemma 2.3.2 If \( g \in C^1(\mathbf{T} \subset \mathbb{R}^2, \mathbb{R}^2) \) is such that \( \|g(x)\| = 1 \) for all \( x \in \partial D \), and if \( \partial D \) has the parametric representation \( \varphi : [0, 2\pi] \rightarrow \mathbb{R}^2 \), then

\[
i_K[g, \partial D] = \frac{1}{2\pi} \int_0^{2\pi} \{g[\varphi(s)] \wedge \partial_\tau g[\varphi(s)]\} \, ds,
\]

(2.12)

where \( \partial_\tau g \) is the tangential derivative of \( g \) along \( \partial D \).

Proof.}

\[
i_K[g, \partial D] = \frac{1}{2\pi} \int_0^{2\pi} \left\{ g_1[\varphi(s)] \frac{d}{ds} g_2[\varphi(s)] - g_2[\varphi(s)] \frac{d}{ds} g_1[\varphi(s)] \right\} \, ds
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left\{ g_1[\varphi(s)] \partial_\tau g_2[\varphi(s)] - g_2[\varphi(s)] \partial_\tau g_1[\varphi(s)] \right\} \, ds
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \{g[\varphi(s)] \wedge \partial_\tau g[\varphi(s)]\} \, ds.
\]

\[
\]

Now, \( \partial_\tau g \circ \varphi \) only depends upon the values of \( g \) on \( \partial D \). Therefore, if \( f \in C^2(\partial D, S^1) \) and \( \partial D \) is parametrized by \( \varphi : [0, 2\pi] \rightarrow \mathbb{R}^2 \), then, by analogy with formula (2.12), we can introduce the following definition.

Definition 2.3.1 The Kronecker index \( i_K[f, \partial D] \) of \( f \) with respect to \( \partial D \) is

\[
i_K[f, \partial D] := \frac{1}{2\pi} \int_0^{2\pi} \{f[\varphi(s)] \wedge \partial_\tau f[\varphi(s)]\} \, ds
\]

(2.13)

where \( \partial_\tau \) denotes the tangential derivative of \( f \) with respect to \( \partial D \).

Combining this definition with Lemma 2.3.1, we then have the following result.

Proposition 2.3.1 If \( g \in C^2(\mathbf{T}, \mathbb{R}^2) \) is any \( C^2 \)-extension of \( f \) to \( \mathbf{T} \), then

\[
i_K[f, \partial D] = \frac{1}{\pi} \int_D [\partial_1 g(x) \wedge \partial_2 g(x)] \, dx.
\]

(2.14)

Proof. Using (2.11) and (2.12), we get

\[
i_K[f, \partial D] = \frac{1}{2\pi} \int_0^{2\pi} \{f[\varphi(s)] \wedge \partial_\tau f[\varphi(s)]\} \, ds = \frac{1}{2\pi} \int_0^{2\pi} \{g[\varphi(s)] \wedge \partial_\tau g[\varphi(s)]\} \, ds
\]

\[
= \frac{1}{2\pi} \int_{\partial D} (g_1 dg_2 - g_2 dg_1) = i_K[g, \partial D] = \frac{1}{\pi} \int_D J_g(x) \, dx
\]

\[
= \frac{1}{\pi} \int_D [\partial_1 g(x) \wedge \partial_2 g(x)] \, dx.
\]

\[
\]
2.4 A minimum problem from Ginzburg-Landau’s theory

Ginzburg-Landau’s theory provides a model very suitable for the study of phase transitions occurring in supraconductors and superfluidity [26]. We suppose throughout the whole section that $D \subset \mathbb{R}^2$ be open, bounded, regular and simply connected, and that $g \in C^2(\partial D, S^1)$. We are looking for a function $u = u_1 + iu_2 : D \to \mathbb{C}$ which minimizes the Dirichlet or energy integral

$$\int_D \|\nabla u(x)\|^2 \, dx$$

(2.15)

(where

$$\nabla u = \partial_1 u + i\partial_2 u,$$

$$\|\nabla u\|^2 = |\partial_1 u|^2 + |\partial_2 u|^2 = (\partial_1 u_1)^2 + (\partial_1 u_2)^2 + (\partial_2 u_1)^2 + (\partial_2 u_2)^2$$

under the restrictions

$$|u(x)| = [u_1^2(x) + u_2^2(x)]^{1/2} = 1 \ \mathrm{a.e. \ in} \ D, \ u = f \ \mathrm{on} \ \partial D.$$ 

If $H^1(D, \mathbb{C})$ denotes the Sobolev space of functions $u \in L^2(D, \mathbb{C})$ such that $\partial_1 u$ and $\partial_2 u$ belong to $L^2(D, \mathbb{C})$, with the usual norm, let

$$H^1_0(D, S^1) = \{ u \in H^1(D, \mathbb{C}) : |u(x)| = 1 \ \mathrm{a.e. \ on} \ D, \ u = f \ \mathrm{on} \ \partial D \}$$

where the boundary value is understood in the sense of trace. Thus, our minimum problem can be expressed in the following way

$$\min_{u \in H^1_0(D, S^1)} \int_D \|\nabla u(x)\|^2 \, dx.$$

(2.15)

If a solution exists, the corresponding Euler-Lagrange equation is the coupled nonlinear elliptic system

$$-\Delta u = \|\nabla u\|^2 u \ \mathrm{in} \ D, \ u = f \ \mathrm{on} \ \partial D.$$ 

However, in order that problem (2.15) makes sense, it is necessary first to prove that $H^1_0(D, S^1) \neq \emptyset$. We will see that the Kronecker index $i_K[f, \partial D]$ plays a fundamental role in this respect. This topological invariant is also responsible of the apparition of quantified vorticity effects comparable to those observed in supraconductors. Let us first give a useful generalization of Proposition 2.3.1.

**Proposition 2.4.1** If $f \in C^2(\partial D, S^1)$ and $u \in H^1(D, \mathbb{C})$ is such that $u = f$ on $\partial D$, then

$$i_K[f, \partial D] = \frac{1}{\pi} \int_D [\partial_1 u(x) \wedge \partial_2 u(x)] \, dx.$$

(2.16)
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Proof. Let \( u_0 \in C^2(D, \mathbb{C}) \) be such that \( u_0 = f \) on \( \partial D \), and \( u \in H^1(D, \mathbb{C}) \) be such that \( u = f \) on \( \partial D \). Clearly \( u - u_0 \in H^1_0(D, \mathbb{C}) \), and therefore there exists a sequence \( (u_n) \) in \( D(D, \mathbb{C}) \) such that \( u_n \to u - u_0 \) strongly in \( H^1(D, \mathbb{C}) \). Then \( \tilde{u}_n := u_n + u_0 \in C^2(D, \mathbb{C}) \) is equal to \( f \) on \( \partial D \). Using Proposition 2.3.1,

\[
i_K[f, \partial D] = \frac{1}{\pi} \int_D [\partial_1 \tilde{u}_n(x) \wedge \partial_2 \tilde{u}_n(x)] \, dx
\]

for all \( n \in \mathbb{N} \), and the result follows by letting \( n \to \infty \).

We need another result, essentially due to Eilenberg [99].

**Proposition 2.4.2** If \( f : \partial D \to S^1 \) is smooth and such that \( i_K[f, \partial D] = 0 \), then there exists a smooth function \( \varphi_0 : \partial D \to \mathbb{R}^2 \) such that \( f = e^{i\varphi_0} \) on \( \partial D \).

**Proof.** If \( i_K[f, \partial D] = 0 \), then

We are now ready to prove the **Bethuel-Brezis-Hélein theorem** showing the role of \( i_K[f, \partial D] \) in the study of minimization problem (2.15).

**Theorem 2.4.1** \( H^1_f(D, S^1) \neq \emptyset \) if and only if \( i_K[f, \partial D] = 0 \).

**Proof.** Suppose that \( H^1_f(D, S^1) \neq \emptyset \), and let \( u \in H^1_f(D, S^1) \), i.e. \( u \in H^1(D, \mathbb{C}) \),

\[
u^2(x) + u_2^2(x) = 1 \quad \text{for a.e. } x \in D,
\]

(2.17)

and \( u = f \) on \( \partial D \) in the sense of trace. It is well known that if \( v \in H^1(D, \mathbb{R}) \cap L^\infty(D, \mathbb{R}) \) and \( G \in C^1(\mathbb{R}, \mathbb{R}) \), then \( G \circ v \in H^1(D, \mathbb{R}) \) and \( \partial_i(G \circ v) = (G' \circ v) \partial_i v \) (\( i = 1, 2 \)). Clearly, \( u_i \in H^1(D, \mathbb{R}) \cap L^\infty(D, \mathbb{R}) \) (\( i = 1, 2 \)) and hence, for \( G(t) = t^2 \), \( u_i^2 \in H^1(D, \mathbb{R}) \) (\( i = 1, 2 \))

\[\partial_i(u_i^2) = 2u_i \partial_i u_i \quad (i, j = 1, 2).\]

By differentiating the identity (2.17), we obtain

\[u_1 \partial_1 u_1 + u_2 \partial_1 u_2 = 0, \quad u_1 \partial_2 u_1 + u_2 \partial_2 u_2 = 0,\]

i.e.

\[
\begin{pmatrix}
\partial_1 u_1 & \partial_1 u_2 \\
\partial_2 u_1 & \partial_2 u_2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} = 0
\]

with \( (u_1, u_2) \neq 0 \). Therefore,

\[\partial_1 u \wedge \partial_2 u = J_u = 0\]

and Proposition 2.4.1 implies that

\[
i_K[f, \partial D] = \frac{1}{\pi} \int_D [\partial_1 u(x) \wedge \partial_2 u(x)] \, dx = 0.
\]

Conversely, suppose that \( i_K[f, \partial D] = 0 \). Using Proposition 2.4.2, we can write \( f = e^{i\varphi_0} \) for some smooth \( \varphi_0 : \partial D \to \mathbb{R} \). Denote by \( \tilde{\varphi} \) the harmonic extension of \( \varphi_0 \) to \( D \), i.e. the unique solution of the linear Dirichlet problem

\[\Delta \tilde{\varphi} = 0 \quad \text{in } D, \quad \tilde{\varphi} = \varphi_0 \quad \text{on } \partial D.
\]

Then \( \tilde{u} = e^{i\tilde{\varphi}} \in H^1(D, \mathbb{C}) \), \( |\tilde{u}(x)| = 1 \) for \( x \in D \), and \( \tilde{u}(x) = e^{i\varphi_0(x)} = f(x) \) for \( x \in \partial D \). Consequently, \( \tilde{u} \in H^1_f(D, \mathbb{C}) \).
2.5 An exact form

Writing \( i_K[f, \partial D] \) as an integral over \( D \) when \( \|f(x)\| \) is not constant over \( \partial D \) is more delicate, and requires some preparation.

Let \( 0 < \varepsilon_0 < \mu_0, a \in C^k([0, +\infty[, \mathbb{R}), \ (k \geq 0) \) be such that

\[
\text{supp } a \subset [\varepsilon_0, \mu_0],
\]

and consider the differential \( n \)-form (with \( n \) a positive integer)

\[
\omega_a := a(\|y\|) dy_1 \wedge \ldots \wedge dy_n = a(\|y\|) \omega.
\]

The following lemma shows that \( \omega_a \) is exact on \( \mathbb{R}^n \).

**Lemma 2.5.1** The equality

\[
\omega_a = d\sigma_A
\]

with

\[
\sigma_A := A(\|y\|) \sigma,
\]

\( A \in C^{k+1}([0, +\infty[, \mathbb{R}) \) and \( \text{supp } A \subset [\varepsilon_0, +\infty[ \), holds if and only if

\[
rA'(r) + nA(r) = a(r) \quad (r \geq \varepsilon_0),
\]

i.e. if and only if \( A \) is given by

\[
A(0) = 0, \quad A(r) = \frac{1}{r^n} \int_0^r a(s)s^{n-1} ds \quad (r > 0).
\]

**Proof.** Let \( A \in C^{k+1}([0, +\infty[, \mathbb{R}) \), with \( \text{supp } A \subset [\varepsilon_0, +\infty[ \). \( \sigma_A \in C^{k+1}(\mathbb{R}^n, \mathbb{R}) \) and \( \sigma_A = 0 \) on \( B[\varepsilon_0] \). For \( y \neq 0 \), we have

\[
\frac{d}{dr}[A(\|y\|) \wedge \sigma + A(\|y\|) d\sigma] = A'(\|y\|) \sum_{k=1}^n \frac{y_k}{\|y\|} dy_k \wedge \sigma + nA(\|y\|) \omega
\]

Consequently, \( \omega_a = d\sigma_A \) if and only if, for all \( r \in [\varepsilon_0, +\infty[ \), \( A \) satisfies (2.21). For \( r > 0 \), this is equivalent to

\[
\frac{d}{dr}[r^A(r)] = r^{n-1}a(r),
\]

i.e. \( A \) given by formula (2.22). Clearly, \( \text{supp } A \subset [\varepsilon_0, +\infty[ \).

**Corollary 2.5.1** If we assume furthermore that

\[
\int_0^{+\infty} a(r)r^{n-1} dr = 0,
\]

then \( \text{supp } A \subset [\varepsilon_0, \mu_0] \).
2.6. AN EQUIVALENT FORMULA

Proof. For \( r > \mu_0 \), we have, by formula (2.22),

\[
A(r) = \frac{1}{r^n} \int_0^r a(s)s^{n-1} \, ds = \frac{1}{r^n} \int_0^{\mu_0} a(s)s^{n-1} \, ds = \frac{1}{r^n} \int_0^{+\infty} a(s)s^{n-1} \, ds = 0.
\]

\[
\]

2.6 An equivalent formula

The Kronecker integral in the right-hand member of (2.5) is associated to the differential \((n - 1)\)-form \(\sigma_A\) with \(A(r) = r^{-n}\), which is not defined at 0 and does not vanish near 0. However, in (2.5), one only has to consider the values of \(f(x)\) near \(\partial D\), i.e. values of \(f(x)\) for which \(\|f(x)\| \geq \mu\). Hence, let

\[
0 < \varepsilon_0 < \mu_0 < \mu,
\]

and define \(B \in C^1([0, +\infty], \mathbb{R})\) by

\[
B(r) = \begin{cases} 
0 & \text{if } r \in [0, \varepsilon_0] \\
\text{positive} & \text{if } r \in [\varepsilon_0, \mu_0] \\
r^{-n} & \text{if } r \in [\mu_0, +\infty].
\end{cases}
\]  

(2.24)

Notice that, for \( r > \mu_0 \), we have

\[
rB'(r) + nB(r) = 0.
\]  

(2.25)

Proposition 2.6.1 If \(B\) is given by (2.24), the function \(b \in C([0, +\infty], \mathbb{R})\) defined by

\[
b(r) = rB'(r) + nB(r)
\]  

(2.26)

is such that

\[
\text{supp } b \subset [\varepsilon_0, \mu_0], \quad \int_0^{+\infty} b(r)r^{n-1} \, dr = 1,
\]

and, if \(f \in C^2(\overline{D}, \mathbb{R}^n)\),

\[
i_K[f, \partial D] = \frac{1}{\mu_{n-1}} \int_D f^* \omega_B = \frac{1}{\mu_{n-1}} \int_D b(||f(x)||)J_f(x) \, dx.
\]  

(2.27)

Proof. For \( x \in \partial D\), we have \(||f(x)|| \geq \mu > \mu_0\), and hence \(B(||f(x)||) = ||f(x)||^{-n}\). Using Lemma 2.5.1 and Stokes formula (1.3), we get

\[
i_K[f, \partial D] = \frac{1}{\mu_{n-1}} \int_{\partial D} f^* [||y||^{-n}\sigma] = \frac{1}{\mu_{n-1}} \int_{\partial D} f^* \sigma_B
\]

\[
= \frac{1}{\mu_{n-1}} \int_D d[f^* \sigma_B] = \frac{1}{\mu_{n-1}} \int_D f^*[d\sigma_B] = \frac{1}{\mu_{n-1}} \int_D f^* \omega_B.
\]
It is clear from its definition and (2.25) that $b(r) = 0$ for $r \in [0, \varepsilon_0 \cup] \cup [\mu_0, +\infty[$, and that
\[
\int_{0}^{+\infty} b(r) r^{n-1} \, dr = \int_{0}^{+\infty} [r^n B'(r) + nr^{n-1} B(r)] \, dr = \int_{0}^{+\infty} \frac{d}{dr} [r^n B(r)] \, dr = 1.
\]

**Remark 2.6.1** In the construction above, the functions $b$ and $B$ can be taken as smooth as we want.
Chapter 3

Brouwer degree

3.1 Smooth mappings

Let now \( n \geq 1 \) be an integer, \( D \) be open and bounded in \( \mathbb{R}^n \) and \( f \in C^2(D, \mathbb{R}^n) \) be such that \( 0 \not\in f(\partial D) \). We again define \( \mu > 0 \) by (2.4), and take \( 0 < \varepsilon_0 < \mu_0 < \mu \).

The following definition is due to Heinz [171].

**Definition 3.1.1** The Brouwer degree \( d_B[f, D] \) is defined by

\[
d_B[f, D] = \int_D f^* \omega_c - \int_D f^* \omega_e = \int_D f^* \omega_a = \int_D f^* d[\sigma_A] = \int_D d[f^* \sigma_A] = 0,
\]

where \( c \in C([0, +\infty], \mathbb{R}) \) is such that \( \operatorname{supp} c \subset [\varepsilon_0, \mu_0] \) and

\[
\int_{\mathbb{R}^n} c(\|x\|) \, dx = 1 \quad \text{(i.e.} \quad \int_0^{+\infty} c(r) r^{n-1} \, dr = \frac{1}{\mu_{n-1}}). \tag{3.2}
\]

To justify this definition, it suffices to notice that if \( \tilde{c} \in C([0, +\infty], \mathbb{R}) \) satisfies the same conditions as \( c \), then \( a = c - \tilde{c} \) is such that

\[
\int_0^{+\infty} a(r) r^{n-1} \, dr = 0.
\]

Hence, by Corollary 2.5.1, \( \operatorname{supp} A \subset [\varepsilon_0, \mu_0] \), with \( A \) given by (2.22), so that,

\[
\int_D f^* \omega_c - \int_D f^* \omega_e = \int_D f^* \omega_a = \int_D f^* d[\sigma_A] = \int_D d[f^* \sigma_A] = 0,
\]

using (1.4), and the fact that \( \|f(x)\| > \mu_0 \) in a neighbourhood of \( \partial D \).

**Remark 3.1.1** It follows immediately from Definition 3.1.1 that \( d_B[f, \emptyset] = 0 \).

**Remark 3.1.2** Proposition 2.6.1 implies that, if \( \partial D \) is smooth enough, one has

\[
i_K[f, \partial D] = d_B[f, D]. \tag{3.3}
\]
**Proposition 3.1.1** If the open set \( D' \subset D \) is such that \( 0 \not\in f(\overline{D} \setminus D') \), then
\[
d_B[f, D] = d_B[f, D'].
\]
In particular, if \( 0 \not\in f(\overline{D}) \), then \( d_B[f, D] = 0 \).

**Proof.** By assumption, if \( \mu' := \min_{\partial D'} \| f \| \), and \( \mu'' := \min_{\partial D} \| f \| \), then \( 0 < \mu'' < \mu \) and \( \mu'' < \mu' \). If we take \( 0 < \varepsilon_0 < \mu_0 < \mu'' \) in the function \( c \) defining \( d_B[f, D] \), we have \( c(\| f(x) \|) = 0 \) if \( x \in \overline{D} \setminus D' \), and
\[
d_B[f, D] = \int_D c(\| f(x) \|) J_f(x) \, dx = \int_{D'} c(\| f(x) \|) J_f(x) \, dx = d_B[f, D'],
\]
because \( c \) is also good for defining \( D_B[f, D'] \).

It is convenient to introduce a new notation.

**Definition 3.1.2** If \( D \subset \mathbb{R}^n \) is open and bounded, \( f \in C^2(\overline{D}, \mathbb{R}^n) \) and \( z \not\in f(\partial D) \), the **Brouwer degree** \( d_B[f, D, z] \) is defined by
\[
d_B[f, D, z] = d_B[f(\cdot) - z, D].
\]

Of course, \( d_B[f, D] = d_B[f, D, 0] \).

**Definition 3.1.3** \( z \in \mathbb{R}^n \) is a **regular value** for \( f \) if \( J_f(x) \neq 0 \) when \( x \in f^{-1}(z) \).

So any \( z \) with \( f^{-1}(z) \) empty is a regular value.

**Proposition 3.1.2** If \( z \not\in f(\partial D) \) is a regular value for \( f \), then
\[
d_B[f, D, z] = \sum_{x \in f^{-1}(z)} \text{sign } J_f(x) = N_+ - N_-,
\]
where \( N_+ \) (resp. \( N_- \)) denotes the number of elements of \( f^{-1}(z) \) with positive (resp. negative) Jacobian.

**Proof.** If \( f^{-1}(z) = \emptyset \), the result is trivial. If \( z \not\in f(\partial D) \) is a regular value for \( f \) and \( f^{-1}(z) \neq \emptyset \), then \( f^{-1}(z) \subset D \) is compact, and is discrete by the inverse function theorem. Hence it is finite, namely \( f^{-1}(z) = \{x^1, \ldots, x^m\} \). By the inverse function theorem again, there exists, for each \( 1 \leq j \leq m \), an open neighbourhood \( U_j \subset D \) of \( x^j \) such that \( f \) is a diffeomorphism on \( U_j \), and \( U_j \cap U_k = \emptyset \) if \( j \neq k \). Thus the sets \( f(U_j) \) are open neighbourhoods of \( z \), as well as \( N = \bigcap_{j=1}^m f(U_j) \), so that \( N \supset B[z; r] \) for some \( r > 0 \). As \( f(x) \neq z \) for all \( x \in K := \overline{D} \setminus \bigcup_{j=1}^m U_j \), Proposition 3.1.1 gives
\[
d_B[f, D, z] = d_B[f, \bigcup_{j=1}^m U_j, z] = \int_{\bigcup_{j=1}^m U_j} c(\| f(x) - z \|) J_f(x) \, dx
\]
\[
= \sum_{j=1}^m \int_{U_j} c(\| f(x) - z \|) J_f(x) \, dx = \sum_{j=1}^m d_B[f, U_j, z],
\]
where we have taken
\[ \mu_0 < \min \{ r, \min_{U_j} \| f(\cdot) - z \| \} \]
in the definition of \( c \). The change of variables formula in an integral gives
\[
d[f, U_j, z] = \int_{U_j} c(\| f(x) - z \|) |J_f(x)| dx
\]
\[
= \text{sign } J_f(x^j) \int_{U_j} c(\| f(x) - z \|) |J_f(x)| dx
\]
\[
= \text{sign } J_f(x^j) \int_{f(U_j) - z} c(\| y \|) dy.
\]
Now, for \( y \not\in f(U^j) - z \), we have \( \| y - z \| \geq r \geq \mu_0 \), and hence \( c(\| y - z \|) = 0 \). Thus
\[
\int_{f(U_j) - z} c(\| y - z \|) dy = \int_{\mathbb{R}^n} c(\| y - z \|) dy = 1,
\]
which gives (3.4).

**Remark 3.1.3** Proposition 3.1.2 explains the name of “algebraic number of zeros” sometimes given to the degree.

**Remark 3.1.4** For each regular value \( z \not\in f(\partial D) \), \( d_B[f, D, z] \) is an integer.

**Corollary 3.1.1** If \( A : \mathbb{R}^n \to \mathbb{R}^n \) is a linear isomorphism, then
\[
d_B[A, D, z] = \begin{cases} 
\text{sign } \det A & \text{if } z \in A(D) \\
0 & \text{if } z \not\in A(D).
\end{cases}
\]

Now this last result can be expressed in terms of the eigenvalues of \( A \). Recall that if \( \lambda_1, \ldots, \lambda_r \) are the real eigenvalues of \( A \), with respective multiplicities \( m_1, \ldots, m_r \), and if \( \mu_1, \mu_2, \ldots, \mu_s, \mu_s \) are the non real eigenvalues of \( A \), with respective multiplicities \( n_1, \ldots, n_s \), then \( r + 2s = n \) and
\[
\det (A - \mu I) = \prod_{j=1}^{r} (\lambda_j - \mu)^{m_j} \prod_{k=1}^{s} (\mu_k - \mu)^{n_k}
\]
and hence
\[
\det A = \prod_{j=1}^{r} \lambda_j^{m_j} \prod_{k=1}^{s} |\mu_k|^{2n_k}.
\]
Consequently, if \( z \in A(D) \),
\[
d_B[A, D, z] = \text{sign } \det A = (-1)^{\sum_{j=1}^{r} m_j}
\]
where the sum is over the negative eigenvalues \( \lambda_1, \ldots, \lambda_p \) of \( A \). Of course one can use as well the sum over the negative characteristic values of \( A \).
CHAPTER 3. BROUWER DEGREE

3.2 An exact form and homotopy invariance

For $D \subset \mathbb{R}^n$ open and bounded, let $w \in C^1(\mathbb{R}^n, \mathbb{R})$, $F \in C^2(D \times [0, 1], \mathbb{R}^n)$, and

$$dF_j = \sum_{j=1}^n \partial_j F(\cdot, \lambda) \, dx_j.$$  

The following lemma shows that $\partial_\lambda \{ F(\cdot, \lambda)^* [w(y) \omega] \}$ is an exact form.

**Lemma 3.2.1** For each $\lambda \in [0, 1]$, we have

$$\partial_\lambda [(w \circ F) \, dF_1 \wedge \ldots \wedge dF_n] = d \left( (w \circ F) \left( \sum_{j=1}^n (\partial_\lambda F_j) \, dF_1 \wedge \ldots \wedge \hat{dF}_j \wedge \ldots \wedge dF_n \right) \right).$$  \hspace{1cm} (3.6)

**Proof.** We first notice that

$$\partial_\lambda [\partial_\lambda F_j \, dx_j] = \sum_{k=1}^n \partial_k \partial_\lambda F_j \, dx_k = \sum_{k=1}^n \partial_k \partial_\lambda F_j \, dx_k = d[\partial_\lambda F_j].$$

Now

$$\partial_\lambda [(w \circ F) \, dF_1 \wedge \ldots \wedge dF_n] = \partial_\lambda [(w \circ F) \, dF_1 \wedge \ldots \wedge dF_n] + (w \circ F) \partial_\lambda [dF_1 \wedge \ldots \wedge dF_n]$$

$$= \sum_{j=1}^n [(\partial_j w \circ F) \partial_\lambda F_j] \, dF_1 \wedge \ldots \wedge dF_n$$

$$+ (w \circ F) \left( \sum_{j=1}^n dF_1 \wedge \ldots \wedge \partial_\lambda F_j \wedge \ldots \wedge dF_n \right)$$

$$= \sum_{j=1}^n (-1)^{j-1} (\partial_j w \circ F) \, dF_j \wedge \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \hat{dF}_j \wedge \ldots \wedge dF_n$$

$$+ (w \circ F) \left\{ \sum_{j=1}^n (-1)^{j-1} d[\partial_\lambda F_j] \wedge dF_1 \wedge \ldots \wedge \hat{dF}_j \wedge \ldots \wedge dF_n \right\}$$

$$= \sum_{j=1}^n (-1)^{j-1} \sum_{k=1}^n [(\partial_k w \circ F) \, dF_k] \wedge \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \hat{dF}_j \wedge \ldots \wedge dF_n$$

$$+ (w \circ F) \left\{ \sum_{j=1}^n (-1)^{j-1} d[\partial_\lambda F_j] \wedge dF_1 \wedge \ldots \wedge \hat{dF}_j \wedge \ldots \wedge dF_n \right\}$$

$$= d[w \circ F] \wedge \left( \sum_{j=1}^n (-1)^{j-1} \partial_\lambda F_j \, dF_1 \wedge \ldots \wedge \hat{dF}_j \wedge \ldots \wedge dF_n \right).$$
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\[
+ (w \circ F) d \left[ \sum_{j=1}^{n} (-1)^{j-1} \partial_{\lambda} F_j \, dF_1 \wedge \ldots \wedge dF_j \wedge \ldots \wedge dF_n \right]
\]
\[= d \left[ (w \circ F) \left( \sum_{j=1}^{n} (-1)^{j-1} \partial_{\lambda} F_j \, dF_1 \wedge \ldots \wedge dF_j \wedge \ldots \wedge dF_n \right) \right].
\]

Define now
\[
I_w[F(\cdot, \lambda)] := \int_D (w \circ F) \, dF_1 \wedge \ldots \wedge dF_n
\]
\[= \int_D F(\cdot, \lambda)^* [w(y) \, dy_1 \wedge \ldots \wedge dy_n] \quad (3.7)
\]
\[= \int_D w[F(x, \lambda)] \, J_{F(\cdot, \lambda)}(x, \lambda) \, dx.
\]

The following result is due to Tartar [130, 382], who stated and proved it without using differential forms.

Lemma 3.2.2 If \( w \in C^1(\mathbb{R}^n, \mathbb{R}) \) and \( F \in C^2(D \times [0, 1], \mathbb{R}^n) \) are such that
\[
F(\partial D \times [0, 1]) \cap \text{supp} \, w = \emptyset,
\]
then \( I_w[F(\cdot, \lambda)] \) is independent of \( \lambda \) on \([0, 1]\).

Proof. Using Lemma 3.2.1, Assumption (3.8) and Stokes formula (1.4), we get
\[
\frac{d}{d\lambda} I_w[F(\cdot, \lambda)] = \int_D \partial_{\lambda} [(w \circ F) \, dF_1 \wedge \ldots \wedge dF_n]
\]
\[= \int_D d \left[ (w \circ F) \left( \sum_{j=1}^{n} (-1)^{j-1} \partial_{\lambda} F_j \, dF_1 \wedge \ldots \wedge dF_j \wedge \ldots \wedge dF_n \right) \right] = 0.
\)

Corollary 3.2.1 If \( F \in C^2(D \times [0, 1], \mathbb{R}^n) \) is such that \( 0 \notin F(\partial D \times [0, 1]) \), then \( d_B[F(\cdot, \lambda), D] \) is independent of \( \lambda \) over \([0, 1]\).

Proof. By assumption,
\[
\mu := \inf_{\partial D \times [0, 1]} \|F\| > 0,
\]
and, if \( 0 < \varepsilon_0 < \mu_0 < \mu \), the corresponding function \( c \) in Definition 3.1.1 (which can always be chosen of class \( C^1 \)) is such that \( w(y) = c(\|y\|) \) has the property (3.8). Thus, by Lemma 3.2.2, \( d_B[F(\cdot, \lambda), D] = I_w[F(\cdot, \lambda)] \) is independent of \( \lambda \) in \([0, 1]\).
Corollary 3.2.2 If $f, g \in C^2(D, \mathbb{R}^n)$ are such that
\[
\nu := \max_{\partial D} \|f - g\| < \mu := \min_{\partial D} \|f\|,
\]
then $d_B[f, D] = d_B[g, D]$.

Proof. Let us define $F : \overline{D} \times [0, 1] \to \mathbb{R}^n$ by
\[
F(x, \lambda) = (1 - \lambda)f(x) + \lambda g(x).
\]
Then, for all $(x, \lambda) \in \partial D \times [0, 1]$, we have
\[
\|F(x, \lambda)\| \geq \|f(x)\| - \|g(x) - f(x)\| \geq \mu - \nu > 0.
\]
The result follows from Corollary 3.2.1.

Corollary 3.2.3 For each $f \in C^2(D, \mathbb{R}^n)$ and $z \notin f(\partial D)$, $d_B[f, D, z]$ is an integer.

Proof. By Sard’s theorem, there exists a regular value $v$ such that
\[
\|v - z\| < \mu = \min_{\partial D} \|f(\cdot) - z\|.
\]
Hence,
\[
\max_{\overline{D}} \|f(\cdot) - z - [f(\cdot) - u]\| = \|z - u\| < \mu,
\]
and, using Corollary 3.2.2 and Proposition 3.1.2 we get
\[
d_B[f, D, z] = d_B[f, D, u] \in \mathbb{Z}.
\]

3.3 Continuous mappings

Let $D \subset \mathbb{R}^n$ be open and bounded, $f \in C(\overline{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, so that
\[
\mu := \min_{\partial D} \|f(\cdot) - z\| > 0.
\]

Let $g$ and $h$ be two mappings in $C^2(\overline{D}, \mathbb{R}^n)$ such that $\max_{\overline{D}} \|g - f\| < \mu/4$ and $\max_{\overline{D}} \|h - f\| < \mu/4$. They exist by Weierstrass’ approximation theorem. Then
\[
\min_{\partial D} \|g(\cdot) - z\| > \mu/2, \quad \max_{\overline{D}} \|h - g\| < \mu/2,
\]
and hence, by Corollary 3.2.2,
\[
d_B[g, D, z] = d_B[h, D, z]. \tag{3.10}
\]

Definition 3.3.1 If $D \subset \mathbb{R}^n$ is open and bounded, $f \in C(\overline{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$, the Brouwer degree $d_B[f, D, z]$ is defined by
\[
d_B[f, D, z] = d_B[g, D, z], \tag{3.11}
\]
where $g \in C^2(\overline{D}, \mathbb{R}^n)$ is such that $\max_{\overline{D}} \|g - f\| < \mu/4$.

The definition is justified by (3.10) showing that the right-side of (3.11) does not depend upon the choice of the approximating mapping $g$. 
3.4 Basic properties

We now easily extend the basic properties of Brouwer degree to the continuous case. First the excision property.

**Theorem 3.4.1** If the open subset \( D' \subset D \) is such that \( z \notin f(D \setminus D') \), then
\[
d_B[f, D, z] = d_B[f, D', z].
\]

**Proof.** Indeed, taking the approximating \( g \) in Definition 3.3.1 such that
\[
\max_D \| g - f \| < \min_{D \setminus D'} \| f(\cdot) - z \|,
\]
one has \( \min_{D \setminus D'} \| g(\cdot) - z \| > 0 \). By Proposition 3.1.1 applied to \( g \), we get
\[
d_B[g, D, z] = d_B[g, D', z],
\]
and hence the result, as \( g \) is clearly a good approximation for defining \( d_B[f, D, z] \) and \( d_B[f, D', z] \).

The existence property follows directly by taking \( D' = \emptyset \).

**Corollary 3.4.1** If \( z \notin f(D) \), then \( d_B[f, D, z] = 0 \). Equivalently, if \( d_B[f, D, z] \neq 0 \), there exists at least one \( x \in D \) such that \( f(x) = z \).

We now extend the homotopy invariance property.

**Theorem 3.4.2** If \( F \in C(D \times [0, 1], \mathbb{R}^n) \) and \( z \notin F(\partial D \times [0, 1]) \), then
\[
d_B[F(\cdot, \lambda), D, z]
\]
is independent of \( \lambda \) on \([0, 1]\).

**Proof.** Let
\[
\mu := \min_{\partial D \times [0, 1]} \| F(\cdot, \cdot) - z \| > 0.
\]
Then there exists \( G \in C^2(D \times [0, 1]) \) such that \( \max_{\partial D \times [0, 1]} \| G - F \| < \mu / 4 \). Consequently, for any \( \lambda \in [0, 1] \), we have, by definition
\[
d_B[F(\cdot, \lambda), D, z] = d_B[G(\cdot, \lambda), D, z],
\]
and, by Proposition 3.2.1, the right-hand member is independent of \( \lambda \) on \([0, 1] \).

**Remark 3.4.1** It is clear that, in Theorem 3.4.2, one can replace \([0, 1]\) by an arbitrary compact interval \([a, b]\).

A direct consequence is Rouché’s property.
Corollary 3.4.2 If \( f, g \in C(\overline{\Omega}, \mathbb{R}^n) \), \( z \notin f(\partial D) \),
\[
\max_{\partial D} \| f - g \| < \mu := \min_{\partial D} \| f(\cdot) - z \|,
\]
then
\[
d_B[f, D, z] = d_B[g, D, z].
\]
Proof. Define \( F(x, \lambda) = (1 - \lambda)f(x) + \lambda g(x) \) and apply Theorem 3.4.2.

Corollary 3.4.3 If \( f, g \in C(\overline{\Omega}, \mathbb{R}^n) \), \( z \notin f(\partial D) \), \( f = g \) on \( \partial D \), then
\[
d_B[f, D, z] = d_B[g, D, z].
\]
Corollary 3.4.4 If \( f \in C(\overline{\Omega}, \mathbb{R}^n) \) \( z \notin f(\partial D) \), then, for any \( y \in B_z(\min_{\partial D} [f(\cdot)] - z) \),
\[
d_B[f, D, y] = d_B[f, D, z].
\]
Proof. Apply Corollary 3.4.2 to \( f(\cdot) - z \) and \( f(\cdot) - y \) at 0.

Corollary 3.4.5 If \( f \in C(\overline{\Omega}, \mathbb{R}^n) \) is such that \( 0 \notin f(\partial D) \) and \( f(\overline{\Omega}) \subset Y \), where \( Y \) is a proper vector subspace of \( \mathbb{R}^n \), then \( d_B[f, D, 0] = 0 \).

Proof. Let \( y \notin Y \) be such that \( \| y \| < \min_{\partial D} \| f \| \). Then, using Corollary 3.4.4 and the existence theorem 3.4.1, we get \( d_B[f, D, 0] = d_B[f, D, y] = 0 \).

Corollary 3.4.6 Let \( f \in C(\overline{\Omega}, \mathbb{R}^n) \). Then \( d_B[f, D, z] \) is constant on each connected component of \( \mathbb{R}^n \setminus f(\partial D) \).

Proof. Each connected component of \( \mathbb{R}^n \setminus f(\partial D) \) is open and connected and hence path-connected. If \( \Delta \) is such a connected component and if \( y \in \Delta, z \in \Delta \), there exists a continuous path \( \theta : [0, 1] \to \Delta \) such that \( \theta(0) = y, \theta(1) = z \). For each \( (x, \lambda) \in \partial D \times [0, 1] \), one has \( f(x) \neq \theta(\lambda) \), and hence
\[
d_B[f, D, y] = d_B[f, D, \theta(0)] = d_B[f - \theta(0), D, 0] = d_B[f - \theta(1), D, 0] = d_B[f - z, D, 0] = d_B[f, D, z].
\]

Remark 3.4.2 The common value of \( d_B[f, D, z] \) for \( z \) belonging to a connected component \( \Delta \) of \( \mathbb{R}^n \setminus f(\partial D) \) is often denoted by
\[
d_B[f, D, \Delta].
\]
Corollary 3.4.7 If \( f \in C(\overline{\Omega}, \mathbb{R}^n) \), \( z \notin f(\partial \Omega) \) and \( d_B[f, D, z] \neq 0 \), then \( f(\Omega) \) is a neighborhood of \( z \).

Proof. By Corollary 3.4.6, \( d_B[f, D, z] \) is constant, and hence non zero, on the (open) connected component of \( \mathbb{R}^n \setminus f(\partial D) \) containing \( z \). Existence property 3.4.1 implies that equation \( f(x) = y \) has at least one solution for each \( y \) belonging to this connected component, which is therefore contained in \( f(D) \).
3.4. BASIC PROPERTIES

We finally prove the additivity property.

**Theorem 3.4.3** Suppose that there exists a sequence \((D_j)\) of open and mutually disjoint subsets of \(D\). If \(z \notin f(D) \cup \bigcup_{j=1}^{\infty} D_j\), then \(d_B[f, D_j, z] = 0\) for all but finitely many \(j\), and

\[
\sum_j d_B[f, D_j, z] = d_B[f, D, z].
\]

**Proof.** By assumption,

\[
0 < \mu_0 := \min_{D \cup \bigcup_{j=1}^{\infty} D_j} \|f(\cdot) - z\|
\]

and

\[
\mu_0 \leq \mu := \min_{\partial D} \|f(\cdot) - z\|, \quad \mu_j := \min_{\partial D_j} \|f(\cdot) - z\| \quad (j = 1, 2, \ldots).
\]

Let \(g \in C^2(D, \mathbb{R}^2)\) be such that \(
\max_{\partial D} \|f - g\| < \mu_0/4,
\)

so that

\[
\frac{3\mu_0}{4} \leq \tilde{\mu} := \min_{\partial D} \|g(\cdot) - z\|, \quad \frac{3\mu_0}{4} \leq \tilde{\mu}_j := \min_{\partial D_j} \|g(\cdot) - z\| \quad (j = 1, 2, \ldots),
\]

and

\[
d_B[f, D, z] = d_B[g, D, z], \quad d_B[f, D_j, z] = d_B[g, D_j, z] \quad (j = 1, 2, \ldots). \tag{3.12}
\]

Let \(0 < r < \tilde{\mu}\). By Sard’s theorem, there exists a regular value \(u \in B[z; r] \subset D\) of \(f\). By Corollary 3.4.2, this implies that

\[
d_B[g, D_j, z] = d_B[g, D_j, u], \quad d_B[g, D_j, z] = d_B[g, D_j, u] \quad (j \geq 1). \tag{3.13}
\]

Now, \(u\) being regular, \(g^{-1}(u)\) is finite, and hence contained in a finite number of the \(D_j\)’s, namely \(D_{j_1}, \ldots, D_{j_m}\). By Theorem 3.4.1 and Definition 3.1.1,

\[
d_B[g, D, u] = d_B[g, \bigcup_{i=1}^{m} D_{j_i}, u] = \sum_{i=1}^{m} d_B[g, D_{j_i}, u]. \tag{3.14}
\]

The result follows from (3.13), (3.14) and (3.12). \(\blacksquare\)

It is now easy to compute the Brouwer degree of a continuous function \(f : [a, b] \to \mathbb{R}^2\) when \(a < b\) and \(f(a)f(b) \neq 0\). Let \(g : \mathbb{R} \to \mathbb{R}\) be defined by

\[
g(x) = f(r) + \frac{f(b) - f(a)}{b - a}(x - a).
\]

\(g\) is affine and equal to \(f\) on the boundary of \([a, b]\), so that, by Corollary 3.4.3,

\[
d_B[f, [a, b], 0] = d_B[g, [a, b], 0].
\]

Consequently, if \(f(a)f(b) > 0\), \(0 \notin g([a, b])\) and \(d_B[g, [a, b], 0] = 0\), if \(f(a) < 0 < f(b)\) (resp. \(f(b) > 0 > f(a)\)), \(g\) has a unique zero in \([a, b]\) at which \(g' > 0\) (resp. \(g' < 0\)), and \(d_B[g, [a, b], 0] = 1\) (resp. \(d_B[g, [a, b], 0] = -1\)). We deduce immediately the following result.
Proposition 3.4.1 Let $a < b$ and $f : [a, b] \to \mathbb{R}$ continuous.

$$d_B[f, [a, b], 0] = \begin{cases} 
0 & \text{if } f(a)f(b) > 0 \\
1 & \text{if } f(a) < 0 < f(b) \\
-1 & \text{if } f(b) > 0 > f(a)
\end{cases}$$

In other words,

$$d_B[f, [a, b], 0] = \frac{1}{2} \left[ \frac{f(b)}{|f(b)|} - \frac{f(a)}{|f(a)|} \right].$$

(3.15)

Corollary 3.4.8 Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial with $a_n \neq 0$. Then, for all sufficiently large $R > 0$,

$$d_B[p, R, R, 0] = \begin{cases} 
0 & \text{if } n \text{ is even} \\
\frac{2^{n-1}}{k_n} & \text{if } n \text{ is odd}.
\end{cases}$$

3.5 The case where $f : \partial D \subset \mathbb{R}^n \to S^{n-1}$

If $D \subset \mathbb{R}^n$ is open and bounded, and if $f : \partial D \to S^{n-1}$ is continuous, it follows from Corollary 3.4.3 that if $g : \overline{D} \to \mathbb{R}^n$ and $h : \overline{D} \to \mathbb{R}^n$ are two continuous extensions of $f$ to $\overline{D}$, $d_B[g, D]$ and $d_B[h, D]$ are defined and equal, as $g$ and $h$ coincide over $\partial D$. Hence the following definition is justified.

Definition 3.5.1 If $D \subset \mathbb{R}^n$ is open and bounded, and if $f : \partial D \to S^{n-1}$ is continuous, the Brouwer degree $d_B[f, \partial D]$ is defined by

$$d_B[f, \partial D] = d_B[g, D],$$

(3.16)

where $g : \overline{D} \to \mathbb{R}^n$ is any continuous extension of $f$ to $\overline{D}$.

Of course $d_B[f, \partial D]$ inherits the properties of the Brouwer degree $d_B[g, D]$, except that the existence property takes the following form.

Proposition 3.5.1 If $D \subset \mathbb{R}^n$ is open and bounded, $f \in C(\partial D, S^{n-1})$ and $f$ is not onto, then $d_B[f, \partial D] = 0$.

Proof. Let $y \in S^{n-1} \setminus f(\partial D)$. Using Tietze-Dugundji’s extension theorem there exists a continuous extension $\tilde{f} : \mathbb{R}^n \to \overline{B}(1)$ of $f$. Consider the homotopy $F : \overline{B}(1) \times [0, 1] \to \mathbb{R}^n$ defined by

$$F(x, \lambda) = (1 - \lambda)\tilde{f}(x) - \lambda y.$$

If there exists $(x, \lambda) \in \partial D \times [0, 1]$ such that $F(x, \lambda) = 0$, then, because $\tilde{f} = f$ on $\partial D$,

$$(1 - \lambda)(1 - \lambda)\|f(x)\| = \lambda\|y\| = \lambda,$$

so that $\lambda = \frac{1}{2}$ and $\frac{1}{2}|f(x) - y| = 0$, which is impossible. Hence, it follows from the homotopy invariance property 3.4.2 and the definition of $d_B[f, \partial D]$ that

$$d_B[f, \partial D] = d_B[\tilde{f}, D] = d_B[F(\cdot, 0), D] = d_B[F(\cdot, 1), D] = d_B[-y, D] = 0.$$

$\blacksquare$
Proposition 3.5.1 can of course be expressed in the equivalent contraposed way.

**Corollary 3.5.1** If $D \subset \mathbb{R}^n$ is open and bounded, $f \in C(\partial D, S^{n-1})$ and $d_B[f, \partial D] \neq 0$, $f$ is onto.

The following formula for the degree of fixed point free mappings of the sphere into itself was obtained in 1912 by Brouwer [39].

**Proposition 3.5.2** If $g \in C(S^{n-1}, S^{n-1})$ has no fixed point, $d_B[g, S^{n-1}] = (-1)^n$.

**Proof.** Let $\tilde{g} : \mathbb{R}^n \to \overline{B}(1)$ be a continuous extension of $f$ given by Tietze-Dugundji’s theorem, and consider the homotopy $F : \overline{B}(1) \times [0, 1] \to \mathbb{R}^n$ defined by

$$F(x, \lambda) = (1 - \lambda)\tilde{g}(x) - \lambda x.$$

If $F(x, \lambda) = 0$ for some $(x, \lambda) \in \partial B(1) \times [0, 1] = S^{n-1} \times [0, 1]$, then, as $\tilde{g} = g$ on $\partial B(1)$,

$$(1 - \lambda)\|g(x)\| = \lambda\|x\| = \lambda,$$

so that $\lambda = \frac{1}{2}$ and $\frac{1}{2}[g(x) - x] = 0$, which is impossible by assumption. Consequently, using the definition of $d_B[g, S^{n-1}]$, the homotopy invariance property 3.4.2, and Corollary 3.1.1, we get

$$d_B[g, S^{n-1}] = d_B[\tilde{g}, B(1)] = d_B[F(\cdot, 0), B(1)] = d_B[F(\cdot, 1), B(1)] = d_B[-I, B(1)] = (-1)^n.$$  

The equivalent contraposed version of Proposition 3.5.2 is **Brouwer’s fixed point theorem on spheres** [39].

**Corollary 3.5.2** Any mapping $g \in C(S^{n-1}, S^{n-1})$ such that $d_B[g, S^{n-1}] \neq (-1)^n$ has at least one fixed point.

This is in particular the case for any $g \in C(S^{n-1}, S^{n-1})$ which is not onto. On the other hand, there exist mappings $g \in C(S^{n-1}, S^{n-1})$ without fixed point, for example, when $n = 2$, any non-trivial rotation, or, for any $n$, the antipodal mapping $x \mapsto -x$, and there exist also mappings $g$ having fixed points and degree $(-1)^n$, for example the identity for $n$ even.

As the same boundary dependence property of course holds for Kronecker index, we can similarly define the **Kronecker index** of a $C^1$ mapping $f : \partial D \to S^{n-1}$.

**Definition 3.5.2** If $D \subset \mathbb{R}^n$ is open and bounded, $\partial D$ is smooth and if $f \in C^1(\partial D, S^{n-1})$, the **Kronecker index** $i_K[f, \partial D]$ is defined by

$$i_K[f, \partial D] = i_K[g, \partial D],$$

where $g : \overline{D} \to \mathbb{R}^n$ is any $C^1$ extension of $f$ to $\overline{D}$.

Hence

$$d_B[f, \partial D] = i_K[f, \partial D]$$

when $f \in C^1(\partial D, S^{n-1})$ and $\partial D$ is sufficiently smooth.
Chapter 4

Uniqueness of degree

4.1 Introduction

The aim of this chapter is to show that there exists only one application $d$ from the set of $(D, f)$ with $D \subset \mathbb{R}^n$ open and bounded, $f : \overline{D} \to \mathbb{R}^n$ continuous and $0 \notin f(\partial D)$, into the set $\mathbb{Z}$ of the integers, which satisfies the following three properties:

(A 1) (Normalization) $d[I, D] = 1$ if $0 \in D$.

(A 2) (Additivity) $d[f, D] = d[f, D_1] + d[f, D_2]$ if $D_1$ and $D_2$ are disjoint open subsets of $D$ such that $0 \notin f(D \setminus (D_1 \cup D_2))$.

(A 3) (Homotopy invariance) If $F \in C(\overline{D} \times [0, 1], \mathbb{R}^n)$ and $0 \notin F(\partial D \times [0, 1])$, then $d_B[F(\cdot, \lambda), D]$ is independent of $\lambda$ on $[0, 1]$.

The existence of such a function, namely the Brouwer degree $d_B$, has been proved in the previous chapter. It remains to prove the uniqueness, and this was first shown independently by Führer [139] and Amann-Weiss [12], by reducing the problem to a very special case.

We first notice that, by taking $D_1 = D$ and $D_2 = \emptyset$ in Property (A 2), we obtain

$$d[f, \emptyset] = 0,$$

and by taking then $D_2 = \emptyset$ in the same property, we get the

(P 1) (Excision property) : If $0 \notin f(D \setminus D_1)$, then $d[f, D] = d[f, D_1]$.

4.2 Reduction to the linear case

Let $f \in C(\overline{D}, \mathbb{R}^n)$ and $0 \notin f(\partial D)$. Then $\mu := \min_{\partial D} \|f\| > 0$. The Weierstrass approximation theorem implies that there exists $g \in C(\overline{D}) \cap C^\infty(D)$ such that

$$\max_{\overline{D}} \|f - g\| < \mu.$$  \hspace{1cm} (4.1)
The function $F \in C(\overline{D} \times [0,1], \mathbb{R}^n)$ defined by

$$F(x, \lambda) = (1-\lambda)f(x) + \lambda g(x)$$

is such that, for each $x \in \partial D$ and $\lambda \in [0,1]$, one has

$$\|F(x, \lambda)\| \geq \|f(x)\| - \lambda\|f(x) - g(x)\| \geq \|f(x)\| - \|f(x) - g(x)\| > 0.$$  

Consequently, Property A 3 implies that $d[f, D] = d[g, D]$ and hence $d$ is uniquely determined by its values on $C^\infty$ functions.

Now, by Sard’s theorem, there exists a regular value $z$ for $g$ such that $\|z\| < \mu$. Hence the function $G \in C(\overline{D} \times [0,1], \mathbb{R}^n)$ defined by

$$G(x, \lambda) = g(x) - \lambda z$$

is such that, for each $x \in \partial D$ and $\lambda \in [0,1]$, one has

$$\|G(x, \lambda)\| \geq \|g(x)\| - \lambda\|z\| \geq \|g(x)\| - \|z\| > 0.$$  

Consequently, Property A 3 implies that $d[g, D] = d[g(\cdot) - z, D]$, and $d$ is uniquely determined by its values on $C^\infty$ functions for which 0 is a regular value.

For such a function, it follows from the argument of the beginning of the proof of Proposition 3.1.2 that $g^{-1}(0) = \{x^1, \ldots, x^m\}$ finite. If $g^{-1}(0) = \emptyset$, we have $d[g, D] = d[g, \emptyset] = 0$. If $g^{-1}(0) \neq \emptyset$, there exists mutually disjoint open balls $U_i$ centered at $x^i$ ($1 \leq i \leq m$), contained in $D$ and, using Property A 2, we obtain

$$d[g, D] = \sum_{j=1}^{m} d[g, U_j].$$

Thus $d[g, D]$ is uniquely determined by the values of $d[g, U_j]$ where $U_j$ is an open ball in $D$ centered at a non-degenerate zero $x^j$ of $g$ and containing no other zero of $g$.

For such an $x^j$, we have

$$g(x) = g'_{x^j}(x-x^j) + r(x) := A(x-x^j) + r(x),$$

with $\lim_{x \to x^j} r(x)/\|x-x^j\| = 0$. As 0 is a regular value for $g$, $\det A \neq 0$, and there exists $c > 0$ such that

$$\|A(y)\| \geq c\|y\|$$

for all $y \in \mathbb{R}^n$. Furthermore, using excision property (P 1), we can reduce if necessary the radius of $U_j$ in such a way that

$$\|r(x)\| \leq \frac{c}{2} \|x-x^j\|.$$
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for all $x \in U_j$. Hence, if $H \in C(\mathcal{T} \times [0,1], \mathbb{R}^n)$ is defined by

$$H(x, \lambda) = (1 - \lambda)g(x) + \lambda A(x - x^j),$$

we have, for all $(x, \lambda) \in \partial U_j \times [0,1],$

$$\|H(x, \lambda)\| \geq \|A(x - x^j)\| - \lambda \|r(x)\| \geq \frac{c}{2}\|x - x^j\| > 0,$$

which implies, using Property A 3, that

$$d[g, U_j] = d[A(-x^j), U_j]. \quad (4.2)$$

Now, as $x^j$ is the only zero of $A(-x^j)$, Property A 2 implies that

$$d[A(-x^j), B(r)] = d[A(-x^j), U_j] \quad (4.3)$$

for any $r > 0$ is such that $U_j \subset B(r)$. Furthermore if $J \in C(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$ is defined by

$$J(x, \lambda) = A(x - \lambda x^j),$$

we have $J(x, \lambda) \neq 0$ for all $(x, \lambda) \in \partial B(r) \times [0,1]$, because $J(x, \lambda) = 0$ if and only if $x = \lambda x^j$, and $x \in B(r)$ for all $\lambda \in [0,1]$. Consequently, by Property A 3, we have

$$d[A(-x^j), B(r)] = d[A, B(r)],$$

which, together with (4.2) and (4.3), shows that $d[g, U_j]$ is uniquely determined by the values of $d[A, B(r)]$ where $A$ is a nonsingular linear mapping.

4.3 Uniqueness for linear mappings

We show in this section that properties A 1 to A 3 imply that if $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear nonsingular mapping and $r > 0$, then

$$d[A, B(r)] = \text{sign } \det A. \quad (4.4)$$

We first recall a classical result of linear algebra.

**Lemma 4.3.1** Let $A$ be a real $n \times n$ matrix such that $\det A \neq 0$, let $\lambda_1, \ldots, \lambda_p$ be its possible negative eigenvalues, with respective multiplicities $m_1, \ldots, m_p$. Then $\mathbb{R}^n$ can be written as the direct sum $\mathbb{R}^n = N \oplus M$ of two vector subspaces, such that

(a) $N$ and $M$ are invariant under $A$.

(b) $A|_N$ only has the eigenvalues $\lambda_1, \ldots, \lambda_p$ and $A|_M$ has no negative eigenvalues.

(c) $\dim N = \sum_{k=1}^{p} m_k$. 


Under the above assumptions, we have
\[
\det (A - \mu I) = (-1)^n \prod_{k=1}^{p} (\mu - \lambda_k)^{m_k} \prod_{j=p+1}^{s} (\mu - \mu_j)^{q_j},
\]
where the \(\mu_j\) are the nonnegative eigenvalues of \(A\) with respective multiplicities \(q_j\). A being real, the complex eigenvalues occur in conjugate pairs, and hence
\[
\det A = (-1)^m \prod_{k=1}^{p} |\lambda_k|^{m_k} \prod_{j=p+1}^{s} |\mu_j|^{q_j},
\]
where \(m = \sum_{k=1}^{p} m_k\). Consequently
\[
\text{sign } \det A = (-1)^m.
\] (4.5)

Assume first that \(A\) has no negative eigenvalues. Then, for all \(\lambda \in [0, 1]\), one has
\[
\det [\lambda A + (1 - \lambda)I] \neq 0,
\]
and, using Properties A 1 and A 3 with \(F(x, \lambda) = \lambda A + (1 - \lambda)I\), we get
\[
d[A, B(r)] = d[I, B(r)] = 1 = (-1)^0,
\]
so that (4.4) follows from (4.5).

Assume now that \(N \neq \{0\}\) and suppose first that \(m = \dim N\) is even. Denote respectively by \(P\) and \(Q = I - P\) the linear projectors \(P : \mathbb{R}^n \to N, \ Q : \mathbb{R}^n \to M\). Then \(A = AP + AQ\), with \(AP(\mathbb{R}^n) \subset N\) and \(AQ(\mathbb{R}^n) \subset M\) by the invariance of \(N\) and \(M\) under \(A\). Furthermore, \(AP\) has only negative eigenvalues et \(AQ\) has no negative eigenvalues. Let us define the homotopy \(F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n\) by
\[
F(x, \lambda) = \lambda Ax + (1 - \lambda)(-Px + Qx).
\]
If \(F(x, 0) = 0\), then \(Px = 0, Qx = 0\), and hence \(x = 0\). If \(F(x, \lambda) = 0\) for some \(\lambda \in [0, 1]\), then by projecting on the subspaces \(N\) and \(M\), and using their invariance under \(A\), we get
\[
APx = \frac{1 - \lambda}{\lambda} Px, \quad AQx = \frac{1 - \lambda}{\lambda} Qx,
\]
which is only possible if \(Px = 0\) and \(Qx = 0\). Consequently, Property A 3 implies that
\[
d[A, B(r)] = d[-P + Q, B(r)].
\] (4.6)
Now, since \(m\) is non zero and even, one can find a linear mapping \(B : N \to N\) such that \(B^2 = -I\mid_N\). The eigenvalues of \(B\) are imaginary, as easily checked. Consider the homotopies \(G\) and \(H\) respectively defined by
\[
G(x, \lambda) = \lambda BPx - (1 - \lambda)Px + Qx, \quad H(x, \lambda) = \lambda BPx + (1 - \lambda)Px + Qx.
\]
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They trivially only vanish at 0 when \( \lambda = 0 \). If \( \lambda \in [0, 1] \) and \( G(x, \lambda) = 0 \), (resp. \( H(x, \lambda) = 0 \), then

\[
BPx = \frac{1 - \lambda}{\lambda} Px, \quad Qx = 0, \quad (\text{resp. } BPx = -\frac{1 - \lambda}{\lambda} Px, \quad Qx = 0),
\]

and hence \( Px = 0, \quad Qx = 0 \). Consequently, Property A 3 implies that

\[
d[-P + Q, B(r)] = d[BP + Q, B(r)] = d[P + Q, B(r)] = 1 = (-1)^m. \tag{4.7}
\]

It follows from (4.5), (4.6) and (4.7) that (4.4) holds.

It remains to consider the case where \( m = \dim N = 2q + 1 \) for some integer \( q \geq 0 \). Then we can write \( N = N_1 \oplus N_2 \), with \( \dim N_1 = 1 \) and \( \dim N_2 = 2r \), and corresponding linear projectors \( P_1 : N \to N_1, \quad Q_1 = I - P_1 : N \to N_2 \). Then, \( P = P_1 P + Q_1 P \), and, following the line of case where \( m \) is even, we introduce the sequence of homotopies

\[
F(x, \lambda) = \lambda Ax + (1 - \lambda)(-Px + Qx),
\]

\[
\tilde{G}(x, \lambda) = \lambda(-P_1 Px + BQ_1 Px) - (1 - \lambda)Px + Qx,
\]

\[
\tilde{H}(x, \lambda) = -P_1 Px + \lambda BQ_1 Px + (1 - \lambda)Q_1 Px + Qx,
\]

where \( B : N_2 \to N_2 \) is linear and such that \( B^2 = -I|_{N_2} \). One checks like above that none of them vanishes on \( \partial B(r) \times [0, 1] \), and hence

\[
d[A, B(r)] = d[-P + Q, B(r)] = d[-P_1 P + BQ_1 P + Q, B(r)]
\]

\[
= d[-P_1 P + Q_1 P + Q, B(r)] = d[-P_2 + Q_2, B(r)], \tag{4.8}
\]

if we set

\[
P_2 = P_1 P, \quad Q_2 = P_2 P + Q,
\]

so that \( P_2 \) and \( Q_2 = I - P_2 \) are the projectors associated to the decomposition \( \mathbb{R}^n = N_1 \oplus (N_2 \oplus M) \). Of course, using Property A 2, we also have

\[
d[-P_2 + Q_2, B(r)] = d[-P_2 + Q_2, B_1(0, r) + B_2(0, r)], \tag{4.9}
\]

where \( B_1(0, r) = B(r) \cap N_1 \) and \( B_2(0, r) = B(r) \cap (N_2 \oplus M) \).

We reduce this situation to a one-dimensional one. Namely, if \( D \subset N_1 \) is open and bounded, and if \( g : \overline{D} \to N_1 \) is continuous and such that \( 0 \notin g(\partial D) \), let us define \( d_1[g, D] \) by

\[
d_1[g, D] = d[g \circ P_2 + Q_2, D + B_2(0, r)],
\]

so that, in particular,

\[
d[-P_2 + Q_2, B_1(0, r) + B_2(0, r)] = d_1[-I, B_1(0, r)]. \tag{4.10}
\]
Then, clearly, using Property A 1,

\[ d_1[I, D] = d[I, D + B_2(0, r)] = 1 \quad \text{if} \quad 0 \in D. \quad (4.11) \]

Now, \( g(P_2x) + Q_2x = 0 \) is and only if \( Q_2x = 0 \) and \( x = P_2x \) is a zero of \( g \). Hence, if \( D_1 \) and \( D_2 \) are disjoint open subsets of \( D \) and \( 0 \notin g(\overline{D} \setminus (D_1 \cup D_2)) \), we have

\[ 0 \notin (g \circ P_2 + Q_2)([D + B_2(0, r)] \setminus [(D_1 + B_2(0, r)) \cup (D_2 + B_2(0, r))]). \]

so that Property A 2 implies that

\[
\begin{align*}
d_1[g, D] &= d[g \circ P_2 + Q_2, D + B_2(0, r)] \\
&= d[g \circ P_2 + Q_2, D_1 + B_2(0, r)] + d[g \circ P_2 + Q_2, D_2 + B_2(0, r)] \\
&= d_1[g, D_1] + d_2[g, D_2]. \quad (4.12)
\end{align*}
\]

Finally, if \( G : \overline{D} \times [0, 1] \to N_1 \) is continuous and such that \( 0 \notin G(\partial D \times [0, 1]) \), then

\[ 0 \notin (G \circ P_2 + Q_2)(\partial(D + B_2(0, r)) \times [0, 1]), \]

and, by Property A 3, we get

\[ d_1[G(\cdot, \lambda), D] = d[G(P_1(\cdot), \lambda) + Q_2, D + B_2(0, r)] \quad \text{is independent of} \quad \lambda. \quad (4.13) \]

Finally we show that

\[ d_1[-I, D] = -1 = (-1)^{2q+1}, \quad (4.14) \]

for any bounded open neighbourhood \( D \) of 0 in \( N_1 \), which, together with (4.8), (4.9) and (4.10), will conclude the proof. We have \( N_1 = \{\alpha \in \mathbb{R} : \alpha = 0\} \) for some \( \varepsilon \in \mathbb{R}^n \) with \( \|\varepsilon\| = 1 \). Let us consider the open bounded sets

\[ D = \{\alpha : \alpha \in [-2, 2]\}, \quad D_1 = \{\alpha : \alpha \in [-2, 0]\}, \quad D_2 = \{\alpha : \alpha \in [0, 2]\}, \]

and let \( h : N_1 \to N_1 \) and \( K : N_1 \times [0, 1] \to N_1 \) be respectively defined by

\[ h(\alpha \varepsilon) = (|\alpha| - 1)\varepsilon, \quad H(\alpha \varepsilon, \lambda) = \lambda(|\alpha| - 2)\varepsilon + \varepsilon. \]

Since \( h(0) = -\varepsilon \neq 0 \), and \( K(\alpha \varepsilon, \lambda) = \varepsilon \neq 0 \) for all \( (\alpha, \lambda) \in \{\pm 2\} \times [0, 1] \), we have, by (4.11), (4.12) and (4.13),

\[
\begin{align*}
0 &= d_1[\varepsilon, D] = d_1[K(\cdot, 0), D] = d_1[K(\cdot, 1), D] \\
&= d_1[h, D] = d_1[h, D_1] + d_1[h, D_2]. \quad (4.15)
\end{align*}
\]

Now, on \( D_1 \), \( h(\alpha \varepsilon) = -(\alpha + 1)\varepsilon := g(\alpha \varepsilon) \), with \( g(\alpha \varepsilon) \neq 0 \) on \( \overline{D} \setminus D_1 \), so that

\[ d_1[h, D_1] = d_1[g, D_1] = d_1[g, D]. \]
4.3. **UNIQUENESS FOR LINEAR MAPPINGS**

Using the homotopy \( L : \mathcal{D} \times [0, 1] \rightarrow N_1 \) defined by

\[
L(\alpha e, \lambda) = -(\alpha + \lambda)e,
\]
we see immediately that \( L(\alpha e, \lambda) \neq 0 \) for \( (\alpha, \lambda) \in \{\pm 2\} \times [0, 1] \), and hence, by (4.13), we get

\[
d_1[g, D] = d_1[-I, D],
\]
so that

\[
d_1[h, D_1] = d_1[-I, D]. \tag{4.16}
\]

In a similar way, on \( D_2 \), \( h(\alpha e) = (\alpha - 1)e := f(\alpha e) \), with \( f(\alpha e) \neq 0 \) on \( \mathcal{D} \setminus D_2 \), so that

\[
d_1[h, D_2] = d_1[f, D_2] = d_1[f, D].
\]

Using the homotopy \( M : \mathcal{D} \times [0, 1] \rightarrow N_1 \) defined by

\[
M(\alpha e, \lambda) = (\alpha - \lambda)e,
\]
we see immediately that \( M(\alpha e, \lambda) \neq 0 \) for \( (\alpha, \lambda) \in \{\pm 2\} \times [0, 1] \), and hence, by (4.13), we get

\[
d_1[f, D] = d_1[I, D] = 1,
\]
so that

\[
d_1[h, D_2] = 1. \tag{4.17}
\]

From (4.15), (4.16) and (4.17), we obtain

\[
0 = d_1[-I, D] + 1,
\]
and hence

\[
d_1[-I, B(r) \cap N_1] = -1.
\]
Chapter 5

Brouwer index

5.1 Definition

One can localize the concept of Brouwer degree in the neighbourhood of an isolated point of $f^{-1}(z)$.

Definition 5.1.1 Let $f \in C(\overline{D}, \mathbb{R}^n)$, $z \in \mathbb{R}^n$, and $y$ be an isolated element of $D \cap f^{-1}(z)$. The Brouwer index of $f$ at $y$ is defined by

$$i_B[f, y] = d_B[f, B_r(y), z] = d_B[f, B_r(y); f(y)],$$

(5.1)

where $r > 0$ is such that $\{y\} = B_r(y) \cap f^{-1}(z)$.

Example 5.1.1 If $L : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible, then, for each $z \in \mathbb{R}^n$, one has, with $y = L^{-1}(z)$,

$$i_B[L, y] = \text{sign } \det L.$$

(5.2)

Using formula (3.5), one can write this result as

$$i_B[L, 0] = \text{sign } \det L = (-1)^m$$

(5.3)

where $m$ is the sum of the multiplicities of the negative eigenvalues of $L$, or, equivalently the sum of the multiplicities of the negative characteristic values of $L$.

If follows easily from Theorem 3.4.1 that the right-hand member of formula (5.1) does not depend upon the choice of $r$. A direct consequence of the definition and Theorem 3.4.3 is the following

Proposition 5.1.1 Let $f \in C(\overline{D}, \mathbb{R}^n)$ and $z \notin f(\partial D)$ be such that $f^{-1}(z)$ is finite. Then

$$d_B[f, D, z] = \sum_{y \in f^{-1}(z)} i_B[f, y].$$

(5.4)

The assumptions are in particular satisfied when $f$ is of class $C^1$ and $z$ is a regular value.
5.2 Computation

The index is easily computed when \( f \) is of class \( C^1 \) in the neighbourhood of \( y \).

**Theorem 5.2.1** Let \( y \in \mathbb{R}^n \), \( D \) an open neighbourhood of \( y \) and \( f \in C(D, \mathbb{R}^n) \) be such that \( f \) is differentiable at \( y \) and \( J_f(y) \neq 0 \). Then

\[
i_B[f, y] = \text{sign } J_f(y). \tag{5.5}\]

**Proof.** As \( f'_y \) is invertible, \( f'_y(u) \neq 0 \) for all \( u \in \partial B(1) \), and hence there exists \( \mu > 0 \) such that \( \|f'_y(x)\| \geq \mu \) for all \( x \in \partial B(1) \). Consequently, if \( x \neq 0 \),

\[
\|f'_y(x)\| = \left\| f'_y \left( \frac{x}{\|x\|} \right) \right\| \geq \mu \|x\|.
\]

On the other hand, there exists \( r_0 > 0 \) such that, whenever \( \|x - y\| \leq r_0 \), one has

\[
\|f(x) - f(y) - f'_y(x - y)\| \leq \frac{\mu}{2} \|x - y\|.
\]

Hence, for all \( 0 < r \leq r_0 \), \( \lambda \in [0, 1] \) and all \( x \in \partial B_r(y) \), one has

\[
\|(1 - \lambda)[f(x) - f(y)] + \lambda f'_y(x - y)\| = \|f'_y(x) - (1 - \lambda)[f(x) - f(y)] - f'_y(x - y)\|
\]

\[
\geq \|f'_y(x - y)\| - \|f(x) - f(y) - f'_y(x - y)\| \geq \frac{\mu}{2} \|x\| > 0.
\]

It then follows from Theorem 3.4.2 that, for all \( 0 < r \leq r_0 \), one has

\[
d_B[f, B_r(y), f(y)] = d_B[f(\cdot) - f(y), B_r(y), 0] = d_B[f'_y(\cdot - y), B_r(y), 0] \tag{5.6}
\]

\[
= d_B[f'_y, B(r), 0], \tag{5.7}
\]

and the result follows from the definition of the Brouwer index and formula (5.2).

Using Theorem 5.2.1 and Proposition 5.1.1, we obtain the following slight generalization of Proposition 3.1.2.

**Corollary 5.2.1** Let \( D \subset \mathbb{R}^n \) be open and bounded, \( f \in C(\overline{D}, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n) \) and \( z \notin f(\partial D) \) be a regular value for \( f \). Then

\[
d_B[f, D, z] = \sum_{x \in f^{-1}(z)} \text{sign } J_f(x),
\]

and the sum is finite.

An important class for which the index at 0 exists is the one of nondegenerate positive homogeneous continuous mappings.
5.2. COMPUTATION

Definition 5.2.1 We say that \( h \in C(\mathbb{R}^n, \mathbb{R}^m) \) is positive homogeneous of order \( \alpha > 0 \) if, for each \( x \in \mathbb{R}^n \) and each \( \lambda > 0 \), one has
\[
h(\lambda x) = \lambda^\alpha h(x).
\] (5.8)

It follows from the definition that \( h(0) = 0 \). Any linear mapping is of course positive homogeneous of order one.

This concept can be generalized as follows. Let \( p \geq 1 \) be an integer, \( n_j \geq 1 \) be integers, \( \alpha_j > 0 \) be positive real numbers \((j = 1, \ldots, p)\), and \( n = \sum_{j=1}^p n_j \).

Definition 5.2.2 We say that the continuous mapping
\[
h : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}, \ x \mapsto h(x) = (h_1(x), \ldots, h_p(x))
\] (5.9)
is positive homogeneous of orders \( \alpha_1, \ldots, \alpha_p \) if, for each \( j = 1, \ldots, p \), the mapping \( h_j : \mathbb{R}^n \to \mathbb{R}^{n_j} \) is positive homogeneous of order \( \alpha_j \), i.e. if for each \( x \in \mathbb{R}^n \) and each \( \lambda > 0 \), one has
\[
h(\lambda x) = [\lambda^{\alpha_1} h_1(x), \ldots, \lambda^{\alpha_p} h_p(x)].
\] (5.10)

Definition 5.2.3 We say that the positive homogeneous mapping (5.9) is nondegenerate if \( h^{-1}(0) = \{0\} \).

Consequently, if (5.9) is nondegenerate, Corollary 3.4.1 implies that \( d_B[h, B(r), 0] \) exists for all \( r > 0 \) and is independent of \( r \). The common value is the Brouwer index \( i_B[h, 0] \).

Proposition 5.2.1 If (5.9) is nondegenerate, there exists \( \mu > 0 \) such that, for all \( u \in \partial B(1) \subset \mathbb{R}^n \), one has
\[
\sum_{j=1}^p \| h_j(u) \| \geq \mu.
\] (5.11)

Proof. By the nondegeneracy assumption, \( h(u) \neq 0 \), and hence \( \sum_{j=1}^p \| h_j(u) \| > 0 \) for all \( u \in \partial B(1) \), and hence there exists \( \mu > 0 \) such that \( \sum_{j=1}^p \| h_j(u) \| \geq \mu \) whenever \( u \in \partial B(1) \).

Proposition 5.2.2 Let \( \Omega \) be an open neighbourhood of \( 0 \), \( f \in C(\Omega, \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}) \) such that
\[
\lim_{x \to 0} \frac{f_j(x) - h_j(x)}{\| x \|^{\alpha_j}} = 0 \quad (j = 1, \ldots, p),
\] (5.12)
for some positive homogeneous non-degenerate mapping \( h \in C(\mathbb{R}^n, \mathbb{R}^n) \) of type (5.9). Then
\[
i_B[f, 0] = i_B[h, 0].
\] (5.13)
Proof. By Proposition 5.2.1, there exists \( \mu > 0 \) be such that \( h \) satisfies condition (5.11). Consequently, for each \( u \in \partial B(1) \), there exists \( j(u) \in \{1, \ldots, p\} \) such that \( \|h_{j(u)}(u)\| \geq \frac{\mu}{2p} \), and, by continuity, there exists \( \delta(u) > 0 \) such that, for all \( x \in \partial B(1) \cap \overline{B}_u(\delta(u)) \), one has

\[
\|h_{j(u)}(x)\| \geq \frac{\mu}{2p}.
\]

By compactness, there exists a finite family \( \{S_1, \ldots, S_m\} \) of closed sets \( S_k \subset \partial B(1) \) and a finite family of points \( \{u^1, \ldots, u^m\} \) such that \( u^k \subset S_k \subset B_{\mu}(\delta(u^k)) \), \( \partial B(1) = \bigcup_{j=1}^m S_k \) and hence such that

\[
\|h_{j(u^k)}(x)\| \geq \frac{\mu}{2p} \quad \text{for all} \quad x \in S_k \quad (1 \leq k \leq m).
\]

By assumption (5.12), there exists \( r_0 \in ]0,1] \) such that, for all \( j = 1, \ldots, p \), and all \( x \in \overline{B}(r_0) \), one has

\[
\|f_j(x) - h_j(x)\| \leq \frac{\mu}{4p} \|x\|^\alpha.
\]

Let \( r \in ]0, r_0] \), \( x \in \partial B(r) \), and \( k \in \{1, 2, \ldots, m\} \) such that \( \frac{x}{r} \in S_k \). Then, for any \( \lambda \in ]0,1] \) one has, letting \( j_k = j(u^k) \),

\[
\|(1 - \lambda)h_{j_k}(x) + \lambda f_{j_k}(x)\| = \|x\|^\alpha_k \left\| h_{j_k} \left( \frac{x}{\|x\|} \right) - \lambda \frac{h_{j_k}(x) - f_{j_k}(x)}{\|x\|^\alpha_k} \right\| \\
\geq \|x\|^\alpha_k \left\| h_{j_k} \left( \frac{x}{\|x\|} \right) - \frac{h_{j_k}(x) - f_{j_k}(x)}{\|x\|^\alpha_k} \right\| \\
\geq \frac{\mu}{4p} \|x\|^\alpha_k > 0,
\]

and hence \( (1 - \lambda)h(x) + \lambda f(x) \neq 0 \). If follows from Theorem 3.4.2 that, for all \( 0 < r \leq r_0 \), one has

\[
d_B[f, B(r), 0] = d_B[h, B(r), 0] = i_B[h, 0],
\]

and the result follows.

\[
\text{■}
\]

### 5.3 The Brouwer index at infinity

Let \( f \in C(\mathbb{R}^n, \mathbb{R}^n) \) and \( z \in \mathbb{R}^n \) be such that there exists \( R_0 > 0 \) for which

\[
f(x) \neq z \quad \text{whenever} \quad \|x\| \geq R_0.
\]

(5.15)

Hence, by Corollary 3.4.1, \( d_B[f, B(R), z] \) is defined for each \( R \geq R_0 \) and does not depend upon \( R \). This justifies the following
5.3. THE BROUWER INDEX AT INFINITY

Definition 5.3.1 Let \( f \in C(\mathbb{R}^n, \mathbb{R}^n) \) and \( z \in \mathbb{R}^n \) be such that condition (5.15) holds for some \( R_0 \). Then the Brouwer index at infinity of \( f \) at \( z \) is defined by

\[
i_B[f, \infty, z] = d_B[f, B(R), z]
\]

for any \( R \geq R_0 \).

We use the notation \( i_B[f, \infty] \) for \( i_B[f, \infty, 0] \).

Proposition 5.3.1 Let \( f \in C(\mathbb{R}^n, \mathbb{R}^n) \) be such that there exists a continuous non-degenerate homogeneous mapping \( h : \mathbb{R}^n \to \mathbb{R}^n \) of type (5.9) for which

\[
\lim_{\|x\| \to \infty} \frac{f_j(x) - h_j(x)}{\|x\|^\alpha} = 0 \quad (j = 1, \ldots, p).
\]

(5.16)

Then, for each \( z \in \mathbb{R}^n \), one has

\[
i_B[f, \infty, z] = i_B[h, 0].
\]

(5.17)

Proof. By assumption and Proposition 5.2.1, there exists \( \mu > 0 \) such that inequality (5.11) holds for all \( x \in \mathbb{R}^n \). By condition (5.16), there exists \( R_0 > 0 \) such that

\[
\|f_j(x) - z_j - h_j(x)\| \leq \frac{\mu}{4p}\|x\|^\alpha \quad (j = 1, \ldots, p)
\]

whenever \( \|x\| \geq R_0 \). Proceeding like in the proof of Proposition 5.2.2, we can show that if \( x \in \mathbb{R}^n \) is such that \( \|x\| \geq R_0 \) and \( \frac{x}{\|x\|} \in S^k \) for some \( k \in \{1, \ldots, m\} \), and if \( \lambda \in [0, 1] \), we have

\[
\|(1 - \lambda)h_j(x) + \lambda[f_j(x) - z_j]\| \geq \frac{\mu}{4p}\|x\|^\alpha \lambda > 0
\]

and hence \((1 - \lambda)h(x) + \lambda[f(x) - z] \neq 0\). So Theorem 3.4.2 implies that, for each \( R \geq R_0 \),

\[
d_B[f, B(R), z] = d_B[f(\cdot) - z, B(R), 0] = d_B[h, B(R), 0] = i_B[h, 0],
\]

and formula (5.17) follows. \( \Box \)

Corollary 5.3.1 Let \( f \in C(\mathbb{R}^n, \mathbb{R}^n) \) be such that there exists a linear invertible mapping \( L : \mathbb{R}^n \to \mathbb{R}^n \) for which

\[
\lim_{\|x\| \to \infty} \frac{f(x) - L(x)}{\|x\|} = 0.
\]

(5.18)

Then, for each \( z \in \mathbb{R}^n \), one has

\[
i_B[f, \infty, z] = \text{sign det } L.
\]

(5.19)
Chapter 6

Degree in finite-dimensional vector spaces

6.1 Composition with linear isomorphisms

Let \( A, B : \mathbb{R}^n \to \mathbb{R}^n \) be linear isomorphisms, \( D \subset \mathbb{R}^n \) be an open bounded set, \( g : \overline{D} \to \mathbb{R}^n \) be continuous and \( z \not\in Ag(\partial D) \). Then the mapping \( f := A \circ g \circ B \) is continuous on \( B^{-1}(\overline{D}) = B^{-1}(D) \) and \( z \not\in f(\partial B^{-1}(D)) = f(B^{-1}(\partial D)) \), which is equivalent to \( A^{-1}z \not\in g(\partial D) \). Consequently, both \( dB[f, B^{-1}(D), z] \) and \( dB[g, D, A^{-1}z] \) are defined. The following Lemma relates those two Brouwer degrees.

**Lemma 6.1.1** Under the above assumptions,

\[
\begin{align*}
  dB[A \circ g \circ B, B^{-1}(D), z] &= \text{sign } \det(AB) \cdot dB[g, D, A^{-1}z].
\end{align*}
\]

(6.1)

**Proof.** From the definition of Brouwer degree for continuous mappings, Sard’s lemma and Weierstrass approximation theorem, we can assume without loss of generality that \( g \in C^2(D, \mathbb{R}^n) \) and that \( z \) is a regular value for \( A \circ g \circ B \). Hence,

\[
\begin{align*}
  dB[A \circ g \circ B, B^{-1}(D), z] &= \sum_{x \in (A \circ g \circ B)^{-1}(z)} \text{sign } J_{A \circ g \circ B}(x) \\
  &= \sum_{x \in (A \circ g \circ B)^{-1}(z)} \text{sign } [\det A \cdot J_g(Bx) \cdot \det B] \\
  &= (\text{sign } \det AB) \sum_{y \in g^{-1}(A^{-1}z)} \text{sign } J_g(y) \\
  &= (\text{sign } \det AB) dB[g, D, A^{-1}z].
\end{align*}
\]

\[\blacksquare\]
6.2 Definitions and basic properties

Let $X$ be an $n$-dimensional real topological vector space. It is well known that if $(\alpha^1, \cdots, \alpha^n)$ is a base in $X$, and $(e^1, \cdots, e^n)$ the canonical base in $\mathbb{R}^n$, then the linear mapping

$$h : X \rightarrow \mathbb{R}^n, x = \sum_{j=1}^{n} x_j \alpha^j \mapsto h(x) = \sum_{j=1}^{n} x_j e^j$$

is a homeomorphism.

Let now $D \subset X$ be open and bounded, $f : \overline{D} \rightarrow X$ continuous and $z \in X$ such that $z \not\in f(\partial D)$. Then $h \circ f \circ h^{-1}$ is a continuous mapping from the closure of the open bounded set $h(D) \subset \mathbb{R}^n$ to $\mathbb{R}^n$ such that $h(z) \not\in h \circ f \circ h^{-1}(\partial h(D))$. Consequently, $d_B[h \circ f \circ h^{-1}, h(D), h(z)]$ is well defined. If now $(\beta_1, \cdots, \beta_n)$ is another base in $X$, and if

$$g : X \rightarrow \mathbb{R}^n, x = \sum_{j=1}^{n} x_j \beta^j \mapsto h(x) = \sum_{j=1}^{n} x_j e^j,$$

is the corresponding linear homeomorphism, then $d_B[g \circ f \circ g^{-1}, g(D), g(z)]$ is well defined. Now

$$g \circ f \circ g^{-1} = g \circ h^{-1} \circ h \circ f \circ h^{-1} \circ h \circ g^{-1},$$

so that, if we set $m = h \circ g^{-1}$, then $m : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear homeomorphism and

$$g \circ f \circ g^{-1} = m^{-1} \circ (h \circ f \circ h^{-1}) \circ m.$$  

We can therefore apply Lemma 6.1.1 to obtain

$$d_B[g \circ f \circ g^{-1}, g(D), g(z)] = d_B[m^{-1} \circ (h \circ f \circ h^{-1}) \circ m, m, g(D), g(z)] = d_B[m^{-1} \circ (h \circ f \circ h^{-1}) \circ m, m^{-1}(m \circ g(D)), g(z)] = d_B[m^{-1} \circ (h \circ f \circ h^{-1}) \circ m, m^{-1}h(D), m^{-1}(h(z))] = d_B[h \circ f \circ h^{-1}, h(D), h(z)].$$

This independence of the Brouwer degree with respect to the choice of the base justifies the following definition.

**Definition 6.2.1** Let $X$ be a $n$-dimensional topological vector space, $D \subset X$ open and bounded, $f : \overline{D} \rightarrow X$ continuous and $z \in X$ such that $z \not\in f(\partial D)$. The **Brouwer degree** $d_B[f, D, z]$ is defined by the formula

$$d_B[f, D, z] = d_B[h \circ f \circ h^{-1}, h(D), h(z)]$$

where

$$h : X \rightarrow \mathbb{R}^n, x = \sum_{j=1}^{n} x_j \alpha_j \mapsto \sum_{j=1}^{n} x_j e^j.$$
is the linear homeomorphism associated to a base \((\alpha^1, \ldots, \alpha^n)\) of \(X\) and the canonical base \((e^1, \ldots, e^n)\) of \(\mathbb{R}^n\).

It is easy to show, from this definition, that the degree in space \(X\) conserves the properties of degree in \(\mathbb{R}^n\).

Suppose now that \(X\) and \(Z\) are two \(n\)-dimensional topological vector spaces, \(D \subset X\) is open and bounded, \(f : \overline{D} \to Z\) is continuous and \(z \notin f(\partial D)\). Choosing bases \((\alpha^1, \ldots, \alpha^n)\) and \((\beta^1, \ldots, \beta^n)\) in \(X\) and \(Z\) respectively, and denoting by \(h : X \to \mathbb{R}^n\) and \(g : Z \to \mathbb{R}^n\) linear homeomorphisms constructed like above, we see that the Brouwer degree \(d_B[g \circ f \circ h^{-1}, h(D), g(z)]\) is well defined. If we change bases, i.e. homomorphisms, then we have, with \(h : X \to \mathbb{R}^n\) and \(\tilde{g} : Z \to \mathbb{R}^n\),

\[
g^{-1} \circ g \circ f \circ h^{-1} \circ h = f = \tilde{g}^{-1} \circ \tilde{g} \circ f \circ \tilde{h}^{-1} \circ \tilde{h},
\]

and hence

\[
\tilde{g} \circ f \circ \tilde{h}^{-1} = m \circ g \circ f \circ h^{-1} \circ \tilde{m},
\]

where \(m := \tilde{g} \circ g^{-1} : \mathbb{R}^n \to \mathbb{R}^n\) and \(\tilde{m} := h \circ \tilde{h}^{-1} : \mathbb{R}^n \to \mathbb{R}^n\) are linear homeomorphisms. Then, by Lemma 6.1.1, we obtain, like above that

\[
d_B[\tilde{g} \circ f \circ \tilde{h}^{-1}, \tilde{h}(D), \tilde{h}(z)] = \text{sign} \ (\det m \cdot \det \tilde{m}) dB[g \circ f \circ h^{-1}, h(D), h(z)],
\]

and this relation can be interpreted as defining a Brouwer degree for \(f\) between oriented vector spaces \(X\) and \(Z\).

### 6.3 The Brouwer index

One can extend the Brouwer index to this more general situation. Let \(X, Z\) be two \(n\)-dimensional topological vector spaces, which we suppose oriented if they are different. Let \(D \subset X\) be an open bounded set.

**Definition 6.3.1** Let \(f : \overline{D} \to Z\) be continuous, \(z \in Z\), and \(y\) be an isolated element of \(D \cap f^{-1}(z)\). The **Brouwer index** of \(f\) at \(y\) is defined by

\[
i_B[f, y] = d_B[f, B_y(r), z] = d_B[f, B_y(r), f(y)],
\]

where \(r > 0\) is such that \(\{y\} = B_y(r) \cap f^{-1}(z)\).

If follows easily from the excision property of degree that the right-hand member of formula (6.3) does not depend upon the choice of \(r\).

**Example 6.3.1** If \(L : X \to Z\) is linear and invertible, then, for each \(z \in Z\), one has, with \(y = L^{-1}(z)\),

\[
i_B[L, y] = \text{sign} \ \det gL^{-1},
\]

where \(h : X \to \mathbb{R}^n\) and \(g : Z \to \mathbb{R}^n\) are linear homeomorphisms of the type introduced above.
A direct consequence of Definition 6.3.1 and of Theorem 5.2.1 is the following formula for computing the index at $y$ when $f$ is of class $C^1$ in a neighborhood of $y$ and its Jacobian is not zero.

**Theorem 6.3.1** Let $y \in X$, $D \subset X$ an open neighborhood of $y$ and $f \in C(D, Z)$ be such that $f$ is differentiable at $y$ and $f'(y) : X \to Z$ is invertible. Then, if $h : X \to \mathbb{R}^n$ and $g : Z \to \mathbb{R}^n$ are linear homeomorphisms of the type introduced above,

$$i_B[f, y] = \text{sign } J_{gf^{-1}}(h(y)).$$  \hfill (6.5)

**Proof.** By definition and excision property 3.4.1,

$$i_B[f, y] = d_B[f, B_y(r), f(y)] = d_B[gfh^{-1}, h[B_y(r)], g(f(h(y)))]
= d_B[gfh^{-1}, B_{h(y)}(\rho), g(f(h(y)))].$$

for sufficiently small $r > 0$, and $\rho > 0$ such that $B_{h(y)}(\rho) \subset h(B_y(r))$. Then, using formulas (5.6) we get

$$d_B[gfh^{-1}, B_{h(y)}(\rho), g(f(h(y)))]
= d_B((gfh^{-1})', B_{h(y)}(\rho), 0]
= \text{sign } J_{gf^{-1}}(h(y)).$$

\[\blacksquare\]

A direct consequence of the definition and of the additivity of degree is the following

**Proposition 6.3.1** Let $f : \overline{D} \subset X \to Z$ be continuous, and $z \notin f(\partial D)$ be such that $f^{-1}(z)$ is finite. Then

$$d_B[f, D, z] = \sum_{y \in f^{-1}(z)} i_B[f, y].$$ \hfill (6.6)

The assumptions are in particular satisfied when $f$ is of class $C^1$ and $z$ is a regular value.

### 6.4 Reduction formulas

Suppose now that $X$ is a $n$-dimensional topological vector space, $Y \subset X$ a vector subspace of dimension $m < n$, $D \subset X$ is open and bounded, $c : \overline{D} \to Y$ is continuous and $z \in Y \setminus c(\partial D)$. Let $f := I - c$. Notice that each solution of equation $f(x) = z$ is such that $x = c(x) + z \in Y$ and hence a relation could be expected between $d_B[f, D, z]$ and $d_B[f|_Y, D \cap Y, z]$. This is the conclusion of the first reduction formula due to Leray and Schauder.

**Theorem 6.4.1** Let $X$ be a $n$-dimensional topological vector space, $Y \subset X$ a vector subspace of dimension $1 \leq m < n$, $D \subset X$ be open and bounded, $c : \overline{D} \to Y$ continuous and $z \in Y \setminus c(\partial D)$. If $f := I - c$, then

$$d_B[f, D, z] = d_B[f|_Y, D \cap Y, z].$$ \hfill (6.7)
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Proof. From the definitions of degrees and Weierstrass approximation theorem, we can assume that \( X = \mathbb{R}^n, Y = \mathbb{R}^m = \{ x \in \mathbb{R}^n : x_{m+1} = \cdots = x_n = 0 \} \), and \( c \) is of class \( C^2 \). Now,

\[
f'(x) = I - e'(x) = \begin{pmatrix} I_m - (\partial_x c_i(x))_{1 \leq i, j \leq m} & -(\partial_x c_i(x))_{1 \leq i \leq m, m+1 \leq j \leq n} \\ 0 & I_{n-m} \end{pmatrix}
\]

so that

\[ J_f(x) = \det \begin{pmatrix} I_m - (\partial_x c_i(x))_{1 \leq i, j \leq m} \end{pmatrix}, \]

and hence, if \( x \in Y \),

\[ J_f(x) = J_{f|_Y}(x). \tag{6.8} \]

Therefore, if \( z \in Y \) is a regular value for \( f|_Y \), we have \( J_{f|_Y}(x) \neq 0 \), and hence \( z \) is also a regular value for \( f \) because if \( f(x) = z \), then \( x \in Y \cap D \) and, by (6.8), \( J_f(x) \neq 0 \). Thus, by Sard lemma applied to \( f|_Y \) and the above reasoning, we can assume that \( z \in Y \) is a regular value for both \( f|_Y \) and \( f \). Consequently,

\[ d_B[f, D, z] = \sum_{x \in f^{-1}(z)} \text{sign } J_f(x) = \sum_{x \in f|_Y^{-1}(z)} \text{sign } J_{f|_Y}(x) = d_B[f|_Y, D \cap Y, z]. \]

We can deduce from the previous result a **second reduction formula**, proved in a more general setting in [265].

**Theorem 6.4.2** Let \( X \) and \( Z \) be \( n \)-dimensional topological vector spaces, \( L : X \to Z \) a linear mapping with \( N(L) \neq \{ 0 \} \), \( Y \subseteq Z \) a vector subspace such that \( Z = Y \oplus R(L) \), \( D \subseteq X \) an open bounded set, \( r : \overline{D} \to Y \) a continuous mapping such that \( 0 \notin (L + r)(\partial D) \). Then, for each isomorphism \( J : N(L) \to Y \), and each projector \( P : X \to X \) such that \( R(P) = N(L) \), one has, if \( f := L + r, \)

\[ d_B[f, D, z] = i_B[L + JP, 0] \cdot d_B[J^{-1}r|_{N(L)}, D \cap N(L), 0]. \tag{6.9} \]

Proof. Let \( Q : Z \to Z \) be the projector such that \( R(Q) = Y \) and \( N(Q) = R(L) \). We first notice that if \((L + JP)x = 0\), then, by applying \( Q \) and \( I - Q \) to the equation, we obtain the equivalent system

\[ Lx = 0, \quad JPx = 0 \]

which immediately implies that \( x = 0 \). Thus \( L + JP : X \to Z \), one-to-one, is onto and \( i_B[L + JP, 0] \) is well defined and has absolute value one. Furthermore, if \( z \in Y \), then, again by projection on \( Y \) and \( R(L) \), one gets

\[ (L + JP)(x) = z \iff Lx = 0, \quad JPx = z \iff x =Px = J^{-1}z, \]

where the above reasoning, we can assume that \( z \in Y \) is a regular value for both \( f|_Y \) and \( f \). Consequently,
so that
\[(L + JP)^{-1}z = J^{-1}z.\]
Consequently,
Using Lemma 6.1.1, Theorem 6.4.1 and the definitions above, we get
\[d_B(f, D, 0) = d_B(g \circ h^{-1}, h(D), 0) = d_B(g \circ (L + JP) \circ h^{-1} \circ h \circ (I - P + J^{-1}r) \circ h^{-1}, h(D), 0) = \text{sign det } [g \circ (L + JP) \cdot d_B(h \circ (I - P + J^{-1}r) \circ h^{-1}, h(D), 0) = i_B[L + JP, 0] \cdot d_B[I - P + J^{-1}r, D, 0] = i_B[L + JP, 0] \cdot d_B[(I - P + J^{-1}r)|_{N(L), D \cap N(L), 0}] = i_B[L + JP, 0] \cdot d_B[J^{-1}r|_{N(L), D \cap N(L), 0}].\]

\[\text{Remark 6.4.1} \quad \text{Formula (6.9) implies in particular that } |d_B[L + r, D, 0]| = |d_B[J^{-1}r|_{N(L), D \cap N(L), 0}|.\]

6.5 Computation of the index in degenerate case

The computation of the index of \(f\) at \(y\) when \(f'_y\) is not invertible is more complicated. We state and prove here a generalization, due to B. Laloux [237], of a result of V.B. Melamed [280].

**Theorem 6.5.1** \(m\) and \(k\) being nonnegative integers, and \(D \subset X\) being an open neighborhood of 0, assume that \(f \in C(D, Z)\) can be written in the form
\[f(x) = Lx + \sum_{i=0}^{k} H_{m+i}(x) + R(x),\]
where \(L : X \to Z\) is linear and not invertible, \(H_{m+i}\) is homogeneous of order \(m + i\) \((i = 0, \ldots, k)\), and \(R\) is such that
\[\lim_{x \to 0} \frac{R(x)}{\|x\|^{m+k}} = 0. \quad (6.10)\]
Assume furthermore that the following conditions hold.
(i) For \(i \neq k\), \(H_{m+i}(N(L)) \subset R(L)\).
(ii) \( H_{m+k}(x) \notin R(L) \) for any \( x \in N(L) \setminus \{0\} \).

(iii) There exists \( \alpha > 0 \) such that, for all \( i \neq k \) and all \( x, y \in D \), one has
\[
\| H_{m+i}(x) - H_{m+i}(y) \| < \alpha \| x - y \| \max\{\| x \|^m, \| y \|^m\}.
\]

(iv) \( m > k + 1 \).

Then \( i_B[f, 0] \) is well defined and
\[
i_B[f, 0] = i_B[L + JP, 0] \cdot i_B[J^{-1}QH_{m+k}N(L), 0]
\]
where \( Q : Z \to Z \) is any projector such that \( N(Q) = R(L) \), and \( J : N(L) \to R(Q) \) any isomorphism.

**Proof.** Let us define the homotopy \( F : D \times [0, 1] \to Z \) by
\[
F(x, \lambda) = Lx + (1 - \lambda)QH_{m+k}(x) + \lambda \left( \sum_{i=0}^{k} H_{m+i}(x) + R(x) \right).
\]

Let \( \rho > 0 \) be such that \( \overline{B}(\rho) \subset D \), \( \lambda \in [0, 1] \), and let \( x \in \partial B(\rho) \) be a possible zero of \( F(\cdot, \lambda) \). Then, by applying \( Q \) and \( (I - Q) \) to both members, we see \( x \) is a solution of the system
\[
(1 - \lambda)QH_{m+k}(x) + \lambda \left( \sum_{i=0}^{k} QH_{m+i}(x) + QR(x) \right) = 0
\]
\[
L(I - P)x + \lambda(I - Q) \left( \sum_{i=0}^{k} H_{m+i}(x) + R(x) \right) = 0,
\]
where \( P : X \to X \) is a projector onto \( N(L) \), or equivalently
\[
QH_{m+k}(x) + \lambda \left( \sum_{i=0}^{k-1} QH_{m+i}(x) + QR(x) \right) = 0
\]
\[
L(I - P)x + \lambda(I - Q) \left( \sum_{i=0}^{k} H_{m+i}(x) + R(x) \right) = 0,
\]
noticing that there exists \( \beta > 0 \) such that \( \| L(I - P)x \| \geq \beta \| (I - P)x \| \) for all \( x \in X \), we deduce from the second equation in (6.12) and from assumptions (6.10) and (iii), that there exist \( \rho_1 > 0 \) and \( \rho > 0 \) such that
\[
\| (I - P)x \| \leq \lambda \rho \rho^m
\]
whenever \( x \) is a solution such that \( \| x \| = \rho \in [0, \rho_1] \). Now, if \( \rho_2 > 0 \) is such that \( 2 \rho_2 \rho_2^{m-1} < 1 \), we also have, for any \( \rho \in [0, \min\{\rho_1, \rho_2\}] \), and \( x \) such that \( \| x \| = \rho \),
\[
\| Px \| \geq \| x \| - \| (I - P)x \| \geq \rho - \frac{\rho}{2} = \frac{\rho}{2} > \rho \rho^m \geq \| (I - P)x \|,
\]

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Consequently, from Assumption (ii),
and hence there exists $\rho_3 \leq \min\{\rho_1, \rho_2\}$ such that

$$
\|L\| \rho \rho^{m-1} + \alpha \rho \left( \sum_{i=0}^{k-1} (3/2)^{m+i-1} \rho^{m+i-k-1} + q(\rho) \right) < \gamma
$$
whenever $\rho \in [0, \rho_3]$ which implies that $F(x, \lambda) \neq 0$ whenever $\|x\| = \rho \in [0, \rho_3]$ and $\lambda \in [0, 1]$. Hence, the homotopy invariance property implies that
\[
i_B[f, 0] = d_B[f, B(\rho), 0] = d_B[F(\cdot, 1), B(\rho), 0] = d_B[F(\cdot, 0), B(\rho), 0] = d_B[L + QH_{m+k}, B(\rho), 0].
\]
Using the second reduction formula 6.4.2, we obtain
\[
d_B[L + QH_{m+k}, B(\rho), 0] = i_B[L + JP, 0] \cdot d_B[J^{-1}QH_{m+k}|_{N(L)}, B(\rho) \cap N(L), 0]
\]
\[
= i_B[L + JP, 0] \cdot i_B[J^{-1}QH_{m+k}|_{N(L)}, 0],
\]
where $J : N(L) \to R(Q)$ is any isomorphism.

The special case of Theorem 6.5.1 where $k = 0$ is of interest, and can be traced to M.A. Krasnosel’skii [213]. For other results in this direction, see [403].

**Corollary 6.5.1** $m > 1$ being an integer, and $D \subset X$ being an open neighborhood of 0, assume that $f \in C(D, Z)$ can be written in the form
\[
f(x) = Lx + H_m(x) + R(x),
\]
where $L : X \to Z$ is linear and not invertible, $H_m$ is homogeneous of order $m$ and $R$ is such that (6.10) holds. If $H_m(x) \not\in R(L)$ for any $x \in N(L) \setminus \{0\}$, $i_B[f, 0]$ is well defined and
\[
i_B[f, 0] = i_B[L + JP, 0] \cdot i_B[J^{-1}QH_m|_{N(L)}, 0] \tag{6.18}
\]
where $Q : Z \to Z$ is any projector such that $N(Q) = R(L)$, and $J : N(L) \to R(Q)$ any isomorphism.

**Example 6.5.1** Consider the mapping $f : \mathbb{R}^4 \to \mathbb{R}^4$ defined, in complex notations ($\mathbb{R}^4 \cong \mathbb{C}^2$) by
\[
f_1(z_1, z_2) = \overline{z_1}^2 + z_2^2, \quad f_2(z_1, z_2) = z_1 + z_1z_2 + z_2^3;
\]
so that, in the notations of Corollary 6.5.1, we have
\[
f(z_1, z_2) = L(z_1, z_2) + H_2(z_1, z_2) + R(z_1, z_2),
\]
where
\[
L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad H_2(z_1, z_2) = \begin{pmatrix} z_1^2 \\ z_1z_2 \end{pmatrix}, \quad R(z_1, z_2) = \begin{pmatrix} z_1^2z_2 \\ z_1z_2^2 \end{pmatrix}.
\]
Consequently, $N(L) = \mathbb{C} \times \{0\}$, $R(L) = \{0\} \times \mathbb{C}$, $P(z_1, z_2) = (z_1, 0) = Q(z_1, z_2)$, $R(Q) = \mathbb{C} \times \{0\}$. As $QH_2(z_1, 0) = (z_1^2, 0) \not\in R(L)$ for $z_1 \neq 0$, all the conditions of Corollary 6.5.1 are satisfied, and, taking $J = I$ we get $L + P = I$ and hence, using example 2.2.3,
\[
i_B[f, 0] = i_B[z_1^2, 0] = i_B[(x_1^2 - x_2^2, 2x_1x_2), 0] = 2.
\]
Chapter 7

Continuation theorems

7.1 Extended homotopy invariance property

If $[a, b]$ is a compact interval, we consider the trace of the usual topology of $\mathbb{R}^{n+1}$ on $\mathbb{R}^n \times [a, b]$. Let $\mathcal{D} \subset \mathbb{R}^n \times [a, b]$ be a bounded set which is open for the relative topology of $\mathbb{R}^n \times [a, b]$, i.e. which is the trace on $\mathbb{R}^n \times [a, b]$ of an open set of $\mathbb{R}^{n+1}$. Its relative boundary is denoted by $\partial \mathcal{D}$, and, for each $\lambda \in [a, b]$, let

$$
\mathcal{D}_\lambda := \{ x \in \mathbb{R}^n : (x, \lambda) \in \mathcal{D} \}, \quad (\partial \mathcal{D})_\lambda := \{ x \in \mathbb{R}^n : (x, \lambda) \in \partial \mathcal{D} \}.
$$

Notice that $\partial \mathcal{D}_\lambda \subset (\partial \mathcal{D})_\lambda$, and the inclusion may be strict. Let $F \in C(\overline{\mathcal{D}}, \mathbb{R}^n)$ satisfying the condition

$$
z \notin F(\partial \mathcal{D}). \quad (7.1)
$$

Then, for each $\lambda \in [a, b]$, $z \notin F(\partial \mathcal{D}_\lambda)$, and $d_B[F(\cdot, \lambda), \mathcal{D}_\lambda, z]$ is well defined. We show that it is independent of $\lambda$.

**Theorem 7.1.1** Let $F \in C(\overline{\mathcal{D}}, \mathbb{R}^n)$ and $z \in \mathbb{R}^n$ verify condition (7.1). Then $d_B[F(\cdot, \lambda), \mathcal{D}_\lambda, z]$ is constant on $[a, b]$.

**Proof.** For $\lambda \in [a, b]$ fixed, let

$$
N_\lambda = \{ x \in \mathcal{D}_\lambda : F(x, \lambda) = z \} \subset \mathcal{D}_\lambda.
$$

By assumption, $N_\lambda \cap (\partial \mathcal{D})_\lambda = \emptyset$ so that $N_\lambda \cap \partial \mathcal{D}_\lambda = \emptyset$. As $N_\lambda$ is compact in $\mathcal{D}$, there exists an open neighbourhood $\mathcal{O}_\lambda$ of $N_\lambda$ in $\mathcal{D}_\lambda$ with

$$
N_\lambda \subset \mathcal{O}_\lambda \subset \overline{\mathcal{O}_\lambda} \subset \mathcal{D}_\lambda
$$

and some $\varepsilon_0 > 0$ such that

$$
\mathcal{O}_\lambda \times [\lambda - \varepsilon_0, \lambda + \varepsilon_0] \subset \mathcal{D}.
$$
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Of course, for, say, \( \lambda = a \), one replaces \([\lambda - \varepsilon_0, \lambda + \varepsilon_0]\) by \([a, a + \varepsilon_0]\). We show that there exists \(\varepsilon_1 \in [0, \varepsilon_0]\) such that for each \(\mu \in [\lambda - \varepsilon_1, \lambda + \varepsilon_1] \cap [a, b]\), and each solution \(x\) of equation \(F(x, \mu) = z\), one has \((x, \mu) \in O_{\lambda} \times ([\lambda - \varepsilon_1, \lambda + \varepsilon_1] \cap [a, b])\).

If it is not the case, we can find a sequence \((\mu_n)\) in \([a, b]\) with \(|\mu_n - \lambda| \leq 1/n\) and a sequence \((x_n)\) such that \(F(x_n, \mu_n) = z\) and \((x_n, \mu_n) \notin O_{\lambda} \times ([\lambda - (1/n), \lambda + (1/n)] \cap [a, b])\).

Going if necessary to a subsequence, we can assume that \(x_n \to x\) and \(\mu_n \to \lambda\), so that, by continuity, \(F(x, \lambda) = z\), and, by the openness of \(O_{\lambda}\), \(x \notin O_{\lambda}\). But then \(x \notin N_{\lambda}\), a contradiction. For each \(\mu \in [\lambda - \varepsilon_1, \lambda + \varepsilon_1] \cap [a, b]\), \(d_B[F(\cdot, \mu), O_{\lambda}, z]\) is well defined, and, by Theorem 3.4.2, independent of \(\mu\). Furthermore, by Theorem 3.4.1, one has, for all \(\mu \in [\lambda - \varepsilon_1, \lambda + \varepsilon_1] \cap [a, b]\),

\[
d_B[F(\cdot, \mu), O_{\lambda}, z] = d_B[F(\cdot, \mu), D_{\mu}, z].
\]

Hence \(d_B[F(\cdot, \mu), D_{\mu}, z]\) is locally constant on \([a, b]\), and hence constant there. \(\blacksquare\)

Remark 7.1.1 Of course, one has a result similar to Theorem 7.1.1 for a continuous mapping \(F : D \subset X \to Y\) where \(X\) and \(Z\) are oriented topological vector spaces of the same finite dimension and \(D\) an open bounded set of \(X \times [a, b]\).

7.2 Leray-Schauder continuation theorem

To complete the above result by some information on the structure of the solution set of the homotopy, we need a result from the theory of metric spaces, often referred as Whyburn’s lemma.

Lemma 7.2.1 Let \(K\) be a compact metric space, and \(K_1, K_2\) two closed disjoined subsets of \(K\). Then

i) either there exists a component (maximal closed connected part) of \(K\) meeting \(K_1\) and \(K_2\);

ii) or there exist two disjoint compacts \(\overline{K_1}\) and \(\overline{K_2}\) such that \(K = \overline{K_1} \cup \overline{K_2}\) and \(K_i \subset \overline{K_i}, i = 1, 2\).

An easy and useful consequence of Whyburn lemma is the following Corollary.

Corollary 7.2.1 Let \(M\) be a metric space, \(K\) a compact subset of \(M\). Assume that \(K_0\) and \(K_1\) are two disjoint closed subsets of \(K\) such that no connected component of \(K\) intersect both. Then there exists an open bounded set \(U\) such that

\[
K_0 \subset U, \overline{U} \cap K_1 = \emptyset, \partial U \cap K = \emptyset.
\]
The following result is called the **Leray-Schauder continuation theorem** [251].

**Theorem 7.2.1** Let $F \in C(\overline{D}, \mathbb{R}^n)$, $z \in \mathbb{R}^n$ verify condition (7.1), and let

$$S := \{ (x, \lambda) \in D : F(x, \lambda) = z \}$$

and, for each $\lambda \in [a, b]$ let

$$S_\lambda = \{ x \in \mathbb{R}^n : (x, \lambda) \in S \}.$$

If $d_B[F(\cdot, \overline{\lambda}), D_\lambda, z] \neq 0$ for some $\overline{\lambda} \in [a, b]$, there exists a connected component $\mathcal{C}$ of $S$ which meets both $S_a \times \{ a \}$ and $S_b \times \{ b \}$.

**Proof.** By Theorem 7.1.1, $d_B[F(\cdot, \lambda), D_\lambda, z]$ is constant, and hence nonzero for each $\lambda \in [a, b]$. As $S$ is compact, for each $\lambda \in [a, b]$, the set $S_\lambda \times \{ \lambda \}$ is a closed subset of $S$. Hence, the sets

$$K_1 := (D_a \times \{ a \}) \cap S, \quad K_2 := (D_b \times \{ b \}) \cap S$$

are disjoint closed non-empty subsets and $S$. If no component of $S$ meets both $D_a \times \{ a \}$ and $D_b \times \{ b \}$ then Lemma 7.2.1 implies the existence of compact sets $\widehat{K}_1$ and $\widehat{K}_2$ such that

$$K_1 \subset \widehat{K}_1, \quad K_2 \subset \widehat{K}_2, \quad \widehat{K}_1 \cap \widehat{K}_2 = \emptyset, \quad S = \widehat{K}_1 \cup \widehat{K}_2.$$

Let

$$\varepsilon := (1/2)\text{dist} (\widehat{K}_1, \widehat{K}_2), \quad \mathcal{O} := \{ (x, \lambda) \in D : \text{dist} [(x, \lambda), \widehat{K}_1] < \varepsilon \}.$$

$\mathcal{O}$ is bounded, open, such that $\mathcal{O} \cap K_2 = \emptyset$ and $S \cap \partial \mathcal{O} = \emptyset$. The conditions of Theorem 7.1.1 are satisfied for $\mathcal{O}$, so that $d_B[F(\cdot, \lambda), \mathcal{O}_\lambda, z]$ is constant for each $\lambda \in [a, b]$. But, by Corollary 3.4.1, we have

$$d_B[F(\cdot, a), \mathcal{O}_a, z] = d_B[F(\cdot, a), D_a, z] \neq 0,$$

because $\mathcal{O}_a$ contains all the solutions of equation $F(x, a) = z$. On the other hand, $\mathcal{O}_b = \emptyset$, and hence $d_B[F(\cdot, b), \mathcal{O}_b, z] = 0$, a contradiction. 

The Leray-Schauder continuation theorem can be written in the form of an alternative.

**Theorem 7.2.2** Let $D \subset \mathbb{R}^n \times [a, b]$ an open bounded set and $F : \overline{D} \to \mathbb{R}^n$ be continuous. Assume that $S_a \cap (\partial D)_a = \emptyset$ and that

$$d_B[F(\cdot, a), D_a, 0] \neq 0.$$

Then $S$ contains a connected component $\mathcal{C}$ intersecting $S_a \times \{ a \}$ and $\partial D \cup (S_b \times \{ b \})$. 

Proof. By Theorem 7.2.1, the result is proved if \( S \cap \partial D = \emptyset \). If \( S_1 := S \cap \partial D \neq \emptyset \), then \( S_1 \) is a closed subset of \( S \) disjoint from \( S_0 = S_1 \times \{ a \} \). Hence, by Whyburn’s lemma, either there exists a connected component \( C \) of \( S \) which connects \( S_0 \) and \( S_1 \), and the result is proved, or there exist two disjoint closed sets \( C_0 \) and \( C_1 \) containing respectively \( S_0 \) and \( S_1 \) and such that \( S = C_0 \cup C_1 \). We can then find an open neighborhood \( V_0 \) of \( C_0 \) contained in \( D \) and such that \( V_0 \cap C_1 = \emptyset \), and in particular \( V_0 \cap \partial D = \emptyset \). Thus all condition of Theorem 7.2.1 with \( D \) replaced by \( C_0 \) are satisfied, and \( C_0 \) contains a connected component \( C_0 \) which meets \( S_0 \times \{ a \} \) and \( S_0 \times \{ b \} \).

Remark 7.2.1 Of course, one has results similar to Theorems 7.2.1 and 7.2.2 for a continuous mapping \( F : \overline{D} \subset X \to Y \) where \( X \) and \( Z \) are oriented topological vector spaces of the same finite dimension and \( D \) an open bounded set of \( X \times [a, b] \).

7.3 The case of an unbounded set

We can now formulate a variant of Theorem 7.2.2 for the case where \( D \) is unbounded (and in particular when \( D = \mathbb{R}^n \times [a, b] \)).

**Theorem 7.3.1** Let \( D \subset \mathbb{R}^n \times [a, b] \) be an open unbounded set and \( F : \overline{D} \to X \) be continuous. Let

\[
\mathcal{S} = \{(x, \lambda) \in \overline{D} : F(x, \lambda) = 0\},
\]

and, for each \( \lambda \in [a, b] \) let

\[
\mathcal{S}_\lambda = \{x \in \mathbb{R}^n : (x, \lambda) \in \mathcal{S}\}.
\]

Assume that \( \mathcal{S}_0 \) is bounded, that \( \mathcal{S}_0 \cap (\partial D)_a = \emptyset \) and that

\[
d_B[F(\cdot, a), D_a, 0] \neq 0.
\]

Then \( \mathcal{S} \) contains a connected component \( C \) intersecting \( \mathcal{S}_0 \times \{ a \} \) and which either intersects \( \partial D \cup (\mathcal{S}_0 \times \{ b \}) \) or is unbounded.

**Proof.** Let \( n_0 \) be a positive integer such that \( \mathcal{S}_0 \subset B(n_0) \). For each integer \( n \geq n_0 \),

\[
D_n = D \cap (B(n) \times [a, b])
\]

is an open bounded set for which the conditions of Theorem 7.2.2 are satisfied and therefore there exists \( x_n \in \mathcal{S}_a \) and a connected component \( C_n \) of \( \mathcal{S} \) containing \( x_n \) and meeting \( \partial D_n \cup (\mathcal{S}_0 \cap B(n)) \times \{ b \} \). By the compactness of \( \mathcal{S}_a \), the sequence \( (x_n) \) has an accumulation point \( x_0 \in \mathcal{S}_a \). Assume that the connected component \( C_0 \) of \( \mathcal{S} \) containing \( x_0 \) does not meet \( \partial D \cup (\mathcal{S}_0 \times \{ b \}) \). We have to prove that \( C_0 \) is unbounded. If it is bounded and \( \mathcal{A} \) is an open bounded subset of \( D \) such that \( \mathcal{A} \supset C_0 \cup (\mathcal{S}_0 \times \{ a \}) \), then Theorem 7.2.2 applied to \( \mathcal{A} \) instead of \( D \) implies that \( \mathcal{S} \cap \partial \mathcal{A} \neq \emptyset \). Thus, \( \{(x_0, a)\} \) and \( \mathcal{S} \cap \partial \mathcal{A} \) satisfy the conditions of Whyburn’s theorem and, by assumption, cannot be connected by a connected component of \( \mathcal{S} \). Consequently, by Corollary 7.2.1, there is an open neighbourhood \( V_0 \) of \( (x_0, a) \) in \( \mathcal{A} \) such that \( \partial V_0 \cap \mathcal{S} = \emptyset \). But there exists \( n_1 \geq n_0 \) such that \( x_{n_1} \in V_0 \) and \( B(n_1) \times [a, b] \supset A \). Consequently, \( C_{n_1} \) meets both \( V_0 \) and \( (\mathbb{R}^n \times [a, b]) \setminus V_0 \), and hence \( C_{n_1} \cap \partial V_0 \neq \emptyset \), a contradiction with \( \mathcal{S} \cap \partial V_0 = \emptyset \).
7.3. THE CASE OF AN UNBOUNDED SET

We notice the following useful consequence of Theorem 7.3.1.

**Corollary 7.3.1** Let $F : \mathbb{R}^n \times [a, b] \to \mathbb{R}^n$ be continuous. Assume that $S_a$ is bounded and that

$$d_B[f(\cdot, a), \Omega, 0] \neq 0,$$

for some open bounded set $\Omega \supset S_a$. Then $S$ contains a connected component $C$ intersecting $S_a \times \{a\}$ and which either intersects $S_b \times \{b\}$ or is unbounded.

The example of $n = 1$, $[a, b] = [0, 1]$, $F(x, \lambda) = \lambda - \frac{1}{2} \sin x$, shows that the conclusion of Corollary 7.3.1 does not necessarily hold if we drop the assumption that $S_a$ is bounded.

**Remark 7.3.1** Of course, one has results similar to Theorem 7.3.1 and Corollary 7.3.1 for a continuous mapping $F : D \subset X \to Y$ where $X$ and $Z$ are oriented topological vector spaces of the same finite dimension and $D$ an open unbounded set of $X \times [a, b]$. 

Chapter 8

Fixed points and zeros

8.1 Semilinear equations

We deduce here some useful existence theorems based on Theorem 7.2.1. Let $X$, $Z$ be oriented topological vector spaces of dimension $n$, $L : X \to Z$ a linear mapping, $\mathcal{D} \subset X \times [0,1]$ an open bounded set and $N : \overline{\mathcal{D}} \to Z$ a continuous mapping. Let

$$S := \{(x, \lambda) \in \overline{\mathcal{D}} : Lx + \lambda N(x, \lambda) = 0\}$$

**Theorem 8.1.1** Assume that there exists a linear mapping $A : X \to Z$ such that the following conditions hold.

1. $L + A$ is invertible;
2. $Lx + (1 - \lambda)A + \lambda N(x, \lambda) \neq 0$ for each $x \in (\partial \mathcal{D})_\lambda$ and each $\lambda \in ]0,1]$;
3. $0 \in \mathcal{D}_0$.

Then

$$d_B[L + (1 - \lambda)A + \lambda N(\cdot, \lambda), \mathcal{D}_\lambda, 0] = d_B[L + A, \mathcal{D}_\lambda, 0] = \text{sign det}(L + A) \quad (\lambda \in [0,1]),$$

and $S$ contains a continuum $C$ connecting $(0,0)$ to $\mathcal{D}_1$, along which $\lambda$ takes all values in $[0,1]$. In particular the equation $Lx + N(x, 1) = 0$ has at least one solution in $\mathcal{D}_1$.

**Proof.** Because $L + A$ is invertible and $0 \in \mathcal{D}_0$, we have

$$d_B[L + A, \mathcal{D}_0, 0] = \text{sign det}(L + A) = \pm 1,$$

$Lx + (1 - \lambda)A + \lambda N(x, \lambda) \neq 0$ for each $(x, \lambda) \in \partial \mathcal{D}$, and the result follows from Theorem 7.2.1 applied to $F : \mathcal{D} \to Z$ defined by

$$F(x, \lambda) = Lx + (1 - \lambda)A + \lambda N(x, \lambda).$$

\[\blacksquare\]
When $L$ is invertible, the simplest choice for $A$ in Theorem 8.1.1 is $A = 0$. As, in this case,

$$\|Lx\| \neq 0 \text{ for } x \neq 0,$$

we have, for all $x \in X$,

$$\|Lx\| \geq l\|x\| \quad \text{with} \quad l := \min_{\|x\|=1} \|L(x)\| > 0. \quad (8.1)$$

**Corollary 8.1.1** Assume that $L : X \to Z$ is linear and invertible, and that $g : \overline{B}(R) \subset X \to Z$ is continuous and such that

$$\|g(x)\| \leq lR \quad \text{for } x \in \overline{B}(R). \quad (8.2)$$

Then equation

$$Lx + g(x) = 0$$

has at least one solution in $\overline{B}(R)$. Furthermore, if $L + g$ has no zero on $\partial B(R)$, then

$$|d_B[L + g, B(R), 0]| = 1.$$ 

**Proof.** Let us take $N(x, \lambda) = g(x)$ in Theorem 8.1.1. If there exists $x \in \partial B(R)$ such that $Lx + g(x) = 0$, the result is proved. If not, let $(x, \lambda) \in \overline{B}(R) \times [0, 1]$ be a possible solution of equation

$$Lx + \lambda g(x) = 0.$$ 

We already know that $x \notin \partial B(R)$ for $\lambda = 1$. For $\lambda \in [0, 1]$ we have

$$l\|x\| \leq \|Lx\| = \|\lambda g(x)\| \leq lR,$$

and hence $\|x\| < R$. The result follows from Theorem 8.1.1 with $D = B(R) \times [0, 1]$. \qed

**Corollary 8.1.2** Assume that $L : X \to Z$ is linear and invertible, and that $g : X \to Z$ is continuous and such that, for some positive $\beta$

$$\|g(x)\| \leq \alpha \|x\| + \beta \quad \text{for } x \in X, \quad (8.3)$$

where

$$0 \leq \alpha < l. \quad (8.4)$$

Then equation

$$Lx + g(x) = 0$$

has at least one solution in $\overline{B}(R)$. Furthermore, for $\rho > \frac{\beta}{1-\alpha}$, we have

$$|d_B[L + g, B(\rho), 0]| = 1.$$
Proof. If we take $R = \frac{\beta}{l - \alpha}$ and $x \in \overline{B(R)}$, then, using (8.3) and (8.4),
\[
\|g(x)\| \leq \frac{\alpha}{l - \alpha} \beta + \beta = \frac{\beta}{l - \alpha} = tR,
\]
and we can apply Corollary 8.1.1 for the existence conclusion. Now, for any possible $(x, \lambda) \in X \times [0, 1]$ such that
\[
Lx + \lambda g(x) = 0,
\]
we have
\[
l\|x\| \leq \|Lx\| = \|\lambda g(x)\| \leq \alpha \|x\| + \beta
\]
and hence $\|x\| \leq \frac{\alpha}{l - \alpha} = R$, so that, for $\rho > R$, the homotopy invariance theorem 3.4.2 implies that
\[
d_B[L + g, B(\rho), 0] = d_B[L, B(\rho), 0] = \pm 1.
\]

An immediate consequence of Corollary 8.1.1 is Brouwer’s fixed point theorem [39].

**Corollary 8.1.3** Any continuous $g : \overline{B}(R) \subset X \to \overline{B}(R) \subset X$ has at least one fixed point.

**Proof.** Apply Corollary 8.1.1 with $L = -I$, so that $l = 1$. \[\blacksquare\]

### 8.2 Brouwer’s fixed point theorem and extensions

We show now that the existence of a fixed point of a continuous mapping $g$ holds under more general conditions upon $g$ and more general sets than closed balls.

**Definition 8.2.1** A Hausdorff topological space $S$ has the fixed point property if any continuous mapping $g : S \to S$ has at least one fixed point.

Thus, Brouwer’s fixed point theorem insures that $\overline{B}(R) \subset \mathbb{R}^n$ has the fixed point property. On the other hand, for an closed annulus $A[r, R] := \{ x \in \mathbb{R}^n : r \leq \|x\| \leq R \}$ in $\mathbb{R}^n$, a non-trivial rotation around 0 is continuous but has no fixed point in $A[r, R]$, and hence $A[r, R]$ has not the fixed point property.

**Proposition 8.2.1** If the Hausdorff topological space $A$ has the fixed point property any Hausdorff topological space $B$ homeomorphic to $A$ has the fixed point property.

**Proof.** Let $g : B \to B$ be continuous, and $h : A \to B$ be a homeomorphism. Then $k = h^{-1} \circ g \circ h$ maps continuously $A$ into itself, and has at least one fixed point $x^*$, namely $x^* = h^{-1} \circ g \circ h(x^*)$. So $h(x^*) = g(h(x^*))$, i.e. $h(x^*)$ is a fixed point of $g$. \[\blacksquare\]
We therefore have the following extended version of Brouwer’s fixed point theorem.

**Corollary 8.2.1** Any Hausdorff topological space homeomorphic to the unit closed ball $\overline{B}(1) \subset \mathbb{R}^n$ has the fixed point property.

In particular any closed ball, closed n-intervall, closed n-simplex, or any compact convex subset of $\mathbb{R}^n$ has the fixed point property.

**Remark 8.2.1** The proof of theorem 8.1.1 only uses the fact that $g$ maps $\partial B(R)$ into $\overline{B}(R)$. Hence, the following generalized version of Brouwer’s fixed point theorem, already noticed by G.D. Birkhoff and Kellogg in 1922 [28], proved by Knaster-Kuratowski-Mazurkiewicz in 1929 [203], but generally known as Rothe’s fixed point theorem holds: Any continuous mapping $g : \overline{B}(R) \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $g(\partial B(R)) \subset \overline{B}(R)$ has at least one fixed point in $\overline{B}(R)$.

The assumption of Brouwer’s fixed point theorem means that, for any $x \in \overline{B}(R)$, the segment joining $x$ to $g(x)$ lies in $\overline{B}(R)$. The assumption of Rothe’s fixed point theorem means that the same is required only for any $x \in \partial B(R)$. We prove now a special case of a result of Leray and Schauder [251] showing that the existence of a fixed point still holds if, for any $x \in \partial B(R)$, the segment joining $x$ to $g(x)$ is not contained in the normal half line $\{\mu x : \mu > 1\}$. This is Leray-Schauder’s fixed point theorem in $\mathbb{R}^n$.

**Theorem 8.2.1** Any continuous mapping $g : \overline{B}(R) \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $g(x) \neq \mu x$ for all $x \in \partial B(R)$ and all $\mu > 1$ has at least one fixed point in $\overline{B}(R)$.

**Proof.** If $g$ has a fixed point on $\partial B(R)$ the result is proved. If not, let us introduce again the homotopy $H : \overline{B}(R) \times [0,1] \to \mathbb{R}^n$ defined by $H(x, \lambda) = x - \lambda g(x)$. Clearly $H(x, 0) = x \neq 0$ for $x \in \partial B(R)$, and, by assumption, $H(x, 1) = x - g(x) \neq 0$ for each $x \in \partial B(R)$. Now, if $H(x, \lambda) = 0$ for some $(x, \lambda) \in \partial B(R) \times [0,1]$, then $\lambda x = g(x)$, a contradiction with the assumption.

**Remark 8.2.2** As shown essentially by H. Schaefer [345] in 1955, Leray-Schauder’s fixed point theorem 8.2.1 follows from Brouwer’s fixed point theorem 8.1.3 applied to the continuous mapping $h : \overline{B}(R) \to \overline{B}(R)$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } \|f(x)\| \leq R, \\ \frac{R}{\|f(x)\|} f(x) & \text{if } \|f(x)\| > R. \end{cases}$$

If the fixed point $x^*$ of $h$ is such that $\|f(x^*)\| > R$, it contradicts Leray-Schauder’s hypothesis with $\lambda = R/\|f(x^*)\|$. Hence, $\|f(x^*)\| \leq R$ and, by construction of $h$, $x^*$ is a fixed point of $f$.
8.3 Non-existence theorems

Brouwer’s degree also provides the non-existence of continuous mappings having specific properties.

**Definition 8.3.1** If $B \subset A \subset H$, where $H$ is a Hausdorff topological space, $B$ is a retract of $A$ if there exists a continuous mapping $r : A \to B$ such that $r|_B = I_B$.

The following result, which can be traced to Bohl [31], was rediscovered independently by Borsuk [35] and is called Borsuk’s no-retraction theorem.

**Theorem 8.3.1** There is no retraction $r : \overline{B}(R) \subset \mathbb{R}^n \to \partial B(R)$.

**Proof.** If such a retraction $r$ exists, then, by Corollary 3.4.3, we have $d_B[r, B(R), 0] = d_B[I, B(R), 0] = 1$ and hence, $r$ has at least one zero in $B(R)$, a contradiction with $\|r(x)\| = R$ for each $x \in B(R)$.

**Remark 8.3.1** Theorem 8.3.1 also follows from Brouwer’s fixed point theorem 8.1.3. Indeed, if a retraction $r : \overline{B}(R) \to \partial B(R)$ exists, then $-r : \overline{B}(R) \to \partial B(R)$ has at least one fixed point $x^*$ by Brouwer’s fixed point theorem 8.1.3. By assumption $R = \|−r(x^*)\| = \|x^*\|$, so that $x^* \in \partial B(R)$, and hence $x^* = r(x^*)$. Thus, $r(x^*) = -r(x^*)$, i.e. $r(x^*) = 0$, a contradiction. On the other hand, Theorem 8.3.1 also implies Theorem 8.1.3, because if $g : \overline{B}(R) \to \overline{B}(R)$ has no fixed point, and if, for each $x \in \overline{B}(R)$, we denote by $r(x)$ the intersection with $\partial B(R)$ of the oriented half-line joining $g(x)$ to $x$, then $r : \overline{B}(R) \to \partial B(R)$ is continuous and equal to identity on $\partial B(R)$, i.e. is a retraction.

When $n = 2m \geq 2$ is even, the continuous mapping $v : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defined by

$$v(x) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{2m}, x_{2m-1})$$

is such that $\|v(x)\| = 1$ and $\langle v(x), x \rangle = 0$ for any $x \in \partial B(1)$. In other words, $v$ is a continuous non-vanishing tangent vector field to $\partial B(1)$.

The following non-existence theorem, which can be traced to to Poincaré [319] and Brouwer [40], is called the hedgehog or hairy ball theorem and states that no such vector field can exist when $n$ is odd.

**Theorem 8.3.2** If $n$ is odd, there exists no continuous mapping $v : \partial B(R) \subset \mathbb{R}^n \to \partial B(1) \subset \mathbb{R}^n$ such that $\langle v(x), x \rangle = 0$ for all $x \in \partial B(R)$. 

Proof. If such a mapping \( v \) exists, let \( \tilde{v} : \overline{B}(R) \to \mathbb{R}^n \) be a continuous extension of \( v \) to \( \overline{B}(R) \), and let us define the homotopy \( H : \overline{B}(R) \times [0, 1] \to \mathbb{R}^n \) by

\[
H(x, \lambda) = (\cos \pi \lambda)x - (\sin \pi \lambda)\tilde{v}(x).
\]

If \( (x, \lambda) \in \partial B(R) \times ([0, 1] \setminus \{1/2\}) \), then

\[
\langle H(x, \lambda), x \rangle = (\cos \pi \lambda)R^2 \neq 0,
\]

and hence \( H(x, \lambda) \neq 0 \). If \( x \in \partial B(R) \) \( H(x, 1/2) = -v(x) \neq 0 \). Hence, using the homotopy invariance theorem 3.4.2 and Corollary 3.1.1, we get

\[
1 = d_B[I, B(R), 0] = d_B[H(:, 0), B(R), 0] = d_B[H(:, 1), B(R), 0]
\]

\[
= d_B[-I, B(R), 0] = (-1)^n = -1,
\]

a contradiction. \( \blacksquare \)

### 8.4 Zeros of continuous mappings

Brouwer degree allows to prove a \( n \)-dimensional version of Bolzano’s intermediate value theorem, first stated and proved, using Kronecker index, in 1883 by H. Poincaré [320, 321], then forgotten, rediscovered by S. Cinquini [63] with an incorrect proof in 1940, and proved to be equivalent to Brouwer’s fixed point theorem by C. Miranda [286] in 1941. It is usually referred as Poincaré-Miranda’s theorem.

**Theorem 8.4.1** Any continuous map \( f : P \subset \mathbb{R}^n \to \mathbb{R}^n \) of a \( n \)-dimensional closed interval \( P = \prod_{i=1}^n [a_i, b_i] \) with faces

\[
P_i^+ : \{ x \in P : x_i = b_i \}, \quad P_i^- : \{ x \in P : x_i = a_i \} \quad (1 \leq i \leq n),
\]

such that

\[
f_i(x) \leq 0 \quad (x \in P_i^-), \quad f_i(x) \geq 0 \quad (x \in P_i^+) \quad (1 \leq i \leq n), \tag{8.5}
\]

has at least one zero in \( P \).

**Proof.** If \( f \) has a zero on \( \partial P = \bigcup_{i=1}^n (P_i^- \cup P_i^+) \), the result is proved. If not, let us define the homotopy \( H : P \times [0, 1] \to \mathbb{R}^n \) by

\[
H_i(x, \lambda) = (1 - \lambda) \left( x_i - \frac{a_i + b_i}{2} \right) + \lambda f_i(x) \quad (1 \leq i \leq n).
\]

We already know that \( H(x, 1) = f(x) \neq 0 \) on \( \partial P \). Now, if \( \lambda \in [0, 1] \) and \( x \in \partial P \), then \( x \in P_i^k \) for some \( 1 \leq i \leq n \), and, if, say, \( x \in P_i^- \), we have

\[
H_i(x, \lambda) = (1 - \lambda) \left( \frac{a_i - b_i}{2} \right) + \lambda f_i(x) < 0,
\]
8.4. ZEROS OF CONTINUOUS MAPPINGS

and hence $H(x, \lambda) \neq 0$. Similarly $H_1(x, \lambda) > 0$ if $x \in P^+$. Consequently, using the homotopy invariance theorem 3.4.2 and Corollary 3.1.1, we get, with $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$,

$$d_B[f, \text{int } P, 0] = d_B[H(\cdot, 1), \text{int } P, 0] = d_B[H(\cdot, 0), \text{int } P, 0] = d_B[I - \frac{a + b}{2}, \text{int } P, 0] = 1$$

and the existence of a zero of $f$ in $\text{int } P$ follows from the existence theorem 3.4.1.

The following result, called Poincaré-Bohl existence theorem following Hadamard [162], was never stated in this way by Poincaré or Bohl, but used by Hadamard in the first published proof of Brouwer’s fixed point theorem.

**Theorem 8.4.2** If $f : \overline{B}(R) \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuous and such that $\langle f(x), x \rangle \geq 0$ for all $x \in \partial B(R)$, then $f$ has at least one zero in $\overline{B}(R)$.

**Proof.** If $f$ has a zero on $\partial B(R)$, the result is proved. If not, let us define the homotopy $H : \overline{B}(R) \times [0, 1] \to \mathbb{R}^n$ by $H(x, \lambda) = (1 - \lambda)x + \lambda f(x)$. By assumption, $H(x, 1) \neq 0$ for all $x \in \partial B(R)$. For $(x, \lambda) \in \partial B(R) \times [0, 1]$, we have

$$\langle H(x, \lambda), x \rangle = (1 - \lambda)R^2 + \lambda \langle f(x), x \rangle \geq (1 - \lambda)R^2 > 0,$$

and hence $H(x, \lambda) \neq 0$. Consequently, using the homotopy invariance theorem 3.4.2, we get

$$d_B[f, B(R), 0] = d_B[H(\cdot, 1), B(R), 0] = d_B[H(\cdot, 0), B(R), 0] = d_B[I, B(R), 0] = 1,$$

and $f$ has at least one zero in $B(R)$.

A useful consequence of Theorem 8.4.2 is the following surjectivity theorem.

**Corollary 8.4.1** If $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and strongly coercive, i.e. such that

$$\frac{\langle g(x), x \rangle}{\|x\|} \to +\infty \quad \text{as} \quad \|x\| \to +\infty,$$

then $g$ is onto.

**Proof.** Let $y \in \mathbb{R}^n$. Using (8.6), we can find $R > 0$ such that

$$\langle g(x) - y, x \rangle \geq \langle g(x), x \rangle - \|y\|\|x\| \geq 0$$

whenever $\|x\| \geq R$. Hence, by Theorem 8.4.2 applied to $f(x) = g(x) - y$, there is at least one $x \in \overline{B}(R)$ such that $g(x) = y$. 

8.5 A Brouwer-Poincaré-Bohl’s theorem

When \( L : X \to Z \) is not invertible, a suitable choice of \( A \) in Theorem 8.1.1 it to take
\[
A = JP,
\]
where \( P \) is a projector onto \( N(L) \) and \( J \) a isomorphism between \( N(L) \) and a supplementary subspace to \( R(L) \).

**Corollary 8.5.1** Let \( L : X \to Z \) be a linear mapping, \( P : X \to X, Q : Z \to Z \) projectors such that
\[
R(P) = N(L), \quad N(Q) = R(L),
\]
\( J : N(L) \to R(Q) \) an isomorphism, \( \mathcal{D} \subset X \times [0,1] \) an open bounded set, and \( N : \overline{\mathcal{D}} \to Z \) a continuous mapping. Assume that the following conditions hold.

(A) \( Lx + (1 - \lambda)JPx + \lambda N(x) \neq 0 \) for each \( (x, \lambda) \in \partial \mathcal{D} \).

(B) \( 0 \in \mathcal{D} \).

Then
\[
S_{JP} = \{(x, \lambda) \in \overline{\mathcal{D}} : Lx + (1 - \lambda)JPx + \lambda N(x) = 0\}
\]
contains a compact connected component \( \mathcal{C}_{JP} \) along which \( \lambda \) takes all values in \([0,1]\). In particular equation
\[
Lx + Nx = 0
\]
has at least one solution in \( \mathcal{D}_1 \).

**Proof.** We have
\[
(L + JP)x = 0 \iff Lx = 0, \quad JPx = 0 \iff x \in N(L), \quad Px = 0
\]
\[
\iff x = 0,
\]
and the result follows from Theorem 8.1.1 with \( A = JP \).

We use Corollary 8.5.1 to obtain a unique statement containing and connecting Brouwer’s fixed point theorem 8.1.3 and Poincaré-Bohl’s theorem 8.4.2.

**Theorem 8.5.1** Let \( X \) be a normed vector space and \( Z \) a Hilbert space of the same finite dimension, with inner product \( \langle \cdot, \cdot \rangle \), \( L : X \to Z \) linear, \( P : X \to X, Q : Z \to Z \) be projectors such that
\[
N(L) = R(P), \quad R(L) = N(Q),
\]
\( J : N(L) \to R(Q) \) an isomorphism, and let \( l > 0 \) be such that
\[
\|L(I - P)x\|_Z \geq l\|(I - P)x\|_X \quad \text{for all } x \in X.
\]
Let \( \rho > 0, \ R > 0, \)
\[
D_{\rho,R} := \{x \in X : \|Px\|_X < \rho, \quad \|(I - P)x\|_X < R\},
\]
and \( N : \overline{D_{\rho,R}} \to Z \) be continuous. Assume that the following conditions hold.
8.5. A BROUWER-POINCARÉ-BOHL’S THEOREM

(i) \[ \|(I - Q)N(x)\|_Z \leq lR \quad \text{for all} \quad x \in \overline{D_{\rho,R}}. \]

(ii) \[ \langle QN(x), JPx \rangle \geq 0 \quad \text{whenever} \quad \|Px\|_X = \rho \quad \text{and} \quad \|(I - P)x\|_X \leq R. \]

Then \( L + N \) has at least one zero in \( \overline{D_{\rho,R}} \).

Proof. If \( L + N \) has a zero such that \( \|Px\|_X = \rho \) and \( \|(I - P)x\|_X \leq R \), or has a zero such that \( \|Px\|_X = \rho \) and \( \|(I - P)x\|_X \geq R \), then the theorem is proved. Then we can assume that \( Lx + Nx = 0 \Rightarrow \|(I - P)x\|_X < R \quad \text{and} \quad \|Px\|_X < \rho. \) (8.10)

We apply Corollary 8.5.1 with \( D = D_{\rho,R} \times [0, 1] \). Let \( \lambda \in [0, 1] \) and \( x \in \overline{D_{\rho,R}} \) be a possible zero of \( L + (1 - \lambda)JPx + \lambda N \). Then

\[ L(I - P)x + \lambda(I - Q)N(x) = 0, \] (8.11)

and

\[ (1 - \lambda)JPx + \lambda QN(x) = 0. \] (8.12)

From (8.11), (8.9) and Assumption (i), we deduce that, for \( \lambda \in [0, 1] \),

\[ l\|(I - P)x\|_X \leq \|L(I - P)x\|_Z = \|\lambda N(x)\|_Z < lR, \]

and hence

\[ \|(I - P)x\|_X < R. \] (8.13)

By (8.10), we can assume that (8.13) also holds for \( \lambda = 1 \). From (8.12), we have

\[ (1 - \lambda)\|JPx\|_Z^2 + \langle QN(x), JPx \rangle = 0, \]

and Assumption (ii) and (8.10) imply that we cannot have \( \|Px\|_X = \rho \), so that \( \|Px\|_X < \rho \). Consequently, for each \( \lambda \in [0, 1] \), each possible zero of \( L + (1 - \lambda)JPx + \lambda N \) belongs to \( D_{\rho,R} \), and the result follows from Corollary 8.5.1.

Remark 8.5.1 If \( X = Z \) is a finite-dimensional Hilbert space, \( L = 0 \), then \( P = Q = I, D_{\rho,R} = B(\rho) \), condition (8.9) and Assumption (i) are trivially satisfied, and Assumption (ii) becomes condition (16.2.2) if we choose \( J = I \) or \( J = -I \). Hence we recover Poincaré-Bohl’s theorem 8.4.2.

Remark 8.5.2 If \( L \) is invertible, then \( P = Q = 0, D_{\rho,R} = B(R) \), Assumption (ii) is trivially satisfied, and the remaining assumptions

\[ \|Lx\| \geq l\|x\| \quad \text{for all} \quad x \in X, \]

\[ \|N(x)\| \leq lR \quad \text{for all} \quad x \in \overline{B}(R) \]

are those of Corollary 8.1.1. Brouwer’s fixed point theorem, which corresponds itself to \( X = Z \) a finite-dimensional normed space, \( L = -I \) and \( l = 1 \).
8.6 Perturbed non-invertible linear mappings

The situation is more complicated when \( L \) is not invertible. Then

\[ \dim N(L) = \text{codim } R(L) > 0 \]

and we denote by \( Q : Y \to Y \) a linear projector such that \( N(Q) = R(L) \).

**Theorem 8.6.1** Assume that the following conditions hold;

1. \( Lx + \lambda N(x, \lambda) \neq 0 \) for each \( x \in (\partial D)_\lambda \) and each \( \lambda \in [0, 1] \);
2. \( QN(x, 0) \neq 0 \) for each \( x \in (\partial D)_0 \);
3. \( dB[QN(\cdot, 0)|_{N(L)}, D_0 \cap N(L), 0] \neq 0 \);

Then

\[ |dB[L + \lambda N(\cdot, \lambda), D_\lambda, 0]| = |dB[QN(\cdot, 0), D_0 \cap N(L), 0]|, \quad (\lambda \in [0, 1]), \]

and \( S \) contains a continuum \( C \) connecting \( D_0 \) to \( D_1 \), along which \( \lambda \) takes all values in \( [0, 1] \). In particular the equation \( Lx + N(x, 1) = 0 \) has at least one solution in \( D_1 \).

**Proof.** Let us define \( F : \overline{D} \to Z \) by

\[ F(x, \lambda) = Lx + (1 - \lambda)QN(x, \lambda) + \lambda N(x, \lambda). \]

For \( \lambda \in [0, 1] \), we have, by projections on the supplementary subspaces \( R(Q) \) and \( N(Q) \),

\[ F(x, \lambda) = 0 \iff QN(x, \lambda) = 0, \quad Lx + \lambda(I - Q)N(x, \lambda) = 0 \]
\[ \iff Lx + \lambda N(x, \lambda) = 0. \]

For \( \lambda = 0 \), we have

\[ F(x, 0) = 0 \iff Lx + QN(x, 0) = 0 \iff Lx = 0, \quad QN(x, 0) = 0 \]
\[ \iff x \in N(L), \quad QN(x, 0) = 0. \]

Consequently, Assumptions 1 and 2 imply that \( 0 \notin F(\partial D) \). On the other hand, it follows from Theorem 6.4.2 that

\[ dB[F(\cdot, 0), D_0, 0] = dB[L + QN(\cdot, 0), D_0, 0] = \pm dB[QN(\cdot, 0)|_{N(L)}, D_0 \cap N(L), 0]. \]

The result follows from Theorem 7.2.1 applied to \( F \).
Chapter 9

Fixed points and variational inequalities

9.1 Browder's fixed point theorem

We show in this chapter that, although Brouwer's fixed point theorem is a finite-dimensional tool, it provides results for mappings in locally convex topological vector spaces. This is due to two classical tools from topology and functional analysis:

1) if $E$ is a Hausdorff topological vector space, the topology induced on any finite-dimensional vector subspace of $E$ by the topology of $E$ coincides with the usual Euclidean topology.

2) for any finite open covering of a compact set one can find a partition of unity subordinated to this covering.

The combination of those ideas was used by Browder [45] in proving a fixed point theorem for multivalued mappings. If $B$ is a set, recall that $2^B$ denotes the set of subsets of $B$. A multivalued mapping $f$ of $A$ into $B$ is a mapping of $A$ into $2^B$. We state and prove Browder's fixed point theorem for multivalued mappings.

**Theorem 9.1.1** Let $K$ be a non-empty compact convex subset of a Hausdorff topological vector space $E$ and $T : K \to 2^K$ a mapping such that

(i) for each $x \in K$, $T(x)$ is a nonempty convex subset of $K$;

(ii) for each $y \in K$, $T^{-1}(y) = \{ x \in K : y \in T(x) \}$ is open in $K$.

Then there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

**Proof.** The family $\{ T^{-1}(y) : y \in K \}$ is a covering of $K$, because, for each $x \in K$, $x \in T^{-1}(T(x))$, and hence an open covering by Assumption (ii). Since $K$ is
compact, there is a finite family \( \{y_1, \ldots, y_n\} \subset K \) such that \( K = \bigcup_{j=1}^n T^{-1}(y_j) \). Let \( \{\beta_1, \ldots, \beta_n\} \) be a partition of unity subordinate to this covering, namely, each \( \beta_j : K \to [0, 1] \) is continuous, vanishes outside of \( T^{-1}(y_j) \), and \( \sum_{j=1}^n \beta_j(x) = 1 \) for all \( x \in K \). Define the continuous mapping \( p : K \to K \) by \( p(x) = \sum_{j=1}^n \beta_j(x)y_j \). Let \( K_0 = \left\{ \sum_{j=1}^n \lambda_j y_j : \lambda_j \geq 0 \ (1 \leq j \leq n), \ \sum_{j=1}^n \lambda_j = 1 \right\} \) be the closed convex set spanned by the \( y_j \). By construction, \( p \) is a continuous mapping from \( K_0 \) into itself. From the version of Brouwer’s fixed point theorem given in Corollary 8.1.3, \( p \) has a fixed point \( x_0 \). To conclude the proof, it suffices to show that, for each \( x \in K \), \( p(x) \in T(x) \). Let \( x \in K \); since \( \beta_j(x) = 0 \) if \( x \notin T^{-1}(y_j) \), and \( T(x) \) is convex, then
\[
p(x) = \sum_{1 \leq j \leq n, x \in T^{-1}(y_j)} \beta_j(x)y_j = \sum_{1 \leq j \leq n, y_j \in T(x)} \beta_j(x)y_j \in T(x).
\]

\[\square\]

### 9.2 Variational inequalities

A consequence of Theorem 9.1.1 is a version of Hartman-Stampacchia’s theorem for variational inequalities [169] in a locally convex topological vector space, also proved independently by Karamardian [193] in the special case of a simplex in \( \mathbb{R}^n \).

**Definition 9.2.1** A locally convex topological vector space is a Hausdorff topological vector space \( E \) such that its origin possesses a neighborhood base consisting of convex sets or, equivalently, a Hausdorff topological vector space in which any neighborhood of any point \( x \in E \) contains a convex neighborhood of \( x \).

**Theorem 9.2.1** Let \( K \) be a compact convex subset of a locally convex topological vector space \( E \), and \( T : K \to E^* \) be continuous. Then there exists \( u_0 \in K \) such that
\[
\langle T(u_0), v - u_0 \rangle \geq 0
\]
for all \( v \in K \).

**Proof.** Suppose that the assertion of the theorem is false. Then, for each \( u \in K \), there exists \( v \in K \) such that \( \langle T(u), v - u \rangle < 0 \). Define \( T : K \to 2^K \) by
\[
T(u) = \{ v \in K : \langle T(u), v - u \rangle < 0 \}.
\]
For each \( u \in K \), \( T(u) \) is non-empty, convex and open, being the inverse image by the continuous map \( v \mapsto \langle T(u), v - u \rangle \) of the open interval \( ]-\infty, 0[ \). By Theorem 9.1.1, there exists \( u_0 \in K \) such that \( u_0 \in T(u_0) \), i.e. such that
\[
0 = \langle T(u_0), u_0 - u_0 \rangle < 0,
\]
a contradiction. \[\square\]
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Remark 9.2.1 A condition of the type (9.1) is called a variational inequality. Such relation occurs for example in the study of the minimum on a closed set $C$ of a function $\varphi \in C^1(C, \mathbb{R})$ Namely, if there exists $u_0 \in C$ such that $\varphi(u_0) = \inf C \varphi$, then

$$\langle \nabla \varphi(u_0), v - u_0 \rangle \geq 0$$

(9.2)

for all $v \in C$, where $\nabla \varphi$ denotes the gradient of $\varphi$. This follows from the fact that, given $v \in C$, the real function $\Phi : [0,1] \to \mathbb{R}$ defined by $\Phi(t) = \varphi(u_0 + t(v - u_0))$ reaches its minimum on $[0,1]$ at $t = 0$, and hence

$$\Phi'(0) = \langle \nabla \varphi(u_0), v - u_0 \rangle \geq 0.$$

The converse is true if $C$ is convex, because, in this case,

$$\varphi(u) \geq \varphi(u_0) + \langle \nabla \varphi(u_0), u - u_0 \rangle \geq \varphi(u_0)$$

for all $u \in C$.

Another elementary occurrence of variational inequalities is in the complementary problem of nonlinear programming: given a continuous mapping $F : \mathbb{R}^n_+ \to \mathbb{R}^n$, where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \ (1 \leq i \leq n)\}$, find $u_0 \in \mathbb{R}^n_+$ such that $F(u_0) \in \mathbb{R}^n_+$ and $\langle F(u_0), u_0 \rangle = 0$. The following characterization is due to Karman-drian [193].

Proposition 9.2.1 $u_0$ is a solution of the complementary problem if and only if

$$u_0 \in \mathbb{R}^n_+, \quad \langle F(u_0), v - u_0 \rangle \geq 0 \quad \text{for} \quad v \in \mathbb{R}^n_+.$$  \hspace{1cm} (9.3)

Proof. If $u_0$ is a solution of the complementary problem, then $F(u_0) \in \mathbb{R}^n_+$ and hence $\langle F(u_0), v \rangle \geq 0$ for all $v \in \mathbb{R}^n_+$, so

$$\langle F(u_0), v - u_0 \rangle = \langle F(u_0), v \rangle \geq 0$$

for all $v \in \mathbb{R}^n_+$. On the other hand, if $u_0$ satisfies the variational inequality (9.3), then, taking $v = u_0 + e^i \in \mathbb{R}^n_+$, with $e^i = (0, \ldots, 1, \ldots, 0)$ the $i^{th}$ unit vector, we obtain

$$0 \leq \langle F(u_0), e^i \rangle = F_i(u_0) \quad (1 \leq i \leq n),$$

i.e. $F(u_0) \in \mathbb{R}^n_+$. In particular, $\langle F(u_0), u_0 \rangle \geq 0$, and, taking $v = 0$ in (9.3), we obtain $\langle F(u_0), -u_0 \rangle \geq 0$, and $u_0$ solves the complementary problem.

The Hilbert space version of Hartman-Stampacchia’s theorem is of interest.

Corollary 9.2.1 Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)$, $K \subset H$ a compact convex set and $T : H \to H$ a continuous mapping. Then there exists $u_0 \in K$ such that

$$\langle T(u_0), v - u_0 \rangle \geq 0$$

for all $v \in K$. 

Proof. By Riesz representation theorem, for each \( u \in H \), there exists \( J u \in H^* \) such that
\[
(J u, v)_{H, H^*} = (u, v) \quad (v \in H),
\]
and \( J : H \to H^* \) is linear, bijective and isometric. Now,
\[
(JT(u), w)_{H, H^*} = (T(u), w)
\]
for all \( u \in H \) and \( w \in H \), and Theorem 9.2.1 applied to \( JT : H \to H^* \) implies the existence of \( u_0 \in H \) such that
\[
0 \leq (JT(u_0), v - u_0)_{H, H^*} = (T(u_0), v - u_0)
\]
for all \( v \in K \).

Remark 9.2.2 A classical result of the theory of Hilbert spaces states that if \( C \) is a non-empty closed convex subset of a real Hilbert space \( H \) with inner product \( (\cdot, \cdot) \), then, for each \( f \in H \) there exists a unique \( u_0 := P_C f \) which minimizes the distance \( v \to \| v - f \| \) for all \( v \in C \). Furthermore, \( u_0 = P_C f \) is characterized by the property
\[
u_0 \in C, \quad (u_0 - f, v - u_0) \geq 0 \quad \text{for all} \quad v \in C.
\]
(9.4)

Corollary 9.2.1 extends this result from the special mapping \( u \mapsto u - f \) to a general continuous mapping \( T : H \to H \), but for compact convex sets only.

In turn, Corollary 9.2.1 implies an extension of Brouwer fixed point to some mappings in a possibly infinite-dimensional space, namely the Hilbert space version of Schauder’s fixed point theorem [348].

Corollary 9.2.2 Let \( H \) be a real Hilbert space, \( K \subset H \) a compact convex set, and \( g : K \to K \) a continuous mapping. Then \( g \) has at least one fixed point.

Proof. If we apply Corollary 9.2.1 to \( T = I - g \), we obtain the existence of \( u_0 \in K \) such that
\[
((I - g)(u_0), v - u_0) \geq 0
\]
(9.5)
for all \( v \in K \). Taking \( v = g(u_0) \), we obtain \((I - g)(u_0) = 0\).

Remark 9.2.3 In an infinite-dimensional Hilbert space, the statement of Corollary 9.2.2 is false if we replace the compact convex set \( K \) by a closed ball. For example, for \( H = l^2 \), the mapping \( g \) defined on \( B(1) \subset l^2 \) by
\[
g(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, \ldots),
\]
(9.6)
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with \( \|x\| = \left( \sum_{j=1}^{\infty} x_j^2 \right)^{1/2} \), is continuous and maps \( \overline{B}(1) \) into itself (indeed into \( \partial \overline{B}(1) \)). But it has no fixed point, because if \( x^* = f(x^*) \), then \( \|x^*\| = \|f(x^*)\| = 1 \) and

\[
x_1^* = 0, \quad x_2^* = x_1^*, \quad x_3^* = x_2^*, \ldots
\]

i.e. \( x^* = 0 \), a contradiction. This example was given by Tychonov [391] as an answer to a question of Ulam.

Using the construction used in the finite-dimensional case to prove the equivalence between Brouwer’s fixed point theorem and the no-retraction theorem, this result implies that the mapping \( R \) defined on \( \overline{B}(1) \subset l^2 \) by

\[
r(x) = \frac{\left(\|x\|^2 - 1\right) g(x) + 2(1 - \langle x, g(x) \rangle)x}{\|x\|^2 - 2\langle x, g(x) \rangle + 1},
\]

with \( g \) given in (8.3.1), is a retraction of \( \overline{B}(1) \subset l^2 \) onto \( \partial B(1) \subset l^2 \).

**Remark 9.2.4** It may be of interest to notice that Corollary 9.2.1 can also be deduced from Schauder’s fixed point theorem in a Hilbert space and the characterization of the projection on a closed convex set given in Remark 9.2.2. Indeed, given \( T : K \to H \), let us define \( g : K \to K \) by \( g(x) = P_K(I - T)(x) \), where \( P_K \) is the projection on \( K \) defined in Remark 9.2.2. By Corollary 9.2.2, there exists \( u_0 \) such that \( u_0 = P_K(I - T)(u_0) \). Then, applying (9.4) to \( f = (I - T)(u_0) \), we get

\[
(u_0 - (I - T)(u_0), v - u_0) \geq 0
\]

for all \( v \in K \), which is nothing but (9.5).

9.3 Hemivariational inequalities

The **hemivariational inequalities** have been introduced in mechanics by P.D. Panagiotopoulos [310]-[311]. Their derivation is based upon the generalized gradient of F.H. Clarke [69] and the corresponding mechanical notion of nonconvex superpotential. In what follows, we give, following [91] and [93], an existence result for noncoercive hemivariational inequalities. The main tools are Hartman-Stampacchia’s theorem 9.2.1 for variational inequalities in finite-dimensional spaces and a limit process.

Let \( D \subset \mathbb{R}^n \) be open and bounded, \( V \) a real Hilbert space such that \( V \subset L^2(D) \subset V^* \), the injection of \( V \) in \( L^2(D) \) being compact. We denote the corresponding duality mapping by \( \langle \cdot, \cdot \rangle_V \). Assume that

(i) \( a : V \times V \to \mathbb{R} \) is a bilinear, continuous form such that \( a(u, u) \geq 0 \) for all \( u \in V \).

(ii) \( \beta \in L^\infty(\mathbb{R}) \) is such that
(a) \( \beta(\xi \pm 0) \) exists for any \( \xi \in \mathbb{R} \).

(b) there exists \( \xi_0 \) such that
\[
\operatorname{ess sup}_{-\infty < \xi < -\xi_0} \beta(\xi) \leq 0 \leq \operatorname{ess inf}_{\xi_0 < \xi < +\infty} \beta(\xi).
\] (9.7)

(iii) \( K \subset V \) is a closed, convex subset such that \( 0 \in K \), and \( l \in V^* \).

Under those assumptions, we consider

Problem \((P)\). Prove the existence of some \( u \in K \) which satisfies
\[
a(u, v - u) + \int_D D_C j(u(x)) \cdot (v(x) - u(x)) \, dx \geq \langle l, v - u \rangle_{V^*} \quad \text{for all } v \in K. \] (9.8)

In (9.8),
\[
j(\xi) := \int_0^\xi \beta(t) \, dt \quad (\xi \in \mathbb{R}),
\] (9.9)

and \( D_C j(u(x)) \cdot (v(x) - u(x)) \) means Clarke’s derivative of \( j \) at \( u(x) \) in direction \((v(x) - u(x))\).

Recall at this point that given a locally Lipschitzian function \( f : X \to \mathbb{R} \), with \( X \) a locally convex Hausdorff topological space, the directional Clarke’s derivative is defined by
\[
D_C f(x) \cdot h = \lim_{y \to x, \lambda \to 0^+} \sup_{\lambda} \frac{f(y + \lambda h) - f(y)}{\lambda},
\]
and Clarke’s gradient of \( f \) at \( x \) is defined by
\[
\partial_C f(x) = \{ x^* \in X^* : D_C f(x) \cdot h \geq \langle x^*, h \rangle_X \quad \text{for all } h \in X \}. \]

According to a result of K.C. Chang [57], Clarke’s generalized gradient of the locally Lipschitzian function (9.9) is given by
\[
\partial_C j(\xi) = [\overline{\beta(\xi)}, \underline{\beta(\xi)}],
\] (9.10)

where
\[
\beta(\xi) = \lim_{\mu \to 0^+} \beta_\mu(\xi), \quad \underline{\beta(\xi)} = \operatorname{ess inf}_{\xi_1 - \xi \leq \mu} \beta(\xi_1), \quad \overline{\beta(\xi)} = \operatorname{ess sup}_{\xi_1 - \xi \leq \mu} \beta(\xi_1), \] (9.11)

\[
\beta_\mu(\xi) = \lim_{\mu \to 0^+} \beta_\mu(\xi), \quad \overline{\beta_\mu(\xi)} = \operatorname{ess inf}_{\xi_1 - \xi \leq \mu} \beta(\xi_1), \quad \underline{\beta_\mu(\xi)} = \operatorname{ess sup}_{\xi_1 - \xi \leq \mu} \beta(\xi_1). \] (9.12)

In order to prove that Problem \((P)\) has a solution, we first consider a regularized finite-dimensional problem. Denote by \( \mathcal{F} \) the family of all finite-dimensional vector...
subspaces of $V$. Given $\varepsilon \in [0, 1]$ and $F \in \mathcal{F}$, Problem $(P_{\varepsilon, F})$ consists in proving the existence of some $u^{\varepsilon, F} \in F \cap K$ such that

$$a(u^{\varepsilon, F}, v - u^{\varepsilon, F}) + \int_D \beta_\varepsilon(u^{\varepsilon, F}(x))(v(x) - u^{\varepsilon, F}(x)) \, dx \geq \langle l, v - u^{\varepsilon, F} \rangle_V \quad \text{for all } v \in F \cap K.$$  \hfill (9.13)

In (9.13), $\beta_\varepsilon$ is defined as follows. If $p \in \mathcal{D}([-1, 1])$, $p \geq 0$, $\int_{\mathbb{R}} p(\xi) \, d\xi = 1$ is a mollifier, $\varepsilon \in [0, 1]$ and $p_\varepsilon(\xi) := \varepsilon^{-1} p(\varepsilon^{-1} \xi)$ for all $\xi \in \mathbb{R}$, then,

$$\beta_\varepsilon := p_\varepsilon \ast \beta,$$

where $\ast$ denotes the convolution product.

**Proposition 9.3.1** Assume that conditions (i) to (iii) above hold and that there exists $R > 0$ such that

$$\{ u \in K : a(u, u) + \int_D u(x)\beta_\varepsilon(u(x)) \, dx \leq \langle l, u \rangle_V \quad \text{for some } \varepsilon \in [0, 1] \} \subset B(R).$$  \hfill (9.14)

Then problem $(P_{\varepsilon, F})$ has at least one solution. Moreover,

$$S := \bigcup_{\varepsilon \in [0, 1]} \{ u^{\varepsilon, F} \text{ solves } (P_{\varepsilon, F}) \} \subset B(R).$$  \hfill (9.15)

**Proof.** The boundedness (9.15) of $S$ is a consequence of the a priori estimate (9.14). Indeed, taking $v = 0$ in (9.13), one gets

$$a(u^{\varepsilon, F}, u^{\varepsilon, F}) + \int_D u^{\varepsilon, F}(x)\beta_\varepsilon(u^{\varepsilon, F}(x)) \, dx \leq \langle l, u^{\varepsilon, F} \rangle_V,$$

and, according to (9.14), $u^{\varepsilon, F} \in B(R)$ for all $\varepsilon \in [0, 1]$ and all $F \in \mathcal{F}$. In order to prove the existence of some $u^{\varepsilon, F} \in F \cap K$ satisfying (9.13), let us first define the continuous operator $T_{\varepsilon, F} : F \to F^*$ by

$$(T_{\varepsilon, F}(u), v)_V = a(u, v) + \int_D \beta_\varepsilon(u(x))v(x) \, dx - \langle l, v \rangle_V.$$  \hfill (9.16)

From Hartman-Stampacchia’s theorem 9.2.1, for any positive $n \in \mathbb{N}$, there exists $u^{\varepsilon, F}_n \in K \cap F \cap B(n)$ such that

$$\langle T_{\varepsilon, F}(u^{\varepsilon, F}_n), v - u^{\varepsilon, F}_n \rangle_V \geq 0 \quad \text{for all } v \in K \cap F \cap B(n).$$  \hfill (9.17)

Taking $v = 0$ in (9.17), we obtain

$$\langle T_{\varepsilon, F}(u^{\varepsilon, F}_n), u^{\varepsilon, F}_n \rangle_V = a(u^{\varepsilon, F}_n, u^{\varepsilon, F}_n) + \int_D u^{\varepsilon, F}_n(x)\beta_\varepsilon(u^{\varepsilon, F}_n(x)) \, dx - \langle l, u^{\varepsilon, F}_n \rangle_F \leq 0,$$
and then, by (9.14), \( u^\varepsilon_F \in K \cap F \cap \overline{B}(R) \). Consequently, passing if necessary to a subsequence, one may assume that \( u^\varepsilon_F \rightharpoonup u^\varepsilon F \in K \cap F \cap \overline{B}(R) \), and \( u^\varepsilon F \) is a solution of problem \( (P_{\varepsilon,F}) \) (for more details, see [93]).

We now prove the main existence theorem of [91]-[93].

**Theorem 9.3.1** If assumptions (i) to (iii) above hold as well as the a priori estimate condition (9.14), problem (9.8) has at least one solution.

**Proof.** For any \( \varepsilon \in \{0,1\} \) and any \( F_0 \in \mathcal{F} \), let us define

\[
V^\varepsilon_{F_0} := \bigcup_{F_0 \subseteq F \in \mathcal{F}} \{ u^\varepsilon_F : u^\varepsilon F \text{ is a solution to } P_{\varepsilon,F} \}.
\]

Since \( V^\varepsilon_{F_0} \subseteq K \cap \overline{B}(R) \), the same is true for its weak closure \( \overline{V^\varepsilon_{F_0}} \). Now, for each \( \varepsilon \in \{0,1\} \), the family \( \{ \overline{V^\varepsilon_{F_0}} : F_0 \in \mathcal{F} \} \) has the finite intersection property, and hence there exists \( u^\varepsilon \in \bigcap_{F_0 \in \mathcal{F}} \overline{V^\varepsilon_{F_0}} \). Clearly, \( \{ u^\varepsilon : \varepsilon \in \{0,1\} \} \subseteq K \cap \overline{B}(R) \), which implies the existence of a sequence \( (\varepsilon_m) \) in \( \{0,1\} \) converging to zero and such that \( u^{\varepsilon_m} \rightharpoonup u^0 \in K \cap \overline{B}(R) \), with \( u_0 \) a solution of problem \( P \). The principal steps in proving this results are the following ones:

(a) for any \( m \in \mathbb{N} \), \( u^{\varepsilon_m} \) satisfies the relation

\[
a(u^{\varepsilon_m}, v - u^{\varepsilon_m}) + \int_D \beta_{\varepsilon_m}(u^{\varepsilon_m}(x))(v(x) - u^{\varepsilon_m}(x)) \, dx \geq (f, v - u^{\varepsilon_m})_V
\]

for all \( v \in K \).

(b) by using the nonnegativity and the continuity of \( a \), one has

\[
\limsup_{m \to \infty} a(u^{\varepsilon_m}, v - u^{\varepsilon_m}) \leq a(u^0, v - u^0).
\]

(c) For all \( v \in K \), one has

\[
\limsup_{m \to \infty} \int_D \beta_{\varepsilon_m}(u^{\varepsilon_m}(x))(v(x) - u^{\varepsilon_m}) \, dx \leq \int_D D_{CJ}(u^0(x)) \cdot (v(x) - u^0(x)) \, dx.
\]

If (a), (b), (c) are true, then the result follows by taking the upper limit in (a). While (a) and (b) are relatively easy to be checked, we proceed as follows to obtain (c). The properties of \( \beta \) imply the existence of two constants \( \rho_1 > 0 \) and \( \rho_2 = \|\beta\|_{L^\infty} \) such that

\[
\xi \beta_1(\xi) \geq 0 \text{ whenever } |\xi| > \rho_1 \text{ and } \varepsilon \in \{0,1\}.
\]  

(9.18)

\[
\xi \beta_2(\xi) \geq -\rho_1 \rho_2 \text{ for all } \xi \in \mathbb{R} \text{ and } \varepsilon \in \{0,1\}.
\]  

(9.19)

We refer to [93] for the details. From (9.18) and (9.19), we deduce that

\[
\beta_{\varepsilon_m}(u^{\varepsilon_m}(x))(v(x) - u^{\varepsilon_m}(x)) \leq \rho_1 \rho_2 + \|\beta\|_{L^\infty} \|v\|
\]
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for a.e. \( x \in D \). Due to Fatou’s lemma, in order to prove (c), it is sufficient to prove that

\[
\int_D \limsup_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x))(v(x) - u^\varepsilon_m(x)) \, dx \leq \int_D D_C j(u^0(x)) \cdot (v(x) - u^0(x)) \, dx. \tag{9.20}
\]

In order to prove (9.20) we shall show that, for any \( y \in \mathbb{R} \),

\[
\limsup_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x))(y - u^\varepsilon_m(x)) \, dx \leq D_C j(u^0(x)) \cdot (y - u^0(x)) \tag{9.21}
\]

for a.e. \( x \in D \). Indeed, since the injection of \( V \) into \( L^2(D) \) is compact and \( u^\varepsilon_m \to u^0 \) in \( V \), it follows that \( u^\varepsilon_m \to u^0 \) in \( L^2(D) \), and, going if necessary to a subsequence, \( u^\varepsilon_m(x) \to u^0(x) \) a.e. in \( D \). Thus, for the proof of (9.21), it is sufficient to show that for any \( x \in D \) such that \( u^\varepsilon_m(x) \to u^0(x) \), and to any convergent subsequence \((y_k)\) of \((\beta_{\varepsilon_m}(u^\varepsilon_m(x))(y - u^\varepsilon_m(x)))\), inequality

\[
\lim_{k \to \infty} y_k \leq D_C j(u^0(x)) \cdot (y - u^0(x)) \tag{9.22}
\]

holds. First, we show that

\[
\liminf_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x)) \quad \text{and} \quad \limsup_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x))
\]

belong to \( \partial_C j(u^0(x)) \). As a consequence, we shall have

\[
\liminf_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x)) z \leq D_C j(u^0(x)) z
\]

\[
\limsup_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x)) z \leq D_C j(u^0(x)) z \tag{9.23}
\]

for all \( z \in \mathbb{R} \). For the proof, let \( x \in D \) with \( u^\varepsilon_m(x) \to u^0(x) \) and \( \mu > 0 \) be given. Then there exists \( m_{\mu,x} \) such that, for any \( m \geq m_{\mu,x} \) one has

\[
|u^\varepsilon_m(x) - u^0(x)| \leq \frac{\mu}{2}, \quad 0 \leq \varepsilon_m \leq \frac{\mu}{2}.
\]

Consequently,

\[
\beta_{\varepsilon_m}(u^\varepsilon_m(x)) \leq \esssup_{|t-u^\varepsilon_m(x)| \leq \varepsilon_m} \beta(t) \leq \esssup_{|t-u^0(x)| \leq \mu} \beta(t) = \overline{\beta}_\mu(u^0(x)).
\]

Therefore,

\[
\limsup_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x)) \leq \overline{\beta}_\mu(u^0(x)),
\]

and, passing to the limit with \( \mu \to 0^+ \), we deduce

\[
\limsup_{m \to \infty} \beta_{\varepsilon_m}(u^\varepsilon_m(x)) \leq \overline{\beta}(u^0(x)).
\]
Analogously, one obtains
\[ \beta(u^0(x)) \leq \liminf_{m \to \infty} \beta_{\varepsilon_m}(u^{\varepsilon_m}(x)). \]
Thus,
\[ \limsup_{m \to \infty} \beta_{\varepsilon_m}(u^{\varepsilon_m}(x)) \quad \text{and} \quad \liminf_{m \to \infty} \beta_{\varepsilon_m}(u^{\varepsilon_m}(x)) \in [\beta(u^0(x)), \overline{\beta}(u^0(x))] = \partial C_j(u^0(x)), \]
and inequalities (9.23) hold. Now we are able to prove (9.22). If \( y = u^0(x) \), then \( y - u^{\varepsilon_m}(x) \to 0 \). Because of \( |\beta_{\varepsilon}(t)| \leq \|\beta\|_{L^\infty} \) for any \( t \in \mathbb{R} \) and any \( \varepsilon > 0 \), it follows that \( y_k \to 0 \). Thus (9.22) is verified. If now \( y > u^0(x) \), then
\[ \lim_{k \to \infty} y_k = \left( \lim_{k \to \infty} \beta_{\varepsilon_m}(u^{\varepsilon_m}(x)) \right) (y - u^0(x)) \leq \limsup_{m \to \infty} \beta_{\varepsilon_m}(u^{\varepsilon_m}(x))(y - u^0(x)) \]
the last inequality being justified by (9.23). A similar procedure can be used if \( y < u^0(x) \), using the lower limit this time. So (9.21) is proved, and then (9.20) is also proved, implying the assertion (c). Taking into account (b) and (c), the result follows by taking the upper limit in (a).

Finally, we formulate sufficient conditions in order that the a priori estimate assumption (9.14) holds.

**Theorem 9.3.2** Let \( K = \text{span}(K) \). Assume that the bilinear form \( a \) is continuous, nonnegative and semicoercive, i.e. that, in addition to assumptions (i) to (iii), one has
\( \text{(iv)} \) \( K \cap \ker(a) \) is finite dimensional and \( a \) is coercive over \( [K \cap \ker(a)]^\perp \), (the orthogonal being considered in \( K \)), namely
\[ a(u, u) \geq \gamma \|u\|^2 \quad \text{for all } u \in [K \cap \ker(a)]^\perp \quad \text{and some } \gamma > 0. \]
Then condition (9.14) is satisfied over \( K \) if one of the following conditions hold:

1. \( K \cap \ker(a) = \{0\} \),
2. \( K \cap \ker(a) \neq \{0\} \) and
\( \text{(v)} \) \( \beta(-\infty) := \lim_{t \to -\infty} \beta(t) < \beta(\infty) := \lim_{t \to +\infty} \beta(t) \)
\( \text{(vi)} \int_D \left[ \beta(\infty)\theta^+(x) - \beta(-\infty)\theta^-(x) \right] dx > \langle l, \theta \rangle_V \quad \text{for all } \theta \in [K \cap \ker(a)] \setminus \{0\}. \)

Condition (vi) is a **Landesman-Lazer-type condition** first introduced by A.C. Lazer and E.M. Landesman in 1969 [241] for bounded nonlinear perturbations of non-coercive linear Dirichlet problems. The proof of Theorem 9.3.2, which can be found in [92], is inspired by the one of P.J. McKenna and J. Rauch [279], who consider a case where \( \text{dim ker}(a) = 1 \).
9.4 Ky Fan fixed point theorem and consequences

We will now state and prove an extension, due to Ky Fan [111], of Brouwer’s fixed point theorem to some multivalued mappings of a compact convex set of a locally convex topological vector space. A few concepts and results about locally convex topological vector spaces will be required [330].

Definition 9.4.1 If $E$ is a vector space, $C \subset E$ is said to be circled if $\lambda C \subset C$ whenever $|\lambda| \leq 1$.

Lemma 9.4.1 If $E$ is a locally convex topological vector space, there is a neighborhood base at 0 of $E$ consisting of open, convex and circled sets.

Lemma 9.4.2 Let $A$ and $B$ be non-empty, disjoint and convex subsets of a real locally convex topological vector space $E$, such that $A$ is closed and $B$ is compact. Then there exists a closed real hyperplane in $E$ strictly separating $A$ and $B$. In other words, there exists a linear and continuous functional $f : E \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $f(x) < \alpha < f(y)$ for each $x \in A$ and $y \in B$.

Letting $H_{<\alpha} := \{x \in E : f(x) < \alpha\}$, $H_{>\alpha} := \{x \in E : f(x) > \alpha\}$, it is clear that $H_{<\alpha}$ and $H_{>\alpha}$ are disjoint, open and convex, and Lemma 9.4.2 says that $A \subset H_{<\alpha}$ and $B \subset H_{>\alpha}$.

Definition 9.4.2 If $X$ and $Y$ are Hausdorff topological spaces, and $T : X \to 2^Y$ is a multivalued mapping from $X$ into $Y$, we say that $T$ is upper semi-continuous on $X$ if, to each point $x_0$ of $X$ and an arbitrary neighborhood $V$ to $T(x_0)$ in $Y$, there exists a neighborhood $U$ of $x_0$ in $X$ such that, for each $x \in U$, $T(x) \subset V$.

The following characterization of upper semi-continuity is of interest (see [45]).

Lemma 9.4.3 Let $K$ and $K_1$ be compact topological spaces and $T : K \to 2^{K_1}$ a multi-valued mapping from $K$ into $K_1$ such that $T(x)$ is closed for each $x \in K$. Then $T$ is upper semi-continuous on $K$ if and only if its graph

$$G(T) := \{[x, y] \in K \times K_1 : x \in K, y \in T(x)\}$$

is closed in $K \times K_1$.

Proof. Suppose first that $T$ is upper semi-continuous on $K$. We shall prove that $(K \times K_1) \setminus G(T)$ is open in $K \times K_1$. Let $[u, w] \in (K \times K_1) \setminus G(T)$. Since $w \in K_1 \setminus T(u)$, the regularity of $K_1$ implies the existence of neighborhoods $V_1$ of $w$ and $V_2$ of $T(u)$ such that $V_1 \cap V_2 = \emptyset$. Using the upper semi-continuity of $T$, there exists a neighborhood $U$ of $u$ in $K$ such that, for all $x \in U$, $T(x) \subset V_2$. Hence the neighborhood $U \times V_1$ of $[u, w]$ does not intersect $G(T)$, and $G(T)$ is closed.
Suppose conversely that $G(T)$ is closed in $K \times K$ and hence compact. Let $u \in K$ and $V$ an open neighborhood of $T(u)$. If $T$ were not upper semi-continuous at $u$, then, for each neighborhood $U$ of $u$ in $K$ we would have

$$G(T) \cap [U \times (K_1 \setminus V)] \neq \emptyset.$$ 

But

$$\{G(T) \cap [U \times (K_1 \setminus V)] : U \text{ neighborhood of } u\}$$

is a family of compact sets having the finite intersection property, and hence

$$G(T) \cap \bigcap_U [U \times (K_1 \setminus V)] \neq \emptyset.$$ 

Any point $[x, y]$ of this intersection is such that $x = u$ and $y \in T(u) \subset V$, a contradiction.

We can now state and prove Ky Fan’s fixed point theorem [111], using an argument due to F. Terkelsen [384].

**Theorem 9.4.1** Let $K$ be a nonempty compact convex subset of a locally convex topological vector space $E$ and let $T : K \to 2^K$ be an upper semi-continuous multi-valued function of $K$ into $K$, such that $T(u)$ is non empty, closed and convex for each $u \in K$. Then there exists $u_0 \in K$ such that $u_0 \in T(u_0)$.

**Proof.** Denote by $\mathcal{U}$ the set of all open convex and circled neighborhoods of 0 and let us introduce on the set $N^* \times \mathcal{U} = \{(i, U) : i \in N^*, U \in \mathcal{U}\}$ the order structure defined as follows:

$$(i, U) \leq (j, V) \iff i \leq j \text{ and } U \supset V.$$ 

Endowed with this order structure, $((N^*, \mathcal{U}), \leq)$ is directed. Indeed, if $(i, U), (j, V) \in N^* \times \mathcal{U}$, then $(k, U \cap V)$ with $k \geq \max(i, j)$ belongs to $N^* \times \mathcal{U}$ too, and

$$(i, U) \leq (k, U \cap V), \quad (j, V) \leq (k, U \cap V).$$

The net

$$N^* \times \mathcal{U} \to \mathcal{U}, \quad (i, U) \mapsto \frac{1}{i} U := U_i$$

is decreasing, in the sense that

$$(i, U) \leq (j, V) \Rightarrow U_i = \frac{1}{i} U \supset \frac{1}{j} V = V_j.$$ 

Let $(i, U)$ be arbitrarily chosen in $N^* \times \mathcal{U}$. Since $\{x + U_j : x \in K\}$ is an open covering of $K$, there exists a finite set $\{x(i, U)_j \in K : j \in J(i, U)\}$ such that

$$K \subset \bigcup_{j \in J(i, U)} (x(i, U)_j + U_i).$$
Let \( \{ f_{(i,U),j} : j \in J(i,U) \} \) be a continuous partition of unity subordinated to this finite covering, so that the continuous functions \( f_{(i,U),j} : K \to [0,1] \) \( (j \in J(i,U)) \) are such that \( f_{(i,U),j}(x) = 0 \) for \( x \notin x_{(i,U),j} + U \) and \( \sum_{j \in J(i,U)} f_{(i,U),j}(x) = 1 \) for all \( x \in K \). Choose \( y_{(i,U),j} \in T(x_{(i,U),j}) \) arbitrarily and denote by \( C_{(i,U)} \) the convex hull of \( \{ y_{(i,U),j} : j \in J(i,U) \} \). Clearly, \( C_{(i,U)} \) is a compact, convex and finite-dimensional subset of \( K \). Finally, define the continuous mapping \( f_{(i,U)} : C_{(i,U)} \to C_{(i,U)} \) by

\[
  f_{(i,U)}(x) = \sum_{j \in J(i,U)} f_{(i,U),j}(x)y_{(i,U),j}.
\]

Since the topology induced on any finite-dimensional subspace of \( E \) by its locally convex topology coincides with the usual Euclidean one, Brouwer’s fixed point theorem 8.2.1 implies the existence of \( u_{(i,U)} \in C_{(i,U)} \) such that

\[
  u_{(i,U)} = f_{(i,U)}(u_{(i,U)}) = \sum_{j \in J(i,U)} f_{(i,U),j}(u_{(i,U)})y_{(i,U),j}.
\]  (9.24)

Since \( K \) is compact, there is a cluster point \( u_0 \in K \) of the net \( \{ u_{(i,U)} : (i,U) \in \mathbb{N}^* \times \mathcal{U} \} \), which means that, for any neighborhood \( V \) of \( u_0 \), and any \( (m,W) \in \mathbb{N}^* \times \mathcal{U} \), there exists \( (i,U) \in \mathbb{N}^* \times \mathcal{U} \) with \( (i,U) \geq (m,W) \) such that \( u_{(i,U)} \in V \).

We prove now that \( u_0 \in T(u_0) \). If \( u_0 \notin T(u_0) \), since \( T(u_0) \) is compact, there exists by separation an open (in \( E \), convex neighborhood \( W \) of \( T(u_0) \)) such that \( u_0 \notin W \). On the other hand, since \( T \) is upper semi-continuous, there exists a neighborhood \( V \) of \( u_0 \) with \( V \cap W = \emptyset \) such that \( T(V \cap K) \subset W \). Since \( V \) is a neighborhood of \( 0 \), and the map \( [x,y] \mapsto x + y \) is continuous, there exists \( m \in \mathbb{N}^* \) and \( U \in \mathcal{U} \) such that \( \frac{1}{m} U + \frac{1}{m} U = U_m + U_m \subset V - u_0 \), i.e. \( u_0 + U_m + U_m \subset V \). Now, \( u_0 + U_m \) is a neighborhood at \( u_0 \) and, by definition of a cluster point, there is some \( (i,W) \geq (m,U) \) such that \( u_{(i,W)} \in u_0 + U_m \subset (u_0 + U_m) + U_m \subset V \). Moreover, \( (i,W) \geq (m,U) \) implies that \( W_i \subset U_m \), and hence

\[
  u_{(i,W)} + W_i \subset u_0 + U_m + W_i \subset u_0 + U_m + U_m \subset V.
\]  (9.25)

Because of \( f_{(i,W),j}(x) = 0 \) if \( x \notin x_{(i,W),j} + W_i \), we deduce from (9.24) that

\[
  u_{(i,W)} = f_{(i,W)}(u_{(i,W)}) = \sum_{j \in J(i,W) : u_{(i,W)} \in x_{(i,W),j} + W_i} f_{(i,W),j}(u_{(i,W)})y_{(i,W),j}.
\]  (9.26)

Since \( W_i \) is circled, one has \( W_i = -W_i \) and hence \( u_{(i,W)} \in x_{(i,W),j} + W_i \) is equivalent with \( x_{(i,W),j} \in u_{(i,W)} + W_i \subset V \), so that \( x_{(i,W),j} \in K \cap V \). It follows that \( y_{(i,W),j} \in T(x_{(i,W),j}) \subset T(K \cap V) \subset W \). By convexity of \( W \), it follows from (9.26) that \( u_{(i,W)} \in W \), which contradicts \( u_{(i,W)} \in V \) and \( V \cap W = \emptyset \).}

Ky Fan’s fixed point theorem has important special cases obtained by specializing \( E \) or \( T \).

The first one, corresponding to \( E = \mathbb{R}^n \), is Kakutani’s fixed point theorem, proved by Kakutani in 1941 [190], and called by the Nobel Prize in Economics G. Debreu ‘the most powerful tool for proofs of existence in economics’ [88].
Corollary 9.4.1 Let $K \subset \mathbb{R}^n$ be non-empty, compact and convex and $T : K \to 2^K$ be upper semi-continuous and such that $T(x)$ is non empty, closed and convex for each $x \in K$. Then there exists some $u_0 \in K$ such that $u_0 \in T(u_0)$.

The second special case corresponds to the case of a single valued mapping in a locally convex topological vector space. It was first proved in 1935 by A.N. Tychonof [391] and is called **Tychonov’s fixed point theorem**. Notice that, for a single-valued mapping, the upper semicontinuity reduces to classical continuity.

Corollary 9.4.2 Any continuous mapping $f : K \to K$ of a non-empty, compact and convex subset $K$ of a locally convex topological vector space $E$ has at least one fixed point.

The special case of Tychonov’s fixed point theorem where $E$ is a normed vector space was first proved by J. Schauder [348] in 1930 and is called **Schauder’s fixed point theorem**. It is most useful in the study of nonlinear differential equations.

Corollary 9.4.3 Any continuous mapping $f : K \to K$ of a non-empty, compact and convex subset $K$ of a normed vector space $E$ has at least one fixed point.

9.5 Applications to game theory

A **n-person-game** is a set of $n$ players, each player $i$ having at his disposal a finite set of pure strategies $\sigma_{i,j}$ ($1 \leq j \leq N_i$), and a pay-off function $\varphi_i$ from the set of all $n$-uples of pure strategies into the real numbers, such that $\varphi_i(\sigma_{1,i_1}, \sigma_{2,i_2}, \ldots, \sigma_{n,i_n})$ represents the pay-off of the $i^{th}$ player if player 1 chooses strategy $\sigma_{1,i_1}$, player 2 choices strategy $\sigma_{2,i_2}$, and so on. More generally, a mixed strategy for the player $i$ is a convex combination of pure strategies $\sum_{j=1}^{N_i} \pi_{i,j} \sigma_{i,j}$ where $0 \leq \pi_{i,j} \leq 1$ and $\sum_{j=1}^{N_i} \pi_{i,j} = 1$. This can be seen as the possibility for the player $i$ to chose randomly his $j^{th}$ strategy with some probability $\pi_{i,j}$. Consider the $\sum_{j=1}^{N_i} \pi_{i,j} \sigma_{i,j}$ as the points of a $N_i$-simplex $K_i$ whose vertices are the $\sigma_{i,j}$, which can be seen as a convex compact subset of a real vector space. The pay-off function has a unique extension to the $n$-uples of mixed strategies $p_i = (\pi_{i,1}, \ldots, \pi_{i,N_i})$, which is linear in the mixed strategy of each player, and which can be written, $f_i(p_1, \ldots, p_n)$ ($i = 1, \ldots, n$).

The following concept of equilibrium was introduced in 1950 by J. Nash [294, 295], and called to-day a **Nash equilibrium**. Nash shared the Nobel Prize of Economics in 1994 for this theory.

**Definition 9.5.1** $(p_1^*, \ldots, p_n^*)$ is an equilibrium for the $n$-person-game with mixed strategies if, for each $i = 1, 2, \ldots, n$, one has

$$f_i(p_1^*, \ldots, p_n^*) = \max_{x \in K_i} f_i(p_1^*, \ldots, p_{i-1}^*, x, p_{i+1}^*, \ldots, p_n^*).$$

The following existence theorem, due to J. Nash [294, 295] gives sufficient conditions upon the $f_i$ in order that an equilibrium exists.
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Theorem 9.5.1 Let $E_i (1 \leq i \leq n)$ be locally convex topological vector spaces, $K_i \subset E_i$ be non empty compact and convex and $f_i : K_1 \times K_2 \times \ldots \times K_n \rightarrow \mathbb{R}$ be continuous functions such that, for any fixed $1 \leq i \leq n$ and any fixed $(p_1, \ldots, p_i-1, p_{i+1}, \ldots, p_n) \in K_i \times \ldots \times K_{i-1} \times K_{i+1} \times \ldots \times K_n$, the mapping $K_i \rightarrow \mathbb{R}, p \mapsto f_i(p_1, \ldots, p_i-1, p_i, p_{i+1}, \ldots, p_n)$ is concave. Then there exists $(p_1^*, \ldots, p_n^*) \in K_1 \times \ldots \times K_n$ such that, for each $i = 1, 2, \ldots, n$,

$$f_i(p_1^*, \ldots, p_i^*, p_{i+1}^*, \ldots, p_n^*) = \max_{x \in K_i} f_i(p_1^*, \ldots, p_i^*, x, p_{i+1}^*, \ldots, p_n^*). \quad (9.27)$$

Proof. The following notation will be used throughout the proof: for any $i = 1, 2, \ldots, n$, let

$$\widehat{K}_i = K_1 \times \ldots \times K_i \times K_{i+1} \times \ldots \times K_n$$

$$\hat{p}_i = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n).$$

The basic idea is to apply Ky Fan’s fixed point theorem 9.4.1 in the following framework: $E = E_1 \times E_2 \times \ldots \times E_n$, $K = K_1 \times K_1 \times \ldots \times K_n$ and $T : K \rightarrow 2^K$ is defined as follows. For $1 \leq i \leq n$,

$$T_i : \widehat{K}_i \rightarrow 2^{K_i}, \hat{p}_i \mapsto \{ p \in K_i : T_i(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n) \} = \max_{x \in K_i} T_i(p_1, \ldots, p_{i-1}, x, p_{i+1}, \ldots, p_n).$$

and $T(p_1, \ldots, p_n) = (T_1(\hat{p}_1), \ldots, T_n(\hat{p}_n))$. Since $E$ is locally convex and $K$ is non empty, convex and compact, it remains to verify that, for any $(p_1, \ldots, p_n) \in K$, $T(p_1, \ldots, p_n)$ is a non empty, closed and convex subset of $K$, and that $T$ is upper semi-continuous on $K$. Indeed, for any $i = 1, 2, \ldots, n$, $T_i(\hat{p}_i)$ is non-empty (Weierstrass theorem), convex (from the concavity assumption) and closed (from the continuity of $f_i$). According to Lemma 9.4.3, the upper semicontinuity of $T$ is equivalent to the closedness of the graph

$$G(T) = \{ ([p_1, \ldots, p_n], [y_1, \ldots, y_n]) : p_i \in K_i, y_i \in T_i(\hat{p}_i) (i = 1, 2, \ldots, n) \}.$$

Let $\{(p_1^{(k)}, \ldots, p_n^{(k)}), (y_1^{(k)}, \ldots, y_n^{(k)})\}$ be a sequence in $G(T)$ such that $p_i^{(k)} \rightarrow p_i \in K_i$, $y_i^{(k)} \rightarrow y_i \in K_i$ for any $i = 1, 2, \ldots, n$. It suffices to show that $y_i \in T_i(\hat{p}_i)$, i.e., by definition of $T_i$, that

$$f_i(p_1, \ldots, p_{i-1}, y_i, p_{i+1}, \ldots, p_n) \geq f_i(p_1, \ldots, p_{i-1}, x, p_{i+1}, \ldots, p_n) \quad (9.28)$$

for each $x \in K_i (i = 1, 2, \ldots, n)$. The condition $y_i^{(k)} \in T_i(p_i^{(k)})$ is equivalent to

$$f_i(p_1^{(k)}, \ldots, p_{i-1}^{(k)}, y_i^{(k)}, p_{i+1}^{(k)}, \ldots, p_n^{(k)}) \geq f_i(p_1^{(k)}, \ldots, p_{i-1}^{(k)}, x, p_{i+1}^{(k)}, \ldots, p_n^{(k)}) \quad (9.29)$$

for any $x \in K_i (i = 1, 2, \ldots, n)$, and it suffices to let $k \rightarrow \infty$ in (9.29) to obtain (9.28). From Ky Fan’s fixed point theorem 9.4.1, there exists $(p_1^*, \ldots, p_n^*) \in K_1 \times \ldots \times K_n$ such that $p_i^* \in T_i(\hat{p}_i) (i = 1, 2, \ldots, n)$, so that, by definition of $T_i$, $(p_1^*, \ldots, p_n^*)$ is a Nash equilibrium. □
A consequence of Nash’s theorem is a result stated and proved in 1928 by J. von Neumann [395, 396], and called von Neumann’s minimax theorem for zero sum 2-person-games.

**Corollary 9.5.1** Let $K_i$ be compact convex subsets of locally convex topological vector spaces $E_i$ ($i = 1, 2$) and $f : K_1 \times K_2 \to \mathbb{R}$ be a continuous mapping such that

1. for each $p_2 \in K_2$, $p_1 \mapsto f(p_1, p_2)$ is concave on $K_1$;
2. for each $p_1 \in K_1$, $p_2 \mapsto f(p_1, p_2)$ is convex on $K_2$.

Then there exists $(p_1^*, p_2^*)$ such that

\[ f(p_1^*, p_2^*) = \max_{p_1 \in K_1} \left[ \min_{p_2 \in K_2} f(p_1, p_2) \right] = \min_{p_2 \in K_2} \left[ \max_{p_1 \in K_1} f(p_1, p_2) \right]. \tag{9.30} \]

**Proof.** Applying Nash equilibrium theorem 9.5.1 with $n = 2$ and $f_1(p_1, p_2) = f(p_1, p_2)$, $f_2(p_1, p_2) = -f(p_1, p_2)$, we obtain $(p_1^*, p_2^*) \in K_1 \times K_2$ such that

\[ f_1(p_1, p_2^*) = \max_{p_1 \in K_1} f(p_1, p_2^*), \quad f_2(p_1^*, p_2) = \max_{p_2 \in K_2} f(p_1^*, p_2) \]

and hence, in term of $f$,

\[ f(p_1^*, p_2^*) = \max_{p_1 \in K_1} f(p_1, p_2^*), \tag{9.31} \]

\[ f(p_1^*, p_2^*) = \min_{p_2 \in K_2} f(p_1^*, p_2). \tag{9.32} \]

From (9.31), one gets

\[ f(p_1^*, p_2^*) \geq \min_{p_2 \in K_2} \left[ \max_{p_1 \in K_1} f(p_1, p_2) \right], \]

while from (9.32), one gets

\[ f(p_1^*, p_2^*) \leq \max_{p_2 \in K_2} \left[ \min_{p_1 \in K_1} f(p_1, p_2) \right], \]

Thus

\[ \min_{p_2 \in K_2} \left[ \max_{p_2 \in K_2} f(p_1, p_2) \right] \leq f(p_1^*, p_2^*) \leq \max_{p_2 \in K_2} \left[ \min_{p_2 \in K_2} f(p_1, p_2) \right]. \]

Since the inequality

\[ \max_{p_1 \in K_1} \left[ \min_{p_2 \in K_2} f(p_1, p_2) \right] \leq \min_{p_2 \in K_2} \left[ \max_{p_2 \in K_2} f(p_1, p_2) \right] \]

is obvious, the result follows. \;
\]
Remark 9.5.1 Analyzing the proof of Nash’s equilibrium theorem 9.5.1, it is easily seen that the assumption of concavity of $f_i$ with respect to the $i^{th}$ variable can be replaced by an assumption of quasi-concavity, i.e., for any $\alpha \in \mathbb{R}$, the set

$$\{x \in K_i : f_i(p_1, \ldots, p_{i-1}, x, p_{i+1}, \ldots, p_n) \geq \alpha\}$$

is convex. This implies that, in von Neumann’s theorem 9.30, it suffices to assume that $f$ is quasi-concave with respect to the first variable and quasi-convex with respect to the second one.

Remark 9.5.2 In [190], Kakutani deduced von Neumann’s theorem 9.30, when $E_1$ and $E_2$ have finite dimension, from the following intersection lemma, originally given by von Neumann [396], which is of independent interest.

Lemma 9.5.1 Let $K \subset \mathbb{R}^m$ and $L \subset \mathbb{R}^n$ be non-empty, compact and convex, and $U \subset K \times L$ and $V \subset K \times L$ be closed and such that

(i) for each $x_0 \in K, U_{x_0} := \{y \in L : (x_0, y) \in U\}$ is closed, convex and nonempty;

(ii) for each $y_0 \in L, V_{y_0} := \{x \in K : (x, y_0) \in V\}$ is closed, convex and nonempty.

Then $U \cap V \neq \emptyset$.

Proof. Define $\phi : K \times L \to 2^{K \times L}$ by $\phi(x, y) = V_y \times U_x$ for each $(x, y) \in K \times L$. Clearly, $K \times L$ is compact and convex in $\mathbb{R}^m \times \mathbb{R}^n$ and, for each $(x, y) \in K \times L$, $\phi(x, y)$ is a closed, convex and non-empty subset of $K \times L$. Moreover, it is easily seen that $G(\phi)$ is closed and hence $\phi$ is upper semi-continuous by Lemma 9.4.3. Then, from Kakutani fixed point theorem 9.4.1, there exists $(x_0, y_0) \in K \times L$ such that $(x_0, y_0) \in \phi(x_0, y_0) = V_{y_0} \times U_{x_0}$, which means that $(x_0, y_0) \in U \cap V$. □

To deduce directly the minimax theorem from the topological lemma, it suffices, with von Neumann, to take $S = K, T = L$,

$$V = \{(x, y) \in K \times L : f(x, y) \leq \min_{v \in L} f(x, v)\},$$

$$W = \{(x, y) \in K \times L : f(x, y) \geq \max_{u \in K} f(u, y)\}.$$

Then there exists $(x^*, y^*) \in V \cap W$, i.e. such that

$$\max_{x \in K} f(x, y^*) \leq f(x^*, y^*) \leq \min_{y \in L} f(x^*, y),$$

which immediately implies that

$$\min_{y \in L} \max_{x \in K} f(x, y) \leq f(x^*, y^*) \leq \max_{x \in K} \min_{y \in L} f(x, y).$$

As the opposed inequality

$$\max_{x \in K} \min_{y \in L} f(x, y) \leq \min_{y \in L} \max_{x \in K} f(x, y)$$

always holds, the result follows.
Chapter 10

Equations in reflexive Banach spaces

10.1 Monotone mappings

Corollary 8.4.1 is at the basis of some results due to F.E. Browder [43] and to G.J. Minty [285] for the existence of zeros of some mappings between a reflexive Banach space and its dual, which verify some monotonicity conditions, and which can be used to obtain general existence theorem for some nonlinear elliptic boundary value problems.

Let $X$ be a real Banach space, $X^*$ its dual space with the pairing between $x^* \in X^*$ and $x \in X$ denoted by $\langle x^*, x \rangle_{X,X^*}$ or, if no confusion appears, by $\langle x^*, x \rangle$.

**Definition 10.1.1** If $T$ is a (in general nonlinear) mapping with domain $D(T)$ in $X$ and range $R(T)$ in $X^*$, $T$ is said to be

(i) **monotone** if, for all $u$ and $v$ in $D(T)$, one has

$$\langle T(u) - T(v), u - v \rangle \geq 0;$$

(ii) **strongly coercive** if

$$\frac{\langle Tu, u \rangle}{\|u\|} \to +\infty \quad \text{as} \quad \|u\| \to \infty;$$

(iii) **demicontinuous at** $u_0 \in D(T)$ if, for each sequence $(u_n)$ in $D(T)$ with $u_n \to u_0$ one has $T(u_n) \rightharpoonup T(u_0)$, where $\rightharpoonup$ denotes the weak convergence in $X^*$.

(iv) **hemicontinuous at** $u_0 \in \text{int} D(T)$ if, for any $y$ and $z$ in $X$,

$$\langle T(u_0 + ty), z \rangle \to \langle T(u_0), z \rangle \quad \text{as} \quad t \to 0^+.$$
Notice that, since \( u_0 \in \text{int } D(T) \), \( u_0 + ty \in D(T) \) for any \( y \in X \) if \( t > 0 \) is sufficiently small.

The first explicit definition of monotone mappings from a Banach space to its dual was given in 1960 by R.I. Kačurovskii [189]. In 1963, F.E. Browder introduced the concepts of weak-coercivity conditions, demicontinuity and of hemicontinuity respectively in [43], [41] and [42].

The following simple result, usually called Minty’s trick [285] is useful in the study of monotone hemicontinuous mappings.

**Lemma 10.1.1** If \( X \) is a Banach space, \( F : D(F) \subset X \to X^* \) is hemicontinuous at \( u_0 \in \text{int } D(T) \), and if

\[
\langle F(v), v - u_0 \rangle \geq 0
\]

for all \( v \in B[u_0, \rho] \) and some \( \rho > 0 \), then \( F(u_0) = 0 \).

**Proof.** Let \( w \in X \) and take \( v = u_0 + tw \) with \( t > 0 \) sufficiently small so that \( u_0 + tw \in B[u_0, \rho] \). We deduce from (10.1) that

\[
\langle F(u_0 + tw), w \rangle \geq 0
\]

and hence, if \( t \to 0^+ \), using hemicontinuity,

\[
\langle F(u_0), w \rangle \geq 0.
\]

The result follows by taking \( w = -F(u_0) \).

It is clear that, if \( X \) has finite dimension, \( T \) is demicontinuous at \( u_0 \in D(T) \) if and only if \( T \) is continuous at \( u_0 \). It is clear also that, for an arbitrary reflexive Banach space \( X \), the demicontinuity of \( T \) at \( u_0 \in \text{int } D(T) \) implies its hemicontinuity at \( u_0 \). It is more surprising that, when \( T \) is monotone, the converse is true. This result is due to the efforts of R.T. Rockafellar [331], T. Kato [197] and F.E. Browder [44].

**Lemma 10.1.2** Let \( X \) be a reflexive Banach space, \( T : D(T) \subset X \to X^* \) be monotone and hemicontinuous at \( u_0 \in \text{int } D(T) \). Then \( T \) is demicontinuous at \( u_0 \).

**Proof.** We first prove that \( T \) is locally bounded at \( u_0 \), i.e. that \( \|T(u)\|_{X^*} \leq c \) on \( B_{\omega_0}(r) \subset D(T) \) for some \( c > 0 \) and some \( r > 0 \). By translation, we can assume without loss of generality that \( u_0 = 0 \). Let \( \overline{B}(\rho) \subset D(T) \) and \( z \in \overline{B}(\rho) \). Then, by monotonicity,

\[
\langle T(x), x - z \rangle \geq \langle T(z), x - z \rangle \geq -n
\]

for all \( x \in \overline{B}(\rho) \) and some \( n \in \mathbb{N} \). If we define

\[
M_n := \{ z \in \overline{B}(\rho) : \langle T(x), x - z \rangle \geq 0 \text{ for all } x \in \overline{B}(\rho) \},
\]

\[
\overline{B}(\rho) \cap M_n \subset \overline{B}(\rho) \cap M_{n+1}
\]

for all \( n \in \mathbb{N} \). Hence, \( \bigcap M_n \neq \emptyset \). Therefore, there exists \( z \) such that

\[
\langle T(z), x \rangle 
\]
then $M_n$ is closed and $\overline{B}(p) = \bigcup_{n \in \mathbb{N}} M_n$. From Baire’s theorem, one of the $M_n$, say $M_q$, has a non-empty interior, i.e. we can find $r > 0$ and $z_0 \in \overline{B}(p)$ such that $z_0 + \overline{B}(r) \subset M_q$. In other words,

$$\langle T(x), x - z_0 - y \rangle \geq -p$$

(10.2)

for all $x \in \overline{B}(p)$ and $y \in \overline{B}(r)$. Since $-z_0 \in M_q$ for some $q \in \mathbb{N}$, we have also

$$\langle T(x), x + z_0 \rangle \geq -q$$

(10.3)

for all $x \in \overline{B}(p)$, and hence, by adding (10.2) and (10.3),

$$\langle T(x), 2x - y \rangle \geq -(p + q)$$

(10.4)

for all $x \in \overline{B}(p)$ and $y \in \overline{B}(r)$. Now, if $x \in \overline{B}(r/4)$ and $z \in \overline{B}(r/2)$, then $y = 2x - z \in \overline{B}(r)$ and (10.4) gives

$$\langle T(x), z \rangle \geq -(p + q)$$

for $x \in \overline{B}(r/4)$ and $z \in \overline{B}(r/2)$. Consequently, for all $x \in \overline{B}(r/4)$,

$$\|T(x)\|_{X^*} = \frac{2}{r} \sup \{ \langle F(x), z \rangle : z \in \overline{B}(r/2) \} \leq \frac{2}{r} (p + q)$$

and $T$ is locally bounded at $u_0$. Now, let $u_n \to u_0$. By the local boundedness of $T$, we may assume that $(T(u_n))$ is bounded, and as $X$ is reflexive, we may assume, going if necessary to a subsequence, that $T(u_n) \to y$ for some $y \in X^*$. Therefore

$$0 \leq \langle T(u_n) - T(v), u_n - v \rangle \to \langle y - T(v), u_0 - v \rangle$$

for every $v \in \overline{B}_{\|\cdot\|}(\rho) \subset D(T)$. Using Lemma 10.1.1 with $F(v) = T(v) - y$, we get $y = T(u_0)$. As $y$ is the same for any converging subsequence of $(T(u_n))$ the conclusion follows.

### 10.2 Browder-Minty’s theorem

To prove Browder-Minty’s theorem, we need a variant of Corollary 8.4.1.

**Lemma 10.2.1** Let $Y$ be a finite-dimensional real normed vector space, and $h : Y \to Y^*$ be continuous and strongly coercive. Then $h$ is onto.

**Proof.** Let $n = \dim Y$, $j : (\mathbb{R}^n, \|\cdot\|) \to (Y, \|\cdot\|_Y)$ be an isomorphism and $j^* : Y^* \to (\mathbb{R}^n)^*$ its adjoint. We identify $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$ through the standard relation

$$\langle f, x \rangle_{(\mathbb{R}^n)^*, \mathbb{R}^n} = \langle \pi f, x \rangle \quad (x \in \mathbb{R}^n)$$
We now prove that \( R > 0 \). We have
\[
\begin{align*}
\frac{(g(x), x)}{\|x\|} &= \frac{((\pi \circ j^* \circ h \circ j)(x), x)}{\|x\|} = \frac{(j^* \circ h \circ j)(x)_{\mathbb{R}^n, \mathbb{R}^n}, \|x\|}
\end{align*}
\]
and hence, using Lemma 10.1.2, continuous on \( Y \). From Corollary 8.4.1, by \( \pi \) the family of all finite-dimensional vector subspaces of \( X \), the same is true for \( j \), and \( j \) is monotone, hemicontinuous and strongly coercive operator.

Proof. Let \( T : X \to X^\ast \) be the canonical injection and \( i_T^F : X^\ast \to F^\ast \) its adjoint, so that
\[
\langle i_T^F(x^\ast), u \rangle_{F, F^\ast} = \langle x^\ast, i_T(u) \rangle_{X, X^\ast} = \langle x^\ast, u \rangle_{X, X^\ast} \quad \text{for all } x^\ast \in F^\ast, u \in X.
\]
Define \( T_F : F \to F^\ast \) by \( T_F = i_T^F \circ T \circ i_F \). It is easy to check that \( T_F \) is hemicontinuous, and hence, using Lemma 10.1.2, continuous on \( F \). On the other side, for any \( x, y \in F \), we have
\[
\begin{align*}
\langle T_F(x), y \rangle_{F, F^\ast} &= \langle i_T^F(T(i_F(x))), y \rangle_{F, F^\ast} = \langle T(i_F(x)), i_F(y) \rangle_{X, X^\ast} \\
&= \langle T(x), y \rangle_{X, X^\ast}.
\end{align*}
\]
Relation (10.5) immediately implies that \( T_F \) is monotone and strongly coercive on \( F \). Then, using Lemma 10.2.1,
\[
W_F := \{ u_F \in F : T_F(u_F) = 0 \} \neq \emptyset \quad (F \in \mathcal{F}).
\]
We now prove that \( \bigcup_{F \in \mathcal{F}} W_F \) is bounded. From the strong coercivity, there exists \( r > 0 \) such that, for any \( F \in \mathcal{F} \), and \( u \in F \),
\[
\langle T_F(u), u \rangle_{X, X^\ast} = \langle T(u), u \rangle_{X, X^\ast} \geq \|u\|_X \geq R \quad \text{if } \|u\|_X \geq r.
\]
and hence \( \|u_F\|_X < r \) when \( u_F \in F \) is such that \( T_F(u_F) = 0 \). Thus \( \bigcup_{F \in \mathcal{F}} W_F \subset \overline{f}(R) \). For \( F_0 \in \mathcal{F} \), let
\[
V_{F_0} := \bigcup_{F \in \mathcal{F}, F \supseteq F_0} W_F.
\]
Clearly \( V_{F_0} \subset \overline{f}(R) \) and, \( \overline{f}(R) \) being compact for the weak topology \( \sigma(X, X^*) \), we have \( \overline{V_{F_0}}^{\sigma(X, X^*)} \subset \overline{f}(R) \). Moreover, the family \( \{\overline{V_F}^{\sigma(X, X^*)} : F \in \mathcal{F}\} \) has the finite intersection property. Indeed, if \( (F_i)_{i \in I} \) is a finite family in \( \mathcal{F} \), and \( F_0 \in \mathcal{F} \) is such that \( F_0 \supseteq F_i \) for all \( i \in I \), then \( V_{F_0} \subset \bigcap_{i \in I} V_{F_i} \) and, consequently \( \overline{V_{F_0}}^{\sigma(X, X^*)} \subset \bigcap_{i \in I} \overline{V_{F_i}}^{\sigma(X, X^*)} \). We conclude that \( \bigcap_{F \in \mathcal{F}} \overline{V_F}^{\sigma(X, X^*)} \neq \emptyset \). Let \( u_0 \in \bigcap_{F \in \mathcal{F}} \overline{V_F}^{\sigma(X, X^*)} \), and let us prove that \( T(u_0) = 0 \). Choose \( u \in X \) arbitrary and \( F_0 \in \mathcal{F} \) such that \( F_0 \supseteq u \). Since \( u_0 \in \overline{V_{F_0}}^{\sigma(X, X^*)} \), \( V_{F_0} \) is bounded and \( X \) is reflexive, it follows from Proposition 7.2 in [47] that there exists a sequence \( (u_n) \) in \( V_{F_0} \) such that \( u_n \rightharpoonup u_0 \). By the monotonicity of \( T \), we have
\[
\langle T(u) - T(u_n), u - u_n \rangle_{X, X^*} \geq 0, \tag{10.6}
\]
and, by definition of \( V_{F_0} \), we have
\[
\langle T(u_n), u - u_n \rangle_{X, X^*} = \langle T_{F_0}(u_n), u - u_n \rangle_{F_0, F_0^*} = 0.
\]
Combined with (10.6), this gives
\[
\langle T(u), u - u_n \rangle_{X, X^*} \geq 0
\]
and hence, letting \( n \to \infty \),
\[
\langle T(u), u - u_0 \rangle_{X, X^*} \geq 0
\]
for any \( u \in X \). Lemma 10.1.1 with \( F = T \) implies that \( T(u_0) = 0 \). ■

The following result was first proved independently by M.M. Vainberg [392] and E.H. Zarantonello [406] in 1960 for Lipschitzian mappings.

**Corollary 10.2.1** If \( X \) is a reflexive Banach space and \( T : X \to X^* \) is hemicontinuous and strongly monotone, i.e. if
\[
\langle T(u) - T(v), u - v \rangle \geq \alpha \|u - v\|^2 \tag{10.7}
\]
for all \( u, v \in X \) and some \( \alpha > 0 \), then \( T : X \to X^* \) is a bijection and \( T^{-1} : X^* \to X \) is Lipschitzian.

**Proof.** \( T : X \to X^* \) is obviously monotone, and it follows from (10.7) that it is strongly coercive. Hence, from Browder-Minty's theorem 10.2.1, \( T \) is onto. Condition (10.7) immediately implies that \( T : X \to X^* \) is one-to-one and that, for \( w, z \in X^* \),
\[
\|T^{-1}(w) - T^{-1}(z)\|^2 \leq \alpha^{-1} \langle w - z, T^{-1}(w) - T^{-1}(z) \rangle
\]
\[
\leq \|w - z\|_{X^*} \|T^{-1}(w) - T^{-1}(z)\|,
\]
such that \( T^{-1} \) is Lipschitzian with Lipschitz constant \( \alpha^{-1} \). ■
10.3 A monotone quasilinear Dirichlet problem

We give in this section a simple application of Browder-Minty’s theorem to some quasilinear Dirichlet problem on a bounded open subset $\Omega$ of $\mathbb{R}^n$. This example is treated in [60] under more restrictive assumptions. Let $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping such that there exist $M > 0$ having the property that

$$\|\alpha(\xi)\| \leq M\|\xi\| \quad (\xi \in \mathbb{R}^n).$$  \hspace{1cm} (10.8)

Let $H^1(\Omega)$ be the Sobolev space of functions $u \in L^2(\Omega)$ whose distributional partial derivatives also belong to $L^2(\Omega)$, with the usual inner product

$$\int_\Omega u(x)v(x) \, dx + \int_\Omega \langle \nabla u(x), \nabla v(x) \rangle \, dx,$$

Let $H^1_0(\Omega)$ be the closure in $H^1(\Omega)$ of the space of infinitely differentiable functions having compact support in $\Omega$, endowed with the equivalent inner product

$$\int_\Omega \langle \nabla u(x), \nabla v(x) \rangle \, dx,$$

and let $f \in H^{-1}(\Omega)$ (the dual space of $H^1_0(\Omega)$) be given. We consider the problem of finding weak solutions for the quasilinear elliptic Dirichlet problem

$$-\text{div} \alpha(\nabla u) = f, \quad u \in H^1_0(\Omega).$$  \hspace{1cm} (10.9)

More explicitly, a weak solution of problem (10.9) is a function $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \langle \alpha(\nabla u(x)), \nabla v(x) \rangle \, dx = \langle f, v \rangle_H$$  \hspace{1cm} (10.10)

for all $v \in H^1_0(\Omega)$, where $\langle \cdot, \cdot \rangle_H$ denotes the duality map between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. Notice that, because of condition (10.8), we have, for $u, v \in H^1_0(\Omega)$

$$|\langle \alpha(\nabla u), \nabla v \rangle| \leq \|\alpha(\nabla u)\|\|\nabla v\| \leq M\|\nabla u\|\|\nabla v\| \in L^1(\Omega),$$

and hence the integral in (10.10) is well defined and problem (10.9) makes sense.

**Theorem 10.3.1** If $\alpha$ satisfies condition (10.8) and if there exists $m > 0$ such that

$$\langle \alpha(\xi) - \alpha(\xi'), \xi - \xi' \rangle \geq m\|\xi - \xi'\|^2$$  \hspace{1cm} (10.11)

for all $\xi, \xi' \in \mathbb{R}^n$, then problem (10.9) has a unique solution.

**Proof.** Let us introduce the operator $T : H^1_0(\Omega) \to H^{-1}(\Omega)$ by the relation

$$\langle T(u), v \rangle_H = \int_\Omega \langle \alpha(\nabla u(x)), \nabla v(x) \rangle \, dx$$  \hspace{1cm} (10.12)
for all $v \in H^1_0(\Omega)$. The operator $T$ is well defined, because, if $u \in H^1_0(\Omega)$, using (10.8) and Cauchy-Schwarz inequality,

$$\|\langle T(u), v \rangle_H \| \leq \int_\Omega \|\alpha(\nabla u(x))\| \|\nabla v(x)\| \, dx$$

$$\leq M \int_\Omega \|\nabla u(x)\| \|\nabla v(x)\| \, dx \leq M \|u\|_{L^2} \|\nabla v\|_{L^2}$$

$$= M \|u\|_{H^1_0} \|v\|_{H^1_0}$$

for all $v \in H^1_0(\Omega)$, and hence $T(u) \in H^{-1}(\Omega)$ and $\|T(u)\|_{H^{-1}} \leq M \|u\|_{H^1_0}$. We now show that $T$ is strongly monotone. If $u, v \in H^1_0(\Omega)$, then, using assumption (10.11),

$$\langle T(u) - T(v), u - v \rangle_H = \int_\Omega \langle \alpha(u(x)) - \alpha(v(x)), \nabla u(x) - \nabla v(x) \rangle \, dx$$

$$\geq m \int_\Omega \|\nabla (u(x) - v(x))\|^2 \, dx = m \|u - v\|^2_{H^1_0}.$$  \hspace{1cm} (10.13)

The inequality (10.13) implies that $T$ is strongly coercive, because

$$\frac{\langle T(u), u \rangle_H}{\|u\|_{H^1_0}} = \frac{\langle T(u) - T(0), u \rangle_H}{\|u\|_{H^1_0}} + \frac{\langle T(0), u \rangle_H}{\|u\|_{H^1_0}} \geq m \|u\|_{H^1_0} - \|T(0)\|$$

$$\rightarrow +\infty \text{ as } \|u\|_{H^1_0} \rightarrow \infty.$$  \hspace{1cm} (10.14)

We now show that $T$ is hemi-continuous at any $u \in H^1_0(\Omega)$. For any positive sequence $(t_k)$ converging to zero and any $h, v \in H^1_0(\Omega)$, one has

$$\langle T(u + t_k h) - T(u), v \rangle_H = \int_\Omega \langle \alpha(\nabla u(x) + t_k \nabla h(x)) - \alpha(\nabla u(x)), \nabla v(x) \rangle \, dx.$$  \hspace{1cm} (10.14)

Set, for $k = 1, 2, \ldots$,

$$g_k(x) = \langle \alpha(\nabla u(x) + t_k \nabla h(x)) - \alpha(\nabla(x)), \nabla v(x) \rangle.$$  \hspace{1cm} (10.14)

Clearly, each $g_k \in L^1(\Omega)$, $(f_k)$ converges to 0 almost everywhere in $\Omega$, and, assuming, without loss of generality, that $t_k \leq 1$ for all $k = 1, 2, \ldots$, we have

$$\|\alpha(\nabla u(x) + t_k \nabla h(x)) - \alpha(\nabla u(x))\| \leq M \|\nabla u(x) + t_k \nabla h(x)\| + M \|\nabla u(x)\|$$

$$\leq M \|\nabla u(x)\| + \|\nabla h(x)\||$$

and hence

$$|g_k(x)| \leq M \|2\nabla u(x)\| + \|\nabla h(x)\||\nabla v(x)\| \quad (k = 1, 2, \ldots)$$

with a right-hand member belonging to $L^1(\Omega)$. Hence, Lebesgue’s dominated convergence theorem applied to (10.14) implies that

$$\lim_{k \to \infty} \langle T(u + t_k h), v \rangle_H = \langle T(u), v \rangle_H.$$  \hspace{1cm} (10.14)

The existence of a unique solution follows then from Browder-Minty’s theorem 10.2.1 and Corollary 10.2.1, with, in addition the Lipschitz character of $T^{-1}$. \hspace{1cm} ■
10.4 A non-monotone quasilinear Dirichlet problem

We show in this section when monotonicity is lacking, some compactness arguments can be used to obtain existence results from finite-dimensional approximations. The corresponding method has been first introduced by Vishik and called the compactness method by J.L. Lions [253]. The example we give is treated in a somewhat different way in [60]. Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set and \( \alpha : \Omega \times \mathbb{R} \to \mathbb{R} \) a Carathéodory function, i.e. for a.e. \( x \in \Omega \), \( \alpha(x, \cdot) \) is continuous and for any \( s \in \mathbb{R} \), \( \alpha(\cdot, s) \) is measurable. Furthermore, let us assume that there exist constants \( 0 < m \leq M \) such that

\[
 m \leq \alpha(x, s) \leq M
\]

for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \). Given \( f \in H^{-1}(\Omega) \), we are interested in finding weak solutions of the following Dirichlet problem for a quasilinear elliptic equation

\[
 -\text{div} \ (\alpha(x, u) \nabla u) = f, \quad u \in H^1_0(\Omega)
\]

i.e. finding \( u \in H^1_0(\Omega) \) such that

\[
 \int_{\Omega} \alpha(x, u(x)) \langle \nabla u(x), \nabla v(x) \rangle \, dx = \langle f, v \rangle_{H^{-1}}
\]

for all \( v \in H^1_0(\Omega) \). We use the same notations as in the previous section.

**Theorem 10.4.1** If condition (10.15) holds, problem (10.16) has at least one solution.

**Proof.** Let \( V \) be a finite-dimensional subspace of \( H^1_0(\Omega) \) endowed with the \( H^1_0 \)-norm, and \( V^* \) its dual. Define the mappings \( H : V \times [0, 1] \to V^* \) by

\[
 \langle H(u, \lambda), v \rangle_{H^{-1}} = \int_{\Omega} \alpha(x, \lambda u) \langle \nabla u(x), \nabla v(x) \rangle \, dx - \langle f, v \rangle_{H^{-1}}
\]

for all \( v \in V \). Because of condition (10.15), \( H \) is well defined. Let us show now that

\[
 \{u \in V : H(u, \lambda) = 0 \text{ for some } \lambda \in [0, 1] \} \subset \overline{B}(m^{-1} \|f\|_{H^{-1}}).
\]

Indeed, if \( H(u, \lambda) = 0 \) for some \( (u, \lambda) \in V \times [0, 1] \), then

\[
 0 = \langle H(u, \lambda), u \rangle_{H^{-1}} \geq m \|u\|_{H^1_0}^2 - \|f\|_{H^{-1}} \|u\|_{H^1_0},
\]

which implies that \( \|u\|_{H^1_0} \leq m^{-1} \|f\|_{H^{-1}} \). Consequently, for any \( R > m^{-1} \|f\|_{H^{-1}} \), we have

\[
 H(u, \lambda) \neq 0 \text{ if } (u, \lambda) \in \partial B^V(R) \times [0, 1].
\]
Now, if \((u, \lambda) \in \overline{B}^V(R) \times [0, 1]\), we have
\[
|\langle H(u, \lambda), v \rangle| \leq M \|u\|_{H^1} \|v\|_{H^1} + \|f\|_{H^{-1}} \|v\|_{H^1} \leq (MR + \|f\|_{H^{-1}}) \|v\|_{H^1},
\]
for all \(v \in H^1_0(\Omega)\), and hence
\[
H(\overline{B}^V(R) \times [0, 1]) \subset \overline{B}^V(MR + \|f\|_{H^{-1}}).
\] (10.20)

We now show that \(H\) is continuous on \(\overline{B}^V(R) \times [0, 1]\). Let \((u_n, \lambda_n) \in \overline{B}^V(R) \times [0, 1]\) converge to \((u, \lambda)\) in \(V \times [0, 1]\), i.e. in \(H^1_0(\Omega) \times [0, 1]\). Since \((H(u_n, \lambda_n))\) is bounded because of (10.20), to prove that \(H(u_n, \lambda_n) \to H(u, \lambda)\), it is sufficient to show that \(H(u, \lambda)\) is the unique cluster point of \((H(u_n, \lambda_n))\). Let \(g \in V^*\) be such a cluster point, \((\lambda_{n_k})\) a subsequence of \((\lambda_n)\), and \((u_{n_k})\) a subsequence of \((u_n)\) such that \(H(u_{n_k}, \lambda_{n_k}) \to g\) in \(V^*\). Since \(u_{n_k} \to u\) in \(H^1_0(\Omega)\), it follows that \(u_{n_k} \to u\) in \(L^2(\Omega)\), and hence, going if necessary to a subsequence, we may assume that \(u_{n_k} \to u\) a.e. in \(\Omega\). This implies that \(\alpha(x, \lambda_{n_k} u_{n_k}) \to \alpha(x, \lambda u)\) a.e. in \(\Omega\), and hence, for any \(v \in V\),
\[
\alpha(x, \lambda_{n_k} u_{n_k}) \nabla v \to \alpha(x, \lambda u) \nabla v
\]
in \(L^2(\Omega)\). On the other hand, \(\partial_t u_{n_k} \to \partial_t u\) in \(L^2(\Omega)\). We conclude that
\[
\begin{align*}
(H(u_{n_k}, \lambda_{n_k}), v)_H &= \int_\Omega [\alpha(x, \lambda_{n_k} u_{n_k})(\nabla u_{n_k}, \nabla v(x))] \, dx \\
&\to \int_\Omega [\alpha(x, \lambda u)(\nabla u(x), \nabla v(x)) \, dx = (H(u, \lambda), v)_H.
\end{align*}
\]
Thus \(g = H(u, \lambda)\). All those properties allow us to apply the homotopy invariance property (3.4.2) and obtain
\[
d_B[H(\cdot, 1), B(R), 0] = d_B[H(\cdot, 0), B(R)].
\] (10.21)

But \(H(u, 0) = 0\) is equivalent to the linear problem
\[
\int_\Omega \alpha(x, 0)(\nabla u(x), \nabla v(x)) \, dx = (f, v)_H
\]
for all \(v \in V\), whose solution is unique because of the boundedness of the set of its possible solutions. Consequently, \(d_B[H(\cdot, 0), B(R), 0] = \pm 1\), and, from (10.21) and the existence property 3.4.1 of degree, there exists \(u \in B^V(R)\) which satisfies
\[
\int_\Omega \alpha(x, u(x))(\nabla u(x), \nabla v(x)) \, dx = (f, v)_H, \quad \|u\|_{H^1_0} \leq m^{-1} \|f\|_{H^{-1}}
\] (10.22)
for all \(v \in V\).

Now, it is well known that one can write \(H^1_0(\Omega) = \bigcup_{k \geq 1} V_k\) where \(V_k \subset V_{k+1}\) \((k \geq 1)\) and \(V_k\) has dimension \(k\). Consequently, given any \(v \in H^1_0(\Omega)\), there exists
a sequence \((v_k)\) with \(v_k \in V_k\) which converges to \(v\). On the other hand, by (10.22) applied to \(V = V_k\), there exists, for each \(k \geq 1\), some \(u_k \in V_k\) such that

\[
\int_\Omega \alpha(x, u_k(x)) \langle \nabla u_k(x), \nabla w(x) \rangle \, dx = \langle f, w \rangle_H, \quad \|u_k\|_{H^1_0} \leq m^{-1} \|f\|_{H^{-1}}
\]

for all \(w \in V_k\). In particular, taking \(w = v_k\) introduced above,

\[
\int_\Omega \alpha(x, u_k(x)) \langle \nabla u_k(x), \nabla v_k(x) \rangle \, dx = \langle f, v_k \rangle_H, \quad \|u_k\|_{H^1_0} \leq m^{-1} \|f\|_{H^{-1}} \tag{10.23}
\]

for all \(k \geq 1\). The estimate in (10.23) implies that, going if necessary to subsequences, we can assume that there exists \(u \in H^1_0(\Omega)\) such that \(u_k \rightharpoonup u\) in \(H^1_0(\Omega)\), \(u_k \to u \) in \(L^2(\Omega)\) and \(u_k \to u \) a.e. in \(\Omega\). As \((\alpha(\cdot, u_k(\cdot)))\) is bounded, and \(\nabla v_k \to \nabla v\) strongly in \(L^2(\Omega)\), one can let \(k \to \infty\) in (10.23) to obtain

\[
\int_\Omega \alpha(x, u(x)) \langle \nabla u(x), \nabla v(x) \rangle \, dx = \langle f, v \rangle_H,
\]

and, \(v \in H^1_0(\Omega)\) being arbitrary, \(u\) is a weak solution of (10.16).

### 10.5 Homogeneous Navier-Stokes equations

Let \(\Omega \subset \mathbb{R}^n\), with \(n \geq 2\), be a bounded open set with Lipschitz boundary, let \(f = (f_1, \ldots, f_n) \in L^2(\Omega) \times \ldots \times L^2(\Omega) := L^n_2(\Omega)\). Similar notations will be used for other products of \(n\) vector spaces. We are looking for a vector function \(u = (u_1, \ldots, u_n)\) and a scalar function \(p\) on \(\Omega\) (representing the velocity and the pressure of the fluid respectively) satisfying the following homogeneous stationary Navier-Stokes equations with Dirichlet boundary conditions, with \(\nu\) a real coefficient (representing the coefficient of kinematic viscosity of the fluid):

\[
-\nu \Delta u + \sum_{i=1}^n (u_i \partial_i)u + \text{grad} \, p = f \quad \text{in} \quad \Omega, \quad \tag{10.24}
\]

\[
div \, u = 0 \quad \text{in} \quad \Omega \quad \tag{10.25}
\]

\[
u \partial_k u_k = 0 \quad \text{on} \quad \Gamma := \partial \Omega. \quad \tag{10.26}
\]

In a scalar form, the system (10.24) can be rewritten as

\[
-\nu \sum_{i=1}^n \partial_i^2 u_k + \sum_{i=1}^n u_i \partial_i u_k + \partial_k p = f_k \quad \text{in} \quad \Omega \quad (k = 1, 2, \ldots, n).
\]

Denote by

\[
\mathcal{V} = \{ u \in \mathcal{D}_a(\Omega) : div \, u = 0 \} \quad \tag{10.27}
\]
the space of test functions, and let $V$ be the closure of $\mathcal{V}$ in the Sobolev space $H^1_{0, n}(\Omega)$ of $(u_1, \ldots, u_n)$ such that $u \in L^2_n(\Omega)$ and $\partial_i u \in L^1_n(\Omega)$ for all $i = 1, 2, \ldots, n$, with the norm

$$
\|u\|_{H^1_{0, n}(\Omega)}^2 = \sum_{i,j=1}^n \int_\Omega |\partial_i u_j(x)|^2 dx = \sum_{j=1}^n \|u_j\|_{H^1_0(\Omega)}^2.
$$

The following result is proved in [383].

**Proposition 10.5.1** Let $\Omega \subset \mathbb{R}^n$ be open with a Lipschitzian boundary. Then

$$
V = \{ u \in H^1_{0,n}(\Omega) : \text{div} u = 0 \}. \tag{10.28}
$$

If $f, p$ and $u$ are smooth functions satisfying (10.24)-(10.26), then, taking the scalar product of (10.24) in $L^1_n(\Omega)$ with some $v \in \mathcal{V}$, integrating by parts and using conditions (10.25) and (10.26), we get

$$
\nu(u, v)_{H^1_{0, n}(\Omega)} + b(u, u, v) = \langle f, v \rangle_{L^2_n(\Omega)}, \quad (v \in \mathcal{V}), \tag{10.29}
$$

where

$$
\langle u, v \rangle_{H^1_{0, n}(\Omega)} = \sum_{i,j=1}^n \int_\Omega \partial_i u_j(x) \partial_i v_j(x) dx, \tag{10.30}
$$

and $b$ is defined by

$$
b(u, v, w) = \sum_{i,j=1}^n \int_\Omega u_i(\partial_i v_j) w_j dx. \tag{10.31}
$$

Consequently, if $f, u$ and $p$ are smooth functions satisfying (10.24)-(10.26) then, for any $v \in \mathcal{V}$, equality (10.29) holds. Continuity and density arguments then show that (10.29) is satisfied for any $v \in V$. Of course a smooth function $u$ satisfying (10.24)-(10.26) is a smooth function in $V$. To prove the converse, we need a condition for a vector-valued distribution to be a gradient given in [87] or in [383].

**Proposition 10.5.2** Let $\Omega \subset \mathbb{R}^n$ be open and $f = (f_1, \ldots, f_n)$ with $f_i \in \mathcal{D}'(\Omega)$ $(i = 1, 2, \ldots, n)$. A necessary and sufficient condition in order that $f = \text{grad} p$ for some $p \in \mathcal{D}'(\Omega)$ is that $\langle \langle f, v \rangle \rangle = 0$ for any $v \in \mathcal{V}$, where $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the duality pairing between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$.

By using Proposition 10.5.2, we deduce from (10.28) the existence of a distribution $p$ such that (10.24) is satisfied, (10.25) and (10.26) being also satisfied since $u \in V$.

### 10.6 Variational formulation

Because of the fact that, for $u$ and $v$ in $V$, the expression $b(u, u, v)$ needs not to make sense, the variational formulation of problem (10.24)-(10.26) cannot be just :
find $u \in V$ such that (10.29) holds for any $v \in V$. Some preparation is required to get the correct variational formulation, and, in particular, we need to recall some results on Sobolev spaces $H^1_0(\Omega) = H^1_{0,1}(\Omega)$. Again, let $\Omega \subset \mathbb{R}^n$ be an open bounded subsets with Lipschitzian boundary. Then the following result is true.

**Proposition 10.6.1**

(i) If $n = 2$, $H^1_0(\Omega) \subset L^q(\Omega)$ with compact injection for any $1 \leq q < +\infty$;

(ii) If $n \geq 3$, $H^1_0(\Omega) \subset L^q(\Omega)$ with continuous injection if $1 \leq q < 2n/(n-2)$ and compact injection if $1 \leq q < 2n/(n-2)$. In particular, $H^1_0(\Omega) \subset L^2(\Omega)$ with compact injection.

In particular, under the assumptions of continuous embedding, there exists $c > 0$ such that

$$\|u\|_{L^q(\Omega)} \leq c \|u\|_{H^1_0(\Omega)}$$

for all $u \in H^1_0(\Omega)$ (Friedrichs-Poincaré’s inequality).

**Corollary 10.6.1**

If $n = 2, 3, 4$, one has $H^1_0(\Omega) \subset L^n(\Omega)$ with continuous embedding, i.e. there exists $c > 0$ such that

$$\|u\|_{L^n(\Omega)} \leq c \|u\|_{H^1_0(\Omega)}$$

for all $u \in H^1_0(\Omega)$.

**Proof.** Follows from Proposition 10.6.1 by noticing that $L^{2n/(n-2)}(\Omega) \subset L^n(\Omega)$ with continuous injection if $2n/(n-2) \geq n$, i.e. if $n \leq 4$.

Skipping from now one ($\Omega$) from the spaces notations, consider the space $H^1_{0,n} \cap L^n_n$ equipped with the norm

$$\|u\|_{H^1_{0,n} \cap L^n_n} = \|u\|_{H^1_{0,n}} + \|u\|_{L^n}.$$  

Notice that

$$V \subset D_n \subset H^1_{0,n} \cap L^n_n,$$

and define the space $\tilde{V}$ as the closure of $V$ in $H^1_{0,n} \cap L^n_n$.

The following result is an immediate consequence of Proposition 10.6.1 and Corollary 10.6.1

**Proposition 10.6.2**

For each integer $n \geq 2$, one has $\tilde{V} \subset V$ and $\tilde{V} = V$ for $n = 2, 3, 4$.

We can now precise the domain where $b$ is defined.

**Lemma 10.6.1**

For any $n \geq 2$, the form $b$ is defined and continuous on $H^1_{0,n} \times H^1_{0,n} \times (H^1_{0,n} \cap L^n_n)$. 
Proof. Let us first suppose that \( n \geq 3 \) and let \((u,v,w) \in H^1_{0,n} \times H^1_{0,n} \times (H^1_{0,n} \cap L^n_0)\). Hence \( \partial_i v_j \in L^2 \), \( w_j \in L^n \), and, by Proposition 10.6.1 we have \( u_i \in L^{2n/(n-2)} \) \((i,j = 1,2,\ldots,n)\). As \( \frac{2n}{2n} + \frac{1}{2} + \frac{1}{n} = 1 \), we get \( u_i (\partial_i v_j) w_j \in L^1 \) and, using Hölder’s inequality

\[
\left| \int_\Omega u_i (x) (\partial_i v_j (x)) w_j (x) \, dx \right| \leq \|u_i\|_{L^{2n/(n-2)}} \|\partial_i v_j\|_{L^2} \|w_j\|_{L^n}. \tag{10.33}
\]

Using again Proposition 10.6.1 we get

\[
\|u_i\|_{L^{2n/(n-2)}} \leq c \|u_i\|_{H^1_0} \leq c \|u\|_{H^1_{0,n}},
\]

Now, trivially

\[
\|\partial_i v_j\|_{L^2} \leq \|v\|_{H^1_{0,n}}, \quad \|w_j\|_{L^n} \leq \|w\|_{L^n} \leq \|w\|_{H^1_{0,n} \cap L^n_0}
\]

which gives, by summing

\[
|b(u,v,w)| \leq \beta \|u\|_{H^1_{0,n}} \|v\|_{H^1_{0,n}} \|w\|_{H^1_{0,n} \cap L^n_0}, \tag{10.34}
\]

for some \( \beta > 0 \) depending only upon \( n \) and Friedrichs-Poincaré’s constant. When \( n = 2 \), we have \( H^1_{0,2} \subset L^q_2 \) for any \( 1 \leq q < \infty \) with continuous injection. Consequently, for \( u, v \in H^1_{0,2} \) and \( w \in H^1_{0,2} \cap L^2_0 = H^1_{0,2} \), we have

\[
u_i \in H^1_0 \subset L^4, \quad \partial_i v_j \in L^2, \quad w_j \in H^1_0 \subset L^4 \quad (i,j = 1,2),
\]

\( \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \), and, using Hölder’s inequality,

\[
\left| \int_\Omega u_i (x) (\partial_i w_j (x)) w_j (x) \, dx \right| \leq \|u_i\|_{L^4} \|\partial_i v_j\|_{L^2} \|w_j\|_{L^4}. \tag{10.35}
\]

To conclude, the procedure used for \( n \geq 3 \) applies if we replace inequalities (10.33) by inequalities (10.35). \( \blacksquare \)

Because \( V \) is a closed subspace of \( H^1_{0,n} \) and \( \tilde{V} \) a closed subspace of \( H^1_{0,n} \cap L^n_0 \), which coincides with \( V \) for \( n = 2,3,4 \), Lemma 1 has the following direct corollary.

**Corollary 10.6.2** The form \( b \) is trilinear and continuous on \( V \times V \times \tilde{V} \). If \( n = 2,3,4 \), it is trilinear and continuous on \( V \times V \times V \).

We need the following further properties of \( b \).

**Lemma 10.6.2** One has

(a) \( b(u,v,w) = 0 \) for all \( u \in V \) and \( v \in H^1_{0,n} \cap L^n_0 \);

(b) \( b(u,v,w) = -b(u,w,v) \) for all \( u \in H^1_{0,n}, v \), \( w \in H^1_{0,n} \cap L^n_0 \).
Proof. (a) The result is true for $u, v \in V$, because, integrating by parts, we have, for all $i, j = 1, \ldots, n$,
\[
\int_{\Omega} u_i(x)(\partial_j v_j(x))v_j(x) \, dx = \int_{\Omega} u_i(x)\partial_i(v_j^2(x)/2) \, dx = \frac{1}{2} \int_{\Omega} (\partial_i u_i(x))v_j^2(x) \, dx,
\]
and hence
\[
b(u,v,v) = -\frac{1}{2} \int_{\Omega} (\text{div } u(x))\|v(x)\|^2 \, dx = 0.
\]
If $u \in V$ and $v \in H_{0,n}^1 \cap L_{n,n}^n$, the result follows by density of $V$ in $V$ (with respect to the $H_{0,n}^1$-norm) and in $H_{0,n}^1 \cap L_{n,n}^n$, (with respect to the induced norm), and from the continuity of $b$.

(b) It is a consequence of (a), namely, for $u \in V$, $v, w \in H_{0,n}^1 \cap L_{n,n}^n$, one has
\[
0 = b(u,v + w, v + w) = b(u,v,v) + b(u,v,w) + b(u,w,v) + b(u,w,w) = b(u,v,w) + b(u,w,v).
\]

Now we are able to give an appropriate variational formulation of problem (10.24)-(10.26) : find $u \in V$ such that
\[
\nu(u,v)_{H_{0,n}^1} + b(u,u,v) = (f,v)_{L_n^2} \quad \text{for all } v \in \tilde{V}.
\]

Remember that $\tilde{V} \subset V$ and that $\tilde{V} = V$ for $n = 2, 3, 4$. The variational formulation of problem (10.24)-(10.26) was first introduced in 1933 by J. Leray [247] for dimensions 2 and 3 in a somewhat different way prefiguring Leray-Schauder’s theory, in which an a priori estimate for the possible solutions was obtained. His reasoning was clarified by E. Hopf [175] in 1950, who used Galerkin’s method. A variant of Leray’s existence theorem was obtained by O.A. Ladyzhenskaia [234] in 1959 for $n = 2$ and 3, and by Shinbrot [365] for $n = 4$. Finally, H. Fujita [140] gave a proof based upon Hopf’s Galerkin’s method, which, modified by R. Finn [125] was valid for all dimensions $n$. See [125] for more details on the history of the problem.

10.7 Existence of a solution

The existence theorem for (10.36) goes as follows. It is another example of application of the compactness method.

Theorem 10.7.1 For each $f \in L_{n,n}^n$, problem (10.36) has at least one solution.

Proof. Step 1. Galerkin approximation of the space $\tilde{V}$. The space $\tilde{V}$ being separable as subspace of $H_{0,n}^1$, there exists a sequence $(w_m)$ $(m = 1, 2, \ldots)$ of linearly
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independent elements which is total in $\tilde{V}$. Because $V$ is dense in $\tilde{V}$, this system can be chosen in $V$. Consequently, if $X_m = \text{span} [w_1, \ldots, w_m]$, $\bigcup_{m \geq 1} X_m$ is dense in $\tilde{V}$.

Of course, for any $m = 1, 2, \ldots$, one has

$$X_m \subset V \subset \tilde{V} \subset V \subset H_{0,n}^1.$$

**Step 2. The operators $P_m : X_m \to X_m$.** Let us consider in $X_m$ the inner product induced by that of $V$, namely $\langle u, v \rangle_{X_m} = \langle u, v \rangle_{H_{0,n}^1}$. Define the operator $P_m : X_m \to X_m$ as follows: for each $u \in X_m$, $P_m(u)$ is the unique element of $X_m$ which satisfies

$$\langle P_m(u), v \rangle_{X_m} = \langle P_m(u), v \rangle_{H_{0,n}^1} = \nu \langle u, v \rangle_{H_{0,n}^1} + b(u, u, v) - (f, v)_{L_2^m}$$

(10.37)

for all $v \in X_m$. The existence and uniqueness of $P_m(u)$ is justified by the fact that, for any fixed $u \in X_m$, the linear form on $X_m$

$$v \mapsto \nu \langle u, v \rangle_{H_{0,n}^1} + b(u, u, v) - (f, v)_{L_2^m}$$

is continuous, and Riesz’ theorem applies. The continuity of $P_m$ follows from the fact that if $u$ and $u'$ are two elements of $X_m$, then, for any $v \in X_m$, one has

$$\langle P_m(u) - P_m(u'), v \rangle_{X_m} = \nu \langle u - u', v \rangle_{H_{0,n}^1} + b(u, u, v) - b(u', u', v).$$

From the continuity of $b$ it follows that if $u \to u'$ in $X_m$, then $P_m(u) \to P_m(u')$ in $X_m$, and hence $P_m(u) \to P_m(u')$ as $X_m$ is finite-dimensional.

**Step 3. Existence of a solution for $P_m(u) = 0$**

It follows from Lemma 10.6.2 that

$$\langle P_m(u), u \rangle_{X_m} = \nu \|u\|_{H_{0,n}^1}^2 - (f, u)_{L_2^m} \geq \nu \|u\|_{H_{0,n}^1}^2 - c\|f\|_{L_2^m} \|u\|_{H_{0,n}^1}$$

where $c$ is the constant of the continuous embedding of $H_{0,n}^1$ into $L_2^m$. Consequently,

$$\langle P_m(u), u \rangle_{H_{0,n}^1} \geq 0 \text{ if } \|u\|_{X_m} = \|u\|_{H_{0,n}^1} \geq R \geq \frac{c}{\nu} \|f\|_{L_2^m}.$$ 

and it follows from Poincaré-Bohl’s existence theorem 8.4.2 that there exists $u_m \in X_m$ such that $\|u_m\|_{X_m} \leq R$ and $P_m(u_m) = 0$, or, equivalently

$$\langle P_m(u_m), v \rangle_{X_m} = \nu \langle u_m, v \rangle_{H_{0,n}^1} + b(u_m, u_m, v) - (f, v)_{L_2^m} = 0$$

(10.38)

for any $v \in X_m$.

**Step 4. Boundedness of $(u_m)$ in $V$.** We deduce in particular from (10.38) and Lemma 10.6.2 that

$$0 = \langle P_m(u_m), u_m \rangle_{X_m} = \nu \|u_m\|_{H_{0,n}^1}^2 - (f, u_m)_{L_2^m} = 0,$$

so that, reasoning like above,

$$\|u_m\|_{H_{0,n}^1} \leq \frac{c}{\nu} \|f\|_{L_2^m}$$

(10.39)
for all \( m = 1, 2, \ldots \). Then, going if necessary to a subsequence, we can assume, using the reflexivity of \( H_{0,n}^1 \) and the compact injection of \( H_{0,n}^1 \) into \( L_n^2 \) (Proposition 10.6.1) that there exists some \( u \in V \) such that \( u_m \rightharpoonup u \) in \( V \) and \( u_n \to u \) in \( L_n^2 \). To show that \( b(u_m, u_m, v) \to b(u, u, v) \) for any \( v \in V \), using the fact that, by Lemma 10.6.2

\[
b(u_m, u_m, v) = -b(u_m, v, u_m) = -\sum_{i,j=1}^{n} \int_{\Omega} u_{m,i}(x) u_{m,j}(x) \partial_i v_j(x) \, dx,
\]

it is sufficient to show that

\[
\sum_{i,j=1}^{n} \int_{\Omega} u_{m,i}(x) u_{m,j}(x) \partial_i v_j(x) \, dx \to \sum_{i,j=1}^{n} \int_{\Omega} u_{i}(x) u_{j}(x) \partial_i v_j(x) \, dx,
\]

which is easy because \( \partial_i v_j \in L^\infty \) for all \( 1 \leq i, j \leq n \).

Step 5. \( u \) is a solution of (10.36). Let \( v \in \tilde{V} \). Since \( \bigcup_{m \geq 1} X_m \) is dense in \( \tilde{V} \), there is a sequence \( (v_k) \) in \( \bigcup_{m \geq 1} X_m \) such that \( v_k \to v \) in \( \tilde{V} \). Because of

\[
\|v_k - v\|_{\tilde{V}} = \|v_k - v\|_{H_{0,n}^1 \cap L_n^2} = \|v_k - v\|_{H_{0,n}^1} + \|v_k - v\|_{L_n^2},
\]

the strong convergence of \( v_k \) to \( v \) in \( \tilde{V} \) implies that \( v_k \to v \) strongly in \( V \) and in \( L_n^2 \). Furthermore, we may assume that for any \( k \in \mathbb{N} \) there exists \( m_k \geq k \) such that \( v_k \in X_{m_k} \). Consequently, it follows from (10.38) that

\[
\nu\langle u_{m_k}, v_k \rangle_{H_{0,n}^1} + b(u_{m_k}, u_{m_k}, v_k) = \langle f, v_k \rangle_{L_n^2},
\]

which, letting \( k \to \infty \), gives (10.36).

10.8 Uniqueness conditions

We now state and prove a result of R. Finn [126] showing that the solution of (10.36) is unique if \( \nu \) is sufficiently large or \( f \) sufficiently small.

**Theorem 10.8.1** If \( 2 \leq n \leq 4 \) and if

\[
\nu^2 > c\beta \|f\|_{L_n^2}
\]

where \( c \) is the constant in Friedrichs-Poincaré’s inequality (10.32) and \( \beta \) is defined in (10.34), there is a unique solution for (10.36).

**Proof.** First we give an estimate, valid for any \( n \geq 2 \), for any solution \( u \) whose existence was proved in Theorem 10.7.1. Indeed, it follows from (10.39) that

\[
\|u\|_{V} \leq \limsup_{m \to \infty} \|u_m\|_{V} \leq \frac{c}{\nu} \|f\|_{L_n^2},
\]

(10.41)
10.8. **UNIQUENESS CONDITIONS**

Now let \( n \leq 4 \), so that \( V = \tilde{V} \), and assume that \( u \in V \) and \( u' \in V \) are two solutions of

\[
\nu(u,v)_{H^1_{0,n}} + b(u,u,v) = (f,v)_{L^2_n}
\]

(10.42)

for any \( v \in \tilde{V} = V \). Then, by substraction of the two equations, we get

\[
\nu(u-u',v)_{H^1_{0,n}} + b(u,u,v) - b(u',u',v) = 0
\]

for any \( v \in V \). Now, a simple computation gives

\[
b(u,u,v) - b(u',u',v) = b(u,u-u',v) + b(u-u',u',v),
\]

so that, by taking \( v = u - u' \) we obtain, using Lemma 10.6.2,

\[
\nu\|u - u'\|_{H^1_{0,n}} = -b(u - u',u',u - u') = b(u - u',u - u',u'). \tag{10.43}
\]

Now, because \( n \leq 4 \), the function \( b \) is continuous on \( V \times V \times V \), so that, using (10.41), we obtain

\[
|b(u-u',u-u',u')| \leq \beta \|u-u'\|_{H^1_{0,n}}^2 \|u'\|_{H^1_{0,n}} \leq \beta \frac{c}{\nu} \|f\|_{L^2_n} \|u-u'\|_{H^1_{0,n}}^2. \tag{10.44}
\]

Combining (10.43) and (10.44) we derive

\[
\left( \nu - \frac{\beta c}{\nu} \|f\|_{L^2_n} \right) \|u - u'\|_{H^1_{0,n}}^2 \leq 0,
\]

and conclude that \( u = u' \) if (10.40) holds.

\[\blacksquare\]
Chapter 11

First order difference equations

11.1 Periodic solutions

Let \( n \geq 2 \) be a fixed integer. For \((x_1,\cdots,x_n)\in \mathbb{R}^n\), define the first order difference operator \((\Delta x_1,\cdots,\Delta x_{n-1})\in \mathbb{R}^{n-1}\) by

\[
\Delta x_m := x_{m+1} - x_m, \quad (1 \leq m \leq n-1).
\]

Let \( f_m : \mathbb{R}^n \to \mathbb{R} \quad (1 \leq m \leq n-1) \) be continuous functions. We study the existence of solutions for the periodic boundary value problem

\[
\Delta x_m + f_m(x_1,\cdots,x_n) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n. \tag{11.1}
\]

Let

\[
U^{n-1} = \{x \in \mathbb{R}^n : x_1 = x_n \} \cong \mathbb{R}^{n-1}, \tag{11.2}
\]

as an element of \( U^{n-1} \) can be characterized by the coordinates \( x_1,\cdots,x_{n-1} \). The restriction \( L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) of \( D \) to \( \mathbb{R}^{n-1} \) is given by

\[
(Lx)_m = x_{m+1} - x_m \quad (1 \leq m \leq n-2), \quad (Lx)_{n-1} = x_1 - x_{n-1}, \tag{11.3}
\]

or, in matrix form, by the circulant matrix

\[
\begin{pmatrix}
-1 & 1 & 0 & \cdots & \cdots & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & -1 & 1 \\
1 & 0 & \cdots & \cdots & 0 & -1
\end{pmatrix}. \tag{11.4}
\]
If we define \( F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by
\[
F_m(x_1, \ldots, x_{n-1}) = f_m(x_1, x_2, \ldots, x_{n-1}, x_1) \quad (1 \leq m \leq n-1),
\]
problem (11.1) is equivalent to study the zeros of the continuous mapping \( H : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) defined by
\[
H_m(x) = (Lx)_m + F_m(x) \quad (1 \leq m \leq n-1).
\]

11.2 Bounded nonlinearities

Let us first consider the linear periodic problem
\[
\Delta x_m + \alpha x_m = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n,
\]
where \( \alpha \in \mathbb{R} \). The solutions of the system are given by
\[
x_m = (1 - \alpha)^{m-1} x_1, \quad (1 \leq m \leq n-1),
\]
and hence (11.6) has a solution if and only if
\[
x_1 = (1 - \alpha)^{n-1} x_1.
\]
This immediately implies the following

**Lemma 11.2.1** Problem (11.6) has only the trivial solution if \( n \) is even and \( \alpha \neq 0 \) or if \( n \) is odd and \( \alpha \notin \{0, 2\} \). When \( \alpha = 0 \), the solutions are of the form \( x_m = c \quad (1 \leq m \leq n-1) \), and when \( n \) is odd and \( \alpha = 2 \), they have the form \( x_m = (-1)^{m-1} c \quad (1 \leq m \leq n-1) \), with \( c \in \mathbb{R} \) arbitrary.

Let
\[
b_m : \mathbb{R}^n \to \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto b_m(x_1, \ldots, x_n) \quad (1 \leq m \leq n-1)
\]
be continuous and bounded, and consider the semilinear periodic problem
\[
\Delta x_m + \alpha x_m + b_m(x_1, \ldots, x_n) = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n.
\]
If \( L \) is defined like above and \( B : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by
\[
B_m(x_1, \ldots, x_{n-1}) = b_m(x_1, x_2, \ldots, x_{n-1}, x_1) \quad (1 \leq m \leq n-1),
\]
then problem (11.8) is equivalent to the semilinear problem in \( \mathbb{R}^{n-1} \)
\[
Lx + \alpha x + B(x) = 0.
\]
An immediate consequence of Lemma 11.2.1 and Corollary 8.1.2 is the following existence result.

**Theorem 11.2.1** Assume \( n \) odd and \( \alpha \neq 0 \) or \( n \) even and \( \alpha \notin \{0, 2\} \) and assume that the functions \( b_m \) in (11.7) are continuous and bounded. Then problem (11.8) has at least one solution and, for all sufficiently large \( R \),
\[
d_B[L + \alpha I + B, B(R), 0] = \pm 1.
\]
11.3 Upper and lower solutions

Let \( f_m : \mathbb{R} \to \mathbb{R} \) (1 ≤ \( m \) ≤ \( n - 1 \)) be continuous functions, and let us consider the periodic problem

\[
\Delta x_m + f_m(x_m) = 0 \quad (1 \leq m \leq n - 1), \quad x_1 = x_n. \tag{11.10}
\]

**Definition 11.3.1** \( \alpha = (\alpha_1, \ldots, \alpha_n) \) (resp. \( \beta = (\beta_1, \ldots, \beta_n) \)) is called a lower solution (resp. upper solution) for (11.10) if

\[
\alpha_1 \geq \alpha_n \quad (\text{resp. } \beta_1 \leq \beta_n),
\]

and the inequalities

\[
\Delta \alpha_m + f_m(\alpha_m) \geq 0 \quad (\text{resp. } \Delta \beta_m + f_m(\beta_m) \leq 0) \tag{11.11}
\]

hold for all 1 ≤ \( m \) ≤ \( n - 1 \). Such a lower or upper solution will be called strict if the inequality (11.11) is strict for all 1 ≤ \( m \) ≤ \( n - 1 \).

The basic theorem for the method of upper and lower solutions goes as follows. The proof given here is taken from [19] and modeled on the corresponding one for differential equations in [267].

**Theorem 11.3.1** If (11.10) has a lower solution \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and an upper solution \( \beta = (\beta_1, \ldots, \beta_n) \) such that \( \alpha_m \leq x_m \leq \beta_m \) (1 ≤ \( m \) ≤ \( n \)), then (11.10) has a solution \( x = (x_1, \ldots, x_n) \) such that \( \alpha_m \leq x_m \leq \beta_m \) (1 ≤ \( m \) ≤ \( n \)). Moreover, if \( \alpha \) and \( \beta \) are strict, then \( \alpha_m < x_m < \beta_m \) (1 ≤ \( m \) ≤ \( n - 1 \)).

**Proof.** 1. A modified problem.

Let \( \gamma_m : \mathbb{R} \to \mathbb{R} \) (1 ≤ \( m \) ≤ \( n - 1 \)) be the continuous functions defined by

\[
\gamma_m(x) = \begin{cases} 
\beta_m & \text{if } x > \beta_m \\
x & \text{if } \alpha_m \leq x \leq \beta_m \\
\alpha_m & \text{if } x < \alpha_m.
\end{cases} \tag{11.12}
\]

We consider the modified problem

\[
\Delta x_m - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) = 0 \quad (1 \leq m \leq n - 1), \quad x_1 = x_n. \tag{11.13}
\]

and show that if \( x = (x_1, \ldots, x_n) \) is a solution of (11.13) then \( \alpha_m \leq x_m \leq \beta_m \) (1 ≤ \( m \) ≤ \( n \)), and hence \( x \) is a solution of (11.10). Suppose by contradiction that there is some 1 ≤ \( i \) ≤ \( n \) such that \( \alpha_i - x_i > 0 \) so that \( \alpha_m - x_m = \max_{1 \leq j \leq n}(\alpha_j - x_j) > 0 \).

If 1 ≤ \( m \) ≤ \( n - 1 \), then

\[
\alpha_{m+1} - x_{m+1} \leq \alpha_m - x_m,
\]

which gives

\[
\Delta \alpha_m \leq \Delta x_m = x_m - \alpha_m - f_m(\alpha_m) \leq x_m - \alpha_m + \Delta \alpha_m < \Delta \alpha_m,
\]

and we are done.
a contradiction. Now the condition $\alpha_1 \geq \alpha_n$ shows that the maximum is reached at $m = n$ only if it is reached also at $m = 1$, a case already excluded. Analogously we can show that $x_m \leq \beta_m$ ($1 \leq m \leq n$). We remark that if $\alpha, \beta$ are strict, the same reasoning gives $\alpha_m < x_m < \beta_m$ ($1 \leq m \leq n - 1$).

II. Solution of the modified problem.
We use Brouwer degree to study the zeros of the continuous map $G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$G_m(x) = (Lx)_m - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) \quad (1 \leq m \leq n-1).$$

(11.14)

As seen above, $L - I : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is invertible. On the other hand the mapping with components $f_m \circ \gamma_m + \gamma_m$ $(1 \leq m \leq n - 1)$ is bounded on $\mathbb{R}^{n-1}$. Consequently, Theorem 11.2.1 implies the existence of $R > 0$ such that, for all $\rho > R$, one has

$$|d_B[G, B(\rho), 0]| = 1,$$

(11.15)

and, in particular, $G$ has a zero $x \in B(\rho)$. Hence, $x$ is a solution of (11.13), which means that $\alpha_m \leq x_m \leq \beta_m$ $(1 \leq m \leq n)$ and $x$ is a solution of (11.10). Moreover if $\alpha, \beta$ are strict, then $\alpha_m < x_m < \beta_m$ $(1 \leq m \leq n - 1)$.

Remark 11.3.1 Suppose that $\alpha$ (resp. $\beta$) is a strict lower (resp. upper) solutions of (11.10). Define the open set

$$\Omega_{\alpha\beta} = \{(x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} : \alpha_m < x_m < \beta_m \quad (1 \leq m \leq n - 1)\}.$$

If $\rho$ is large enough, then, using the additivity-excision property of Brouwer degree, we have


On the other hand, if $H$ is the continuous mapping defined in (11.5), $H$ is equal to $G$ on $\Omega_{\alpha\beta}$, and then

$$|d_B[H, \Omega_{\alpha\beta}, 0]| = 1.$$

(11.16)

A simple but useful consequence of Theorem 11.3.1, goes as follows.

Corollary 11.3.1 Assume that there exists numbers $\alpha \leq \beta$ such that

$$f_m(\alpha) \geq 0 \geq f_m(\beta) \quad (1 \leq m \leq n).$$

Then problem (11.10) has at least one solution with $\alpha \leq x_m \leq \beta$ $(1 \leq m \leq n - 1)$.

Proof. Just observe that $(\alpha, \cdots, \alpha)$ is a lower solution and $(\beta, \cdots, \beta)$ an upper solution for (11.10).
Example 11.3.1 For each $p > 0$, $a_m > 0$ and $b_m \in \mathbb{R}$ ($1 \leq m \leq n - 1$) the problem
\[ \Delta x_m - a_m |x_m|^p x_m + b_m = 0 \quad (1 \leq m \leq n - 1), \quad x_1 = x_n \]
has at least one solution.

Remark 11.3.2 When $\beta_m \leq \alpha_m$ ($1 \leq m \leq n - 1$), one can try to repeat the argument of Theorem 11.3.1 by defining
\[ \delta_m(x) = \begin{cases} 
\alpha_m & \text{if } x > \alpha_m \\
\beta_m & \text{if } x < \beta_m,
\end{cases} \]
and considering the modified problem
\[ (Lx)_m + x_m + f_m \circ \delta_m(x_m) - \delta_m(x_m) = 0 \quad (1 \leq m \leq n - 1), \quad x_1 = x_n. \]
As $L + I$ is invertible, the degree argument still gives the existence of at least one solution for (11.17). If one tries to show that, say, $\beta_m \leq x_m$ ($1 \leq m \leq n - 1$), and assume by contradiction that $\beta_i - x_i > 0$ for some $1 \leq i \leq n$, so that $\beta_m - x_m = \max_{1 \leq j \leq n}(\beta_j - x_j) > 0$, one has for $1 \leq m \leq n - 1$
\[ \beta_{m+1} - x_{m+1} \leq \beta_m - x_m, \]
which gives
\[ \Delta \beta_m + f_m(\beta_m) \leq \Delta x_m + f_m \circ \gamma_m(x_m) = -x_m + \beta_m > 0, \]
implicating no contradiction with $\Delta \beta_m + f_m(\beta_m) \leq 0$. This is in contrast with the ordinary differential equation case, for which the method of upper and lower solutions works independently of their order. The reason of this difference comes from the fact that a local extremum is characterized by an equality (vanishing of the first derivative) in the differential case and by two inequalities (with only one usable in the argument) in the difference case. We show now that, for difference equations, the existence theorem based upon lower and upper solutions does not work when the upper solution is smaller than the lower solution.

11.4 Spectrum of the linear part

The construction of the counter-example proving the last assertion above is made clear by analyzing the spectral properties of the first order difference operator with periodic boundary conditions.

Definition 11.4.1 An eigenvalue of the first order difference operator with periodic boundary conditions is any $\lambda \in \mathbb{C}$ such that the problem
\[ \Delta x_m = \lambda x_m \quad (1 \leq m \leq n - 1), \quad x_1 = x_n \]
has a nontrivial solution.
Explicitly, system (11.18) can be written as

\[
\begin{align*}
   x_1 - x_n &= 0 \\
   x_2 - (1 + \lambda)x_1 &= 0 \\
   & \vdots \\
   x_n - (1 + \lambda)x_{n-1} &= 0 \\
\end{align*}
\]  

(11.19)

or

\[
\begin{pmatrix}
   -1 - \lambda & 1 & 0 & \cdots & 0 \\
   0 & -1 - \lambda & 1 & 0 & \cdots & 0 \\
   \vdots & \ddots & \ddots & \ddots & \ddots \\
   0 & \cdots & 0 & -1 - \lambda & 1 \\
   1 & 0 & \cdots & 0 & -1 - \lambda \\
\end{pmatrix}
\begin{pmatrix}
   x_1 \\
   x_2 \\
   \vdots \\
   x_{n-2} \\
   x_{n-1} \\
\end{pmatrix}
= 0.
\]

Looking for solutions of the form \( x_m = e^{im\theta} (1 \leq m \leq n-1) \) for some \( \theta \in \mathbb{R} \), one easily finds that \( \theta \) must verify the equations

\[
\lambda = -1 + e^{i\theta}, \quad e^{i(n-1)\theta} = 1,
\]

which gives the \( n - 1 \) distinct eigenvalues

\[
\lambda_k = -1 + e^{\frac{2\pi i k}{n-1}} \quad (0 \leq k \leq n-2),
\]  

(11.20)

with the corresponding eigenvectors \( \varphi^k \) \( (0 \leq k \leq n-2) \) having components \( \varphi^k_m = e^{\frac{2\pi i m k}{n-1}} \) \( (1 \leq m \leq n) \). In particular, \( \lambda_0 = 0 \) is always a real eigenvalue, and all the other eigenvalues have negative real part. If \( n = 2, 0 \) is the unique eigenvalue; if \( n > 2 \) is even, 0 is the unique real eigenvalue; if \( n \) is odd, \( \lambda_{n-1} = -2 \) is the unique nonzero real eigenvalue.

**Remark 11.4.1** The \( \mu_k = \lambda_k + 1 \) are the eigenvalues of the (permutation, unitary, circulant) matrix

\[
\begin{pmatrix}
   0 & 1 & 0 & \cdots & 0 \\
   0 & 0 & 1 & 0 & \cdots & 0 \\
   \vdots & \ddots & \ddots & \ddots & \ddots \\
   0 & \cdots & 0 & 0 & 1 \\
   1 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}
\]

11.5 Reversing the order of upper and lower solutions

For \( n \geq 2 \) odd and \( \lambda = -2 \), system (11.19) becomes

\[
x_1 - x_n = 0
\]
11.5. REVERSING THE ORDER OF UPPER AND LOWER SOLUTIONS

\[ x_2 + x_1 = 0 \]
\[ \ldots \ldots \ldots \]
\[ x_n + x_{n-1} = 0 \]  
(11.21)

and has the eigenvector with components \( \varphi_m^{(n-1)/2} = (-1)^{m-1} \) (1 \( \leq m \leq n \)). The adjoint system

\[ x_1 + x_2 = 0 \]
\[ \ldots \ldots \ldots \]
\[ x_{n-1} + x_n = 0 \]
\[ -x_1 + x_n = 0 \]
(11.22)

has the nontrivial solution with components \( \psi_m = (-1)^{m-1} \) (1 \( \leq m \leq n \)). As \( b_m = \delta_{nm} \) (1 \( \leq m \leq n \)) (Kronecker symbol) is not orthogonal to the kernel of the adjoint system (11.22), the problem

\[ x_1 - x_n = 0 \]
\[ x_2 + x_1 = 0 \]
\[ \ldots \ldots \ldots \]
\[ x_{n-1} + x_{n-2} = 0 \]
\[ x_n + x_{n-1} = 1 \]

has no solution, or, equivalently the problem

\[ \Delta x_m + 2x_m = 0 \quad (1 \leq m \leq n - 2), \quad \Delta x_{n-1} + 2x_{n-1} = 1, \quad x_1 = x_n \]  
(11.23)

has no solution. However, \( \alpha = (1,1,\cdots,1) \) is a lower solution and \( \beta = (0,0,\cdots,0) \) is an upper solution of (11.23) such that \( \beta_m \leq \alpha_m \) (1 \( \leq m \leq n \)).

If now \( n > 2 \) is even, the problem

\[ \Delta x_m + 2x_m = 0 \quad (1 \leq m \leq n - 3), \quad \Delta x_{n-2} + 2x_{n-2} = 1, \quad \Delta x_{n-1} = 0, \quad x_1 = x_n \]
(11.24)

is of course equivalent to the problem

\[ \Delta x_m + 2x_m = 0 \quad (1 \leq m \leq n - 3), \quad \Delta x_{n-2} + 2x_{n-2} = 1, \quad x_1 = x_{n-1}. \]

As \( n-1 \) is odd, it follows from the counter-example (11.23) that problem (11.24) has no solution. However \( \alpha = (1,1,\cdots,1) \) is a lower solution and \( \beta = (0,0,\cdots,0) \) is an upper solution of (11.24) such that \( \beta_m \leq \alpha_m \) (1 \( \leq m \leq n \)). Those counter-examples were first given in [20].

For \( n = 2 \), problem (11.10) is equivalent to the unique scalar equation

\[ f_1(x_1) = 0 \]
and, in this case, the validity of the method of upper and lower solutions, independently of their order, follows from its equivalence with Bolzano’s theorem applied to the real function $f_1$.

**Remark 11.5.1** Notice that, in contrast to the periodic problem for difference equations, whose eigenvalues are in the left half-plane, all the eigenvalues $\lambda_k = \frac{2k\pi}{T}$ ($k \in \mathbb{Z}$) of the differential operator $\frac{d}{dt}$ with periodic boundary conditions on $[0, T]$ are on the imaginary axis. This explains the indifference of the result with respect to the order of the lower and the upper solution.

### 11.6 Ambrosetti-Prodi type multiplicity result

Let $f_1, \ldots, f_{n-1} : \mathbb{R} \to \mathbb{R}$ be continuous functions, $s \in \mathbb{R}$. Consider the problem, with $n \geq 2$,  
\begin{equation}
\Delta x_m + f_m(x_m) = s \quad (1 \leq m \leq n - 1), \quad x_1 = x_n, \tag{11.25}
\end{equation}

with the coercivity condition
\begin{equation}
f_m(x) \to \infty \quad \text{as} \quad |x| \to \infty \quad (1 \leq m \leq n - 1). \tag{11.26}
\end{equation}

When $n = 2$, problem (11.25) is equivalent to the scalar equation
\begin{equation}
f_1(x_1) = s \tag{11.27}
\end{equation}

and, under condition (11.26) with $m = 1$, it is clear that there exists $s_1 (= \min f_1)$ such that for $s < s_1$, equation (11.27) has no solution, for $s = s_1$, equation (11.27) has at least one solution, and for $s > s_1$, equation (11.27) has at least two solutions.

We show that a similar result holds for any $n \geq 2$. Problems of this type were initiated by Ambrosetti- Prodi for second order semilinear Dirichlet problems and the results given here, which can be found in [19], are modeled on the approach given in [268] for periodic solutions of first order ordinary differential equations.

**Lemma 11.6.1.** Let $b \in \mathbb{R}$. If condition (11.26) holds, there is $\rho = \rho(b) > 0$ such that each possible solution $x$ of (11.25) with $s \leq b$ is such that $\|x\| < \rho$.

**Proof.** Let $s \leq b$ and $(x_1, \ldots, x_n)$ be a solution of (11.25). We see that
\begin{equation}
\sum_{m=1}^{n-1} f_m(x_m) = (n - 1)s \leq (n - 1)b. \tag{11.28}
\end{equation}

From (11.26),
\begin{equation}
(\exists c \in \mathbb{R})(\forall u \in \mathbb{R})(\forall m \in \{1, \ldots, n - 1\} : f_m(u) \geq c, \tag{11.29}
\end{equation}

and
\begin{equation}
(\forall R > 0)(\exists r' > 0)(\forall u \in \mathbb{R} : |u| \geq r')(\forall m \in \{1, \ldots, n - 1\} : f_m(u) \geq R - (n - 2)c. \tag{11.30}
\end{equation}
If \( r = \sqrt{n-1}r' \) and if \( x \in \mathbb{R}^{m-1} \) is such that \( \|x\| \geq r \), then, for at least one \( j \in \{1, \ldots, n-1\} \), one has \( |x_j| \geq r' \), so that, using (11.29) and (11.30),
\[
\sum_{m=1}^{n-1} f_m(x_m) = \sum_{m=1}^{n-1} [f_m(x_m) - c] + (n-1)c \geq f_j(x_j) - c + (n-1)c \geq R - (n-2)c - c + (n-1)c = R.
\]

Consequently, \( \sum_{m=1}^{n-1} f_m(x_m) \to +\infty \) if \( \|x\| \to \infty \), and the result follows from (11.28).

**Theorem 11.6.1** If the functions \( f_m \) \((1 \leq m \leq n-1)\) satisfy (11.26), there exists \( s_1 \in \mathbb{R} \) such that (98) has zero, at least one or at least two solutions according to \( s < s_1, s = s_1, s > s_1 \).

**Proof.** Let
\[
S_j = \{ s \in \mathbb{R} : (11.25) \text{ has at least } j \text{ solutions } \} \quad (j \geq 1).
\]

(a) \( S_1 \neq \emptyset \).

Take \( s^* = \max_{1 \leq m \leq n-1} f_m(0) \) and use (11.26) to find \( R^*_j < 0 \) such that
\[
\min_{1 \leq m \leq n-1} f_m(R^*_j) > s^*.
\]

Then \( \alpha \) with \( \alpha_j = R^*_j < 0 \) \((1 \leq j \leq n)\) is a strict lower solution and \( \beta \) with \( \beta_j = 0 \) \((1 \leq j \leq n)\) is a strict upper solution for (11.25) with \( s = s^* \). Hence, using Theorem 11.3.1, \( s^* \in S_1 \).

(b) If \( \tilde{s} \in S_1 \) and \( s > \tilde{s} \) then \( s \in S_1 \).

Let \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \) be a solution of (11.25) with \( s = \tilde{s} \), and let \( s > \tilde{s} \). Then \( \tilde{x} \) is a strict upper solution for (11.25). Take now \( R^- = \min_{1 \leq m \leq n} x_m \) such that \( \min_{1 \leq m \leq n-1} f_m(R^-) > s \). It follows that \( \alpha \) with \( \alpha_j = R^- \) \((1 \leq j \leq n)\) is a strict lower solution for (11.25), and hence, using Theorem 11.3.1, \( s \in S_1 \).

(c) \( s_1 = \inf S_1 \) is finite and \( S_1 \supset ]s_1, \infty[ \).

Let \( s \in \mathbb{R} \) and suppose that (11.25) has a solution \((x_1, \ldots, x_n)\). Then (11.28) holds, from where we deduce that \( s \geq c \), with \( c \in \mathbb{R} \) such that \( f_m(x) \geq c \) for all \( x \in \mathbb{R} \), and \( 1 \leq m \leq n - 1 \). To obtain the second part of claim (c) \( S_1 \supset ]s_1, \infty[ \) we apply (b).

(d) \( S_2 \supset ]s_1, \infty[ \).

We reformulate (11.25) to apply Brouwer degree theory. Consider the continuous mapping \( G : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) defined by
\[
G_m(s, x) = (Lx)_m + f_m(x_m) - s \quad (1 \leq m \leq n - 1).
\]

Then \((x_1, \ldots, x_n)\) is a solution of (11.25) if and only if \((x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) is a zero of \( G(s, \cdot) \). Using Lemma 11.6.1 we find \( \rho > 0 \) such that each possible zero of \( G(s, \cdot) \) with \( s \in [s_1, s_2] \) is such that \( \max_{1 \leq m \leq n-1} |x_m| < \rho \).
Consequently, \( d_B[\mathcal{G}(s, \cdot), B(\rho), 0] \) is well defined and does not depend upon \( s \in [s_1, s_2] \). However, using (c), we see that \( \mathcal{G}(s_3, x) \neq 0 \) for all \( x \in \mathbb{R}^{n-1} \). This implies that \( d_B[\mathcal{G}(s_1, \cdot), B(\rho), 0] = 0 \), so that \( d_B[\mathcal{G}(s_2, \cdot), B(\rho), 0] = 0 \) and, by excision property, \( d_B[\mathcal{G}(s_2, \cdot), B(\rho'), 0] = 0 \) if \( \rho' > \rho \). Let \( s \in [s_1, s_2] \) and \( \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n) \) be a solution of (11.25) (using (c)). Then \( \tilde{x} \) is a strict upper solution of (11.25) with \( s = s_2 \). Let \( R < \min_{1 \leq j \leq n} \tilde{x}_j \) be such that \( \min_{1 \leq m \leq n-1} f_m(R) > s_2 \). Then \((R, \cdots, R) \in \mathbb{R}^n \) is a strict lower solution of (11.25) with \( s = s_2 \). Consequently, using Remark 11.3.1, (11.25) with \( s = s_2 \) has a solution in \( \Omega_{R\varepsilon} \) and

\[
|d_B[\mathcal{G}(s_2, \cdot), \Omega_{R\varepsilon}, 0]| = 1.
\]

Taking \( \rho' \) sufficiently large, we deduce from the additivity property of Brouwer degree that

\[
|d_B[\mathcal{G}(s_2, \cdot), B(\rho') \setminus \Omega_{R\varepsilon}, 0]| = |d_B[\mathcal{G}(s_2, \cdot), B(\rho'), 0] - d_B[\mathcal{G}(s_2, \cdot), \Omega_{R\varepsilon}, 0]| = |d_B[\mathcal{G}(s_2, \cdot), \Omega_{R\varepsilon}, 0]| = 1,
\]

and (11.25) with \( s = s_2 \) has a second solution in \( B(\rho') \setminus \Omega_{R\varepsilon} \).

(e) \( s_1 \in S_1 \).

Taking a decreasing sequence \((\sigma_k)_{k \in \mathbb{N}}\) in \([s_1, \infty)\) converging to \( s_1 \), a corresponding sequence \((x_1^k, \cdots, x_n^k)\) of solutions of (11.25) with \( s = \sigma_k \) and using Lemma 11.6.1, we obtain a subsequence \((x_1^k, \cdots, x_n^k)\) which converges to a solution \((x_1, \cdots, x_n)\) of (11.25) with \( s = s_1 \).

Example 11.6.1 There exists \( s_1 \in \mathbb{R} \) such that the periodic problem for Verhulst equation with harvesting

\[
\Delta x_m + x_m(x_m - 1) = s \quad (1 \leq m \leq n - 1), \quad x_1 = x_n
\]

has no solution if \( s < s_1 \), at least one solution if \( s = s_1 \) and at least two solutions if \( s > s_1 \).

Similar arguments allow to prove the following result.

Theorem 11.6.2 If the functions \( f_m \) satisfy condition

\[
f_m(x) \to -\infty \quad \text{as} \quad |x| \to \infty \quad (1 \leq m \leq n - 1).
\]

\[
(11.31)
\]

then there is \( s_1 \in \mathbb{R} \) such that (11.25) has zero, at least one or at least two solutions according to \( s > s_1, s = s_1 \) or \( s < s_1 \).

11.7 One-sided bounded nonlinearities

The nonlinearity in Ambrosetti-Prodi type problems is bounded from below and coercive or bounded from above and anticoercive. In this section, we consider
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nonlinearities which are bounded from below or above but have different limits at $+\infty$ and $-\infty$.

Let $n \geq 2$ be an integer and $f_m : \mathbb{R} \to \mathbb{R}$ continuous functions ($1 \leq m \leq n - 1$). Consider the problem

$$\Delta x_m + f_m(x_m) = 0 \quad (1 \leq m \leq n - 1), \quad x_1 = x_n.$$  (11.32)

Let $H : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the continuous mapping defined in (11.5), whose zeros correspond to the solutions of (11.32). It is easy to check that the linear mapping $L$ defined in (11.3) is such that

$$N(L) = \{(c, \cdots, c) \in \mathbb{R}^{n-1} : c \in \mathbb{R}\},$$

$$R(L) = \{(y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1} : \sum_{m=1}^{n-1} y_m = 0\}.$$

The projectors $P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $Q : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$P(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_1),$$

$$Q(y_1, \cdots, y_{n-1}) = \left(\frac{1}{n-1} \sum_{m=1}^{n-1} y_m, \cdots, \frac{1}{n-1} \sum_{m=1}^{n-1} y_m\right)$$

are such that $N(Q) = R(L), R(P) = N(L)$. Let us finally define $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(x_1, \cdots, x_{n-1}) = (f_1(x_1), \cdots, f_{n-1}(x_{n-1})).$$

so that $H = L + F$. To study the zeros of $G$ using Brouwer degree, we use Theorem 8.6.1. The following Lemma, taken from [19], adapts to difference equations an argument of Ward [398] for ordinary differential equations.

**Lemma 11.7.1** If the functions $f_m$ ($1 \leq m \leq n - 1$), are all bounded from below or all bounded from above, and if for some $R > 0$

$$\sum_{m=1}^{n-1} f_m(x_m) \neq 0 \quad \text{whenever} \quad \min_{1 \leq j \leq n-1} x_j \geq R \quad \text{or} \quad \max_{1 \leq j \leq n-1} x_j \leq -R, \quad (11.33)$$

then there exists $\rho \geq R$ such that, for each $\lambda \in [0, 1]$ and each possible zero $x$ of $L + \lambda F$, one has $\max_{1 \leq j \leq n-1} |x_j| < \rho$.

**Proof.** Let $(\lambda, x) \in [0, 1] \times \mathbb{R}^{n-1}$ be a possible zero of $L + \lambda N$. It is a solution of the equivalent system

$$\sum_{m=1}^{n-1} f_m(x_m) = 0, \quad \Delta x_m + \lambda f_m(x_m) = 0, \quad x_1 = x_n, \quad (1 \leq m \leq n - 1). \quad (11.34)$$
On the other hand, if we assume that each \( f_m \) (\( 1 \leq m \leq n - 1 \)) is bounded from below, say by \( c \), we have, for all \( 1 \leq m \leq n - 1 \),

\[
|f_m(u)| \leq f_m(u) + 2|c| \quad (u \in \mathbb{R}).
\]  

(11.35)

Hence, using (11.34) and (11.35), we obtain

\[
\sum_{m=1}^{n-1} |\Delta x_m| = \lambda \sum_{m=1}^{n-1} |f_m(x_m)| \leq \sum_{m=1}^{n-1} |f_m(x_m)| \leq \sum_{m=1}^{n-1} f_m(x_m) + 2(n-1)|c|.
\]  

(11.36)

We deduce

\[
\max_{1 \leq m \leq n-1} x_m \leq \min_{1 \leq m \leq n-1} x_m + \sum_{m=1}^{n-1} |\Delta x_m| \leq \min_{1 \leq m \leq n-1} x_m + 2(n-1)|c|.
\]  

(11.37)

Using (11.34) and assumption (11.33), we obtain \( \min_{1 \leq m \leq n-1} x_m < R \) and \( -R < \max_{1 \leq m \leq n-1} x_m \). Combined with (11.38), this gives

\[-R + 2(n-1)|c| < \min_{1 \leq m \leq n-1} x_m.\]

It follows that we can take any \( \rho \geq R + 2(n-1)|c| \). If the \( f_m \) are bounded from above, it suffices to consider the equivalent problem \( -Lx - F(x) = 0 \) with all function \( -f_m \) bounded from below, as \( -L \) has the same null-space and range than \( L \).

Define \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\varphi(x) = \frac{1}{n-1} \left( \sum_{m=1}^{n-1} f_m(x) \right)
\]

The following result is taken from [19].

**Theorem 11.7.1** Suppose that the functions \( f_m \) (\( 1 \leq m \leq n - 1 \)) satisfy the conditions of Lemma 11.7.1 and that

\[
\varphi(-\rho)\varphi(\rho) < 0.
\]  

(11.40)

Then, problem (11.32) has at least one solution.

**Proof.** The first assumption of Theorem 8.6.1 is satisfied for \( \mathcal{D} = B(\rho) \times [0, 1] \). Now, \( N(L) \simeq \mathbb{R} \simeq R(Q) \), and

\[
QF(c, \cdots, c) = \varphi(c) \quad (c \in \mathbb{R}).
\]

Hence the second assumption of Theorem 8.6.1 is satisfied, and, furthermore,

\[
|d_{B}[QF, B(\rho) \cap N(L), 0]| = |d_{B}[\varphi, -\rho, \rho, 0]| = 1,
\]

so that the first assumption of Theorem 8.6.1 holds as well.
Example 11.7.1 Given positive $a_m$ and $b_m$ ($1 \leq m \leq n - 1$), the periodic problem

$$\Delta x_m + a_m \exp x_m - b_m = 0 \quad (1 \leq m \leq n - 1), \quad x_1 = x_n,$$  \hfill (11.41)

related to the study of positive periodic solutions of Verhulst equation with variable coefficients has at least one solution.
Chapter 12

Second order difference equations

12.1 Dirichlet problem

For a fixed integer \( n \geq 2 \) and \((x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\), define \((\Delta^2 x_0, \ldots, \Delta^2 x_{n-2}) \in \mathbb{R}^{n-1}\) by

\[
\Delta^2 x_{m-1} = x_{m+1} - 2x_m + x_{m-1}, \quad (1 \leq m \leq n-1).
\]

If \( f_m : \mathbb{R}^{n+1} \to \mathbb{R} \) \((1 \leq m \leq n-1)\) are continuous functions, we study the existence of solutions for the Dirichlet boundary value problem

\[
\Delta^2 x_{m-1} + f_m(x_0, x_1, \ldots, x_n) = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n. \tag{12.1}
\]

We have

\[
V^{n-1} := \{ x \in \mathbb{R}^{n+1} : x_0 = 0 = x_n \} \simeq \mathbb{R}^{n-1}, \tag{12.2}
\]

as its elements can be associated to the coordinates \((x_1, \ldots, x_{n-1})\) and correspond to the elements of \(\mathbb{R}^{n+1}\) of the form \((0, x_1, \ldots, x_{n-1}, 0)\). The restriction \( L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) of \( D^2 \) to \( \mathbb{R}^{n-1} \) is well defined in terms of \((x_1, \ldots, x_{n-1})\) by

\[
(Lx)_1 = -2x_1 + x_2,
\]

\[
(Lx)_m = x_{m+1} - 2x_m + x_{m-1} \quad (2 \leq m \leq n-1), \tag{12.3}
\]

\[
(Lx)_{n-1} = -2x_{n-1} + x_{n-2},
\]

or, in matrix form, by the symmetric Jacobi matrix

\[
L = \begin{pmatrix}
-2 & 1 & 0 & \cdots & \cdots & 0 \\
1 & -2 & 1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
0 & \cdots & 0 & 1 & -2 & \cdots \\
\end{pmatrix}. \tag{12.4}
\]

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We define the continuous mapping \( F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by
\[
F_m(x_1, \ldots, x_{n-1}) = f_m(0, x_1, \ldots, x_{n-1}, 0) \quad (1 \leq m \leq n - 1),
\]
(12.5)
and the continuous mapping \( G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by \( G = L + F \). It is clear that the solutions of (12.1) correspond to the zeros of \( G \) in \( \mathbb{R}^{n-1} \).

### 12.2 Spectrum of the linear part

The linear Dirichlet eigenvalue problem
\[
\Delta^2 x_{m-1} - \lambda x_m = 0 \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n,
\]
(12.6)
where \( \lambda \in \mathbb{R} \), is equivalent to the linear eigenvalue problem in \( \mathbb{R}^{n-1} \)
\[
\begin{pmatrix}
-2 - \lambda & 1 & 0 & \cdots & 0 \\
1 & -2 - \lambda & 1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 1 & -2 - \lambda \\
0 & \cdots & \cdots & 0 & 1 & -2 - \lambda
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-2} \\
x_{n-1}
\end{pmatrix} = 0.
\]
(12.7)
If we search solutions of (12.6) of the form \( x_m = \sin m\theta \) (\( m = 0, 1, \ldots, n \)), with \( \theta \in ]0, 2\pi[\setminus\{\pi\} \), we see that we must have
\[
\sin(m + 1)\theta - (\lambda + 2) \sin m\theta + \sin(m - 1)\theta = 0 \quad (1 \leq m \leq n - 1),
\]
or, equivalently
\[
\sin m\theta [2 \cos \theta - \lambda - 2] = 0 \quad (1 \leq m \leq n - 1),
\]
which holds if
\[
\lambda = 2 \cos \theta - 2.
\]
(12.8)
The boundary condition \( x_0 = 0 \) is satisfied for all \( \theta \), and the boundary condition \( x_n = \sin n\theta = 0 \) gives
\[
\theta = \frac{k\pi}{n} \quad (1 \leq k \leq n - 1),
\]
which, together with (12.8) implies
\[
\lambda_k = 2 \cos \frac{k\pi}{n} - 2 = -4 \sin^2 \frac{k\pi}{2n} \quad (1 \leq k \leq n - 1).
\]
(12.9)
This immediately leads to the following
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Lemma 12.2.1 Problem (12.6) has only the trivial solution if and only if

\[ \lambda \neq 2 \cos \frac{k\pi}{n} - 2 \quad (1 \leq k \leq n - 1). \]

If \( \lambda = \lambda_k \) for some \( 1 \leq k \leq n - 1 \), the corresponding eigenvector \( \varphi^k \in \mathbb{R}^{n-1} \) has the form

\[ \left( \sin \frac{k\pi}{n}, \sin \frac{2k\pi}{n}, \ldots, \sin \frac{(n-1)k\pi}{n} \right). \]

In particular, the eigenvector \( \varphi^1 \) associated to the eigenvalue \( \lambda_1 = 2 \left( \cos \frac{\pi}{n} - 1 \right) = -4 \sin^2 \frac{\pi}{2n} \)

(12.10)

given by

\[ \varphi_1 = \left( \sin \frac{\pi}{n}, \sin \frac{2\pi}{n}, \ldots, \sin \frac{(n-1)\pi}{n} \right) \]

(12.11)

has all its components positive. Furthermore, as the \( \varphi^k \) constitute a system of eigenvectors of the symmetric matrix \( L \), they satisfy the orthogonality conditions \( \langle \varphi^j, \varphi^k \rangle = 0 \) for \( j \neq k \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^{n-1} \).

12.3 Bounded nonlinearities

Let \( b_m : \mathbb{R}^{n+1} \to \mathbb{R}, \quad (x_0, x_1, \ldots, x_n) \mapsto b_m(x_0, x_1, \ldots, x_n) \quad (1 \leq m \leq n - 1) \) be continuous and bounded, and consider the semilinear Dirichlet problem

\[ \Delta^2 x_{m-1} - \lambda x_m + b_m(x_0, x_1, \ldots, x_n) = 0 \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n \]

(12.12)

If \( L \) is defined in (12.4) and \( B : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by

\[ B_m(x_1, \ldots, x_{n-1}) = b_m(0, x_1, x_2, \ldots, x_{n-1}, 0) \quad (1 \leq m \leq n - 1), \]

problem (12.12) is equivalent to the semilinear problem in \( \mathbb{R}^{n-1} \)

\[ Lx - \lambda x + B(x) = 0. \]

(12.13)

An immediate consequence of Lemma 12.2.1 and Corollary 8.1.2 is the following existence result.

Theorem 12.3.1 Assume that \( \lambda \neq 2 \cos \frac{k\pi}{n} - 2 \quad (1 \leq k \leq n - 1) \), and that the functions \( b_m \) are continuous and bounded. Then problem (12.12) has at least one solution and, for all sufficiently large \( R \),

\[ d_B[L - \lambda I + B, B(R), 0] = \pm 1. \]

In particular, the result is true for any \( \lambda \in ] - \infty, -2] \cup [0, +\infty[. \)
12.4 Upper and lower solutions

Let $f_m : \mathbb{R} \to \mathbb{R}$ ($1 \leq m \leq n - 1$) be continuous functions. We study the existence of solutions for the Dirichlet boundary value problem

$$\Delta^2 x_{m-1} + f_m(x_m) = 0 \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n. \quad (12.14)$$

**Definition 12.4.1** $\alpha = (\alpha_0, \cdots, \alpha_n)$ (resp. $\beta = (\beta_0, \cdots, \beta_n)$) is called a lower solution (resp. upper solution) for (12.14) if

$$\alpha_0 \leq 0, \quad \alpha_n \leq 0 \quad (\text{resp. } \beta_0 \geq 0, \quad \beta_n \geq 0) \quad (12.15)$$

and the inequalities

$$\Delta^2 \alpha_{m-1} + f_m(\alpha_m) \geq 0 \quad (\text{resp. } \Delta^2 \beta_{m-1} + f_m(\beta_m) \leq 0) \quad (1 \leq m \leq n - 1) \quad (12.16)$$

hold. Such a lower or upper solution will be called **strict** if the inequalities (12.16) are strict.

The proof of the following result, taken from [20], is modeled upon the one given in [267] for the case of second order ordinary differential equations.

**Theorem 12.4.1** If (12.14) has a lower solution $\alpha = (\alpha_0, \cdots, \alpha_n)$ and an upper solution $\beta = (\beta_0, \cdots, \beta_n)$ such that $\alpha_m \leq \beta_m$ ($1 \leq m \leq n - 1$), then (12.14) has a solution $x = (x_0, \cdots, x_n)$ such that $\alpha_m \leq x_m \leq \beta_m$ ($1 \leq m \leq n - 1$). Moreover, if $\alpha$ and $\beta$ are strict, then $\alpha_m < x_m < \beta_m$ ($1 \leq m \leq n - 1$).

**Proof.** I. A modified problem.

Let $\gamma_m : \mathbb{R} \to \mathbb{R}$ ($1 \leq m \leq n - 1$) be the continuous functions defined by

$$\gamma_m(x) = \begin{cases} 
\beta_m, & x > \beta_m \\
\alpha_m, & \alpha_m \leq x \leq \beta_m \\
x, & x < \alpha_m.
\end{cases}$$

We consider the modified problem

$$\Delta^2 x_{m-1} - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) = 0 \quad (1 \leq m \leq n - 1),$$

$$x_0 = 0 = x_n. \quad (12.17)$$

and show that if $x = (x_0, \cdots, x_n)$ is a solution of (12.17) then $\alpha_m \leq x_m \leq \beta_m$ ($1 \leq m \leq n - 1$), and hence $x$ is a solution of (12.14). Suppose by contradiction that there is some $1 \leq i \leq n - 1$ such that $\alpha_i - x_i > 0$ so that $\alpha_m - x_m = \max_{1 \leq j \leq n-1} (\alpha_j - x_j) > 0$. Hence

$$\Delta^2 (\alpha_{m-1} - x_{m-1}) = (\alpha_{m+1} - x_{m+1}) - 2(\alpha_m - x_m) + (\alpha_{m-1} - x_{m-1}) \leq 0,$$

and

$$\Delta^2 \alpha_{m-1} \leq \Delta^2 x_{m-1} = -F_m(x_m) + (x_m - \gamma_m(x_m)) = -f_m(\alpha_m) + (x_m - \alpha_m) \leq -f_m(\alpha_m) \leq \Delta^2 \alpha_{m-1},$$
a contradiction. Analogously we can show that \( x_m \leq \beta_m \) (1 \( \leq \) \( m \leq \) \( n-1 \)). The same reasoning shows that, if \( \alpha, \beta \) are strict, then \( \alpha_m < x_m < \beta_m \) (1 \( \leq \) \( m \leq \) \( n-1 \)).

II. Solution of the modified problem.

We use Brouwer degree to study the zeros of the continuous mapping \( G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) defined by

\[
G_m(x) = (Lx)_m - x_m + f_m \circ \gamma_m(x_m) + \gamma_m(x_m) \quad (1 \leq m \leq n-1).
\]

(12.18)

As seen above, \( L - I : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) is invertible. On the other hand the mapping with components \( f_m \circ \gamma_m + \gamma_m \) (1 \( \leq \) \( m \leq \) \( n-1 \)) is bounded on \( \mathbb{R}^{n-1} \). Consequently, Theorem 11.2.1 implies the existence of \( R > 0 \) such that, for all \( \rho > R \), one has

\[
|d_B[G, B(\rho), 0]| = 1,
\]

(12.19)

and, in particular, \( G \) has a zero \( x \in B(\rho) \). Hence, \( x \) is a solution of (12.17), which means that \( \alpha_m \leq x_m \leq \beta_m \) (1 \( \leq \) \( m \leq \) \( n-1 \)), and \( x \) is a solution of (12.16).

Moreover if \( \alpha, \beta \) are strict, then \( \alpha_m < x_m < \beta_m \) (1 \( \leq \) \( m \leq \) \( n-1 \)).

Remark 12.4.1 Suppose that \( \alpha \) (resp. \( \beta \)) is a strict lower (resp. upper) solution of (12.16). As we have already seen, (12.16) admits at least one solution \( x \) such that \( \alpha_m < x_m < \beta_m \) (1 \( \leq \) \( m \leq \) \( n-1 \)). Define the open set

\[
\Omega_{\alpha,\beta} = \{ (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} : \alpha_m < x_m < \beta_m \quad (1 \leq m \leq n-1) \}.
\]

If \( \rho \) is large enough, using the additivity-excision property of Brouwer degree, we have

\[
\]

On the other hand, if we define the continuous mapping \( \tilde{G} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by

\[
\tilde{G}_m(x) = (Lx)_m + f_m(x_m) \quad (1 \leq m \leq n-1),
\]

(12.20)

\( \tilde{G} \) is equal to \( G \) on \( \Omega_{\alpha,\beta} \), and then

\[
|d[\tilde{G}, \Omega_{\alpha,\beta}, 0]| = 1.
\]

(12.21)

A simple but useful consequence of Theorem 12.4.1, goes as follows.

Corollary 12.4.1 Assume that there exists numbers \( \alpha \leq \beta \) such that

\[
f_m(\alpha) \geq 0 \geq f_m(\beta) \quad (1 \leq m \leq n-1).
\]

Then problem (12.16) has at least one solution with \( \alpha \leq x_m \leq \beta \) (1 \( \leq \) \( m \leq \) \( n-1 \)).

Proof. Just observe that \((0, \alpha, \cdots, \alpha, 0)\) is a lower solution and \((0, \beta, \cdots, \beta, 0)\) an upper solution for (12.16).

Example 12.4.1 For each \( p > 0 \), \( a_m > 0 \) and \( b_m \in \mathbb{R} \) (1 \( \leq \) \( m \leq \) \( n-1 \)) the problem

\[
\Delta^2 x_{m-1} - a_m|x_m|^p x_m + b_m = 0 \quad (1 \leq m \leq n-1), \quad x_1 = x_n
\]

has at least one solution.
12.5 Ambrosetti-Prodi type multiplicity results

In this section we are interested in problems of the type

\[ \Delta^2 x_m - \lambda_1 x_m + f_m(x_m) = s\varphi_m^1 \quad (1 \leq m \leq n-1), \]

\[ x_0 = x_n = 0, \quad (12.22) \]

where \( n \geq 2 \) is fixed, \( f_1, \ldots, f_{n-1} : \mathbb{R} \to \mathbb{R} \) are continuous, \( s \in \mathbb{R} \), \( \lambda_1 \) is given in (12.10), \( \varphi_1 \) is given in (12.11) and

\[ f_m(x) \to \infty \text{ as } |x| \to \infty \quad (1 \leq m \leq n-1). \]

We prove an Ambrosetti-Prodi type result for (12.22) first proved in [20], which is reminiscent of a multiplicity theorem for second order differential equations with Dirichlet boundary conditions given in [59].

We need an identity for finite sequences satisfying the Dirichlet boundary conditions, which is a type of summation by parts.

**Lemma 12.5.1** If \((x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\) and \((y_0, \ldots, y_n) \in \mathbb{R}^{n+1}\) are such that \(x_0 = 0 = x_n\) and \(y_0 = 0 = y_n\), the identity

\[ \sum_{m=1}^{n-1} x_m D^2 y_m = \sum_{m=1}^{n-1} y_m \Delta^2 x_m - 1 \]

holds.

**Proof.** We have

\[ \sum_{m=1}^{n-1} x_m (y_{m+1} - 2y_m + y_{m-1}) - \sum_{m=1}^{n-1} y_m (x_{m+1} - 2x_m + x_{m-1}) \]

\[ = \sum_{m=1}^{n-2} x_m y_{m+1} + \sum_{m=2}^{n-1} x_m y_{m-1} - \sum_{m=1}^{n-2} y_m x_{m+1} - \sum_{m=2}^{n-1} y_m x_{m-1} = 0 \]

The next lemma provides \textit{a priori bounds} for the possible solutions of (12.22).

**Lemma 12.5.2** Let \( b \in \mathbb{R} \). Then there is \( \rho = \rho(b) > 0 \) such that any possible solution \( x \) of (12.22) with \( s \leq b \) belongs to the open ball \( B(\rho) \).

**Proof.** Let \( s \in [a, b] \) and \((x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\) be a solution of (12.22), or, equivalently let \((x_1, \ldots, x_{n-1})\) be a solution of

\[ (Lx)_m - \lambda_1 x_m + f_m(x_m) = s\varphi_m^1. \]
12.5. AMBROSETTI-PRODI TYPE MULTIPlicity REsULTS

Multiplying each equation by $ϕ^1_m$ and adding, we obtain

$$s \left[ \sum_{m=1}^{n-1} (ϕ^1_m)^2 \right] = \sum_{m=1}^{n-1} [ϕ^1_m (Lx)_m - λ_1ϕ^1_m x_m] + \sum_{m=1}^{n-1} ϕ^1_m f_m(x_m).$$

But, using Lemma 12.5.1,

$$\sum_{m=1}^{n-1} [ϕ^1_m (Lx)_m - λ_1ϕ^1_m x_m] = \sum_{m=1}^{n-1} \{x_m [(Lϕ^1)_m - λ_1ϕ^1_m]\} = 0,$$

so that

$$\sum_{m=1}^{n-1} ϕ^1_m f_m(x_m) = s ||ϕ^1||^2 \leq b ||ϕ^1||^2. \tag{12.25}$$

Using (12.23), there exists $c ∈ \mathbb{R}$ such that

$$f_m(u) ≥ c \text{ for all } u ∈ \mathbb{R} \text{ and all } m ∈ \{1, \ldots, n-1\}, \tag{12.26}$$

and

$$(∀ R > 0)(∃ r' > 0)(∀ u ∈ \mathbb{R} : |u| ≥ r')(∀ m ∈ \{1, \ldots, n-1\}) :$$

$$ϕ^1_m f_m(u) ≥ R - c \left( \sum_{k≠m} ϕ^1_k \right). \tag{12.27}$$

Let $r = \sqrt{n-1}r'$ and $x ∈ \mathbb{R}^{n-1}$ be such that $||x|| ≥ r$. Then for at least one $j$ one has $|x_j| ≥ r'$, so that, using (12.26) and (12.27), one obtains

$$\sum_{m=1}^{n-1} ϕ^1_m f_m(x_m) = \sum_{m=1}^{n-1} ϕ^1_m [f_m(x_m) - c] + c \left[ \sum_{m=1}^{n-1} ϕ^1_m \right]$$

$$≥ ϕ^1_j [f_j(x_j) - c] + c \left[ \sum_{m=1}^{n-1} ϕ^1_m \right]$$

$$≥ R - c \left( \sum_{k≠j} ϕ^1_k \right) - cϕ^1_j + c \left[ \sum_{m=1}^{n-1} ϕ^1_m \right] = R.$$

Hence $\sum_{m=1}^{n-1} ϕ^1_m f_m(x_m) → +∞$ if $||x|| → ∞$, and the result follows from (12.25).

\[\Box\]

Theorem 12.5.1 If the functions $f_m (1 ≤ m ≤ n-1)$ satisfy (12.23), then there is $s_1 ∈ \mathbb{R}$ such that (12.22) has zero, at least one or at least two solutions according to $s < s_1$, $s = s_1$ or $s > s_1$. 
Consequently, the Brouwer degree
We reformulate (12.22) to apply Brouwer degree theory. Cons
holds, from where we deduce that, with
Then (12.25)
(b) If \( s \in S_1 \) and \( s > \hat{s} \) then \( s \in S_1 \).
Let \( \hat{x} = (\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_n) \) be a solution of (12.22) with \( s = \hat{s} \), and let \( s > \hat{s} \). Then \( \hat{x} \) is a strict upper solution for (12.22). Take now \( R_- < \min_{1 \leq m \leq n-1} \frac{m}{\phi_m} \) such that
Then \( m(R_- \varphi_m^1) > s \varphi_m^1 \) (1 \( \leq m \leq n - 1 \)). It follows that \( \alpha \) with \( \alpha_0 = 0 = \alpha_n \) and \( \alpha_j = R_- \varphi_j^1 \) (1 \( \leq j \leq n - 1 \)) is a strict lower solution for (12.22), and hence, using Theorem 12.4.1, \( s \in S_1 \).
(c) \( S_1 \) is finite and \( S_1 \supset [s_1, \infty[ \).
Let \( s \in \mathbb{R} \) and suppose that (12.22) has a solution \((x_0, x_1, \ldots, x_n)\). Then (12.25) holds, from where we deduce that, with \( \varphi \) defined in (12.26), \( s \geq c \sum_{m=1}^{n-1} \varphi_m^1 \). To obtain the second part of claim (c), we apply (b).
(d) \( S_2 \supset [s_1, \infty[ \).
We reformulate (12.22) to apply Brouwer degree theory. Consider the continuous mapping \( G : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1} \) defined by
Then \((x_0, \ldots, x_n)\) is a solution of (12.22) if and only if \((x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\) is a zero of \( G(s, \cdot) \). Let \( s_1 < s < s_2 \). Using Lemma 12.5.2 we find \( \rho > 0 \) such that each possible zero of \( G(s, \cdot) \) with \( s \in [s_1, s_2] \) is such that \( \max_{1 \leq m \leq n-1} |x_m| < \rho \). Consequently, the Brouwer degree \( d_B[G(s, \cdot), B(\rho), 0] \) is well defined and does not depend upon \( s \in [s_1, s_2] \). However, using (c), we see that \( G(s_1, x) \neq 0 \) for all \( x \in \mathbb{R}^{n-1} \). This implies that \( d_B[G(s_2, \cdot), B(\rho), 0] = 0 \), so that \( d_B[G(s_2, \cdot), B(\rho'), 0] = 0 \) and, by excision property, \( d_B[G(s_2, \cdot), B(\rho'), 0] = 0 \) if \( \rho' > \rho \). Let \( s \in [s_1, s_2] \) and \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \) be a solution of (12.22) (using (c)). Then \( \tilde{x} \) is a strict upper solution of (12.22) with \( s = s_2 \). Let \( R < \min_{1 \leq m \leq n} \frac{m}{\varphi_m} \) be such that \( m(R \varphi_m^1) > s_2 \varphi_m^1 \) (1 \( \leq m \leq n - 1 \)). Then \((0, R \varphi_1^1, \ldots, R \varphi_{n-1}^1, 0) \in \mathbb{R}^{n+1} \) is a strict lower solution of (12.22) with \( s = s_2 \). Consequently, using Remark 12.4.1, (12.22) with \( s = s_2 \) has a solution in \( \Omega_{R, \tilde{x}} \) and
Taking \( \rho' \) sufficiently large, we deduce from the additivity property of Brouwer degree that
\[
\begin{align*}
|d_B[G(s_2, \cdot), B(\rho'), \Omega_{R, \tilde{x}}, 0]| & = |d_B[G(s_2, \cdot), B(\rho'), 0] - d_B[G(s_2, \cdot), \Omega_{R, \tilde{x}}, 0]| \\
& = |d_B[G(s_2, \cdot), \Omega_{R, \tilde{x}}, 0]| = 1,
\end{align*}
\]
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and (12.22) with \( s = s_2 \) has a second solution in \( B(\rho') \setminus \bigcup_{s \in \mathbb{R}} B(\rho, \phi_1, b, x) \).

(e) \( s_1 \in S_1 \).

Taking a decreasing sequence \( (s_k)_{k \in \mathbb{N}} \) in \( ]s_1, \infty[ \) converging to \( s_1 \), a corresponding sequence \( (x_{k}^0, x_{k}^1, \ldots, x_{k}^{n}) \) of solutions of (12.22) with \( s = s_k \) and using Lemma 12.5.2, we obtain a subsequence \( (x_{j}^0, x_{j}^1, \ldots, x_{j}^{n}) \) which converges to a solution \( (x_0, x_1, \ldots, x_n) \) of (12.22) with \( s = s_1 \).

Similar arguments allow to prove the following result.

**Theorem 12.5.2** If the functions \( f_m \) satisfy condition

\[
 f_m(x) \to -\infty \text{ as } |x| \to \infty, \quad (2 \leq m \leq n - 1).
\]

then there is \( s_1 \in \mathbb{R} \) such that (12.22) has zero, at least one or at least two solutions

according to \( s > s_1, s = s_1 \) or \( s < s_1 \).

**Example 12.5.1** There exists \( s_1 \in \mathbb{R} \) such that the problem

\[
 \Delta^2 x_{m+1} - \lambda_1 x_m + |x_m|^{1/2} = s \varphi_m^1 \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n
\]

has no solution if \( s < s_1 \), at least one solution if \( s = s_1 \) and at least two solutions

if \( s > s_1 \).

**Example 12.5.2** There exists \( s_1 \in \mathbb{R} \) such that the problem

\[
 \Delta^2 x_{m+1} - \lambda_1 x_m - \exp x_m^2 = s \varphi_m^1 \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n
\]

has no solution if \( s > s_1 \), at least one solution if \( s = s_1 \) and at least two solutions

if \( s < s_1 \).

12.6 One-sided bounded nonlinearities

Let \( n \geq 2 \) be a fixed integer, and \( f_m : \mathbb{R} \to \mathbb{R} \) continuous functions \( (2 \leq m \leq n - 1) \).

Consider the problem

\[
 \Delta^2 x_m - \lambda_1 x_m + f_m(x_m) = 0 \quad (1 \leq m \leq n - 1), \quad x_0 = 0 = x_n
\]

(12.29)

We define the continuous mapping \( G : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by

\[
 G_m(x) = (Lx)_m - \lambda_1 x_m + f_m(x_m) \quad (1 \leq m \leq n - 1),
\]

where \( L \) is given in (12.3), so that \((0, x_1, \ldots, x_{n-1}, 0)\) is a solution of (12.29) if and

only if \((x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) is a zero of \( G \). We also define \( F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) by

\[
 F(x) = (f_1(x_1), \ldots, f_{n-1}(x_{n-1})),
\]

and set \( L_1 := L - \lambda_1 I \). We have

\[
 N(L_1) = \{ c \varphi^1 : c \in \mathbb{R} \},
\]
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and, by the properties of symmetric matrices,

\[ R(L_1) = \{ y \in \mathbb{R}^{n-1} : \langle y, \varphi^1 \rangle = 0 \}. \]

Consider the projector \( P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) defined by

\[ P(x) = \langle x, \varphi^1 \rangle \frac{\varphi^1}{\| \varphi^1 \|^2}, \]

so that \( N(P) = R(L_1), R(P) = N(L_1) \).

We will need the following inequality. For \( x = (x_1, \ldots, x_{n-1}) \), define

\[ \overline{x} := \langle x, \varphi^1 \rangle \frac{\varphi^1}{\| \varphi^1 \|^2} = Px, \quad \tilde{x} = x - \overline{x}, \quad (12.31) \]

so that

\[ \overline{x}_m = \left( \sum_{m=1}^{n-1} x_m \sin \frac{mn}{n} \right) \frac{\varphi^1_m}{\| \varphi^1 \|^2} \quad (1 \leq m \leq n-1), \quad \langle \tilde{x}, \varphi^1 \rangle = 0. \]

Notice that, with \( L \) defined in (12.3) or (12.4),

\[ (Lx)_m - \lambda_1 x_m = (L\overline{x})_m - \lambda_1 \overline{x}_m + (L\tilde{x})_m - \lambda_1 \tilde{x}_m = (L\tilde{x})_m - \lambda_1 \tilde{x}_m. \quad (12.32) \]

**Lemma 12.6.1** If \( x = (x_1, \ldots, x_{n-1}) \), then there exists a constant \( c_n > 0 \), which depends only on \( n \), such that

\[ \max_{1 \leq m \leq n-1} |\overline{x}_m| \leq c_n \left[ \sum_{m=1}^{n-1} |(L\tilde{x})_m - \lambda_1 \tilde{x}_m| \varphi^1_m \right]. \quad (12.33) \]

**Proof.** The application

\[ (x_0, \ldots, x_n) \mapsto \sum_{m=1}^{n-1} |(Lx)_m - \lambda_1 x_m| \varphi^1_m \]

defines a norm on the subspace \( V = \{ x \in \mathbb{R}^{n-1} : \langle x, \varphi^1 \rangle = 0 \} \), because \( L - \lambda_1 I : V \to V \) is one-to-one. It is therefore equivalent to the norm \( \max_{1 \leq m \leq n-1} |x_m| \), and inequality (12.33) holds.

We now obtain a priori estimates for the possible zeros of \( L_1 + \lambda F \) when \( \lambda \in [0, 1] \).

**Lemma 12.6.2** If all functions \( f_m \) (\( 1 \leq m \leq n-1 \)) are bounded from below or are bounded from above, and if, for some \( R > 0 \), one has

\[ \sum_{m=1}^{n-1} f_m(x_m) \varphi^1_m \neq 0 \quad \text{whenever} \quad \min_{1 \leq j \leq n-1} x_j \geq R \quad \text{or} \quad \max_{1 \leq j \leq n-1} x_j \leq -R \quad (12.34) \]

then there exists \( \rho > R \) such that each possible zero \((\lambda, x) \in [0, 1] \times \mathbb{R}^{n-1} \) of \( L_1 + \lambda F \) is such that \( \|x\| < \rho \).
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Proof. Assume first that each \( f_m \) is bounded from below by \( c \). Let \( (\lambda, x) \in [0, 1] \times \mathbb{R}^{n-1} \) be such that \( L_1 x + \lambda F(x) = 0 \). Applying \( P \) and \((I - P)\) to the equation, we obtain

\[ PF(x) = 0, \quad L_1 x + \lambda(I - Q)F(x) = 0, \]

or, equivalently

\[ \sum_{m=1}^{n-1} f_m(x_m) \varphi^1_m = 0, \tag{12.35} \]

\[ (L \tilde{x})_m - \lambda_1 \tilde{x}_m + \lambda f_m(x_m) = 0 \quad (1 \leq m \leq n - 1). \tag{12.36} \]

We can then repeat the reasoning of the proof of Lemma 12.5.2 to obtain that

\[ \sum_{m=1}^{n-1} |f_m(x_m)| \varphi^1_m \leq (|c| - c) \sum_{m=1}^{n-1} \varphi^1_m := R_1. \]

Hence

\[ \sum_{m=1}^{n-1} |(L \tilde{x})_m - \lambda_1 \tilde{x}_m| \varphi^1_m = \lambda \sum_{m=1}^{n-1} |f_m(x_m)| \varphi^1_m \leq R_1. \]

Consequently, using Lemma 12.6.1, we obtain

\[ \max_{1 \leq m \leq n-1} |(\tilde{x})_m| \leq c_n R_1 := R_2. \]

Then, by (12.34) and (12.35), there exists \( 1 \leq k \leq n - 1 \) and \( 1 \leq l \leq n - 1 \) such that \( x_k < R \) and \( x_l > -R \). Consequently, \( \overline{x}_k = x_k - \tilde{x}_k < R + R_2 \) and \( \overline{x}_l = x_l - \tilde{x}_l > -R - R_2 \). Therefore, for each \( 1 \leq m \leq n - 1 \),

\[ \overline{x}_m = \frac{\overline{x}_k}{\varphi^1_k} \varphi^1_m < (R + R_2) \max_{1 \leq m \leq n-1} \frac{\varphi^1_m}{\varphi^1_k} := R_3, \]

\[ (\overline{x})_m = \frac{(\overline{x})_l}{\varphi^1_l} \varphi^1_m > -(R + R_2) \max_{1 \leq m \leq n-1} \frac{\varphi^1_m}{\varphi^1_k} := -R_3. \]

Consequently \( ||x|| < \rho \) for some \( \rho > 0 \).

In the case where the \( f_m \) are bounded from above, if suffices to write the problem

\[ \tilde{L}_1 x + \tilde{F}(x) = 0, \]

with \( \tilde{L}_1 = -L_1 \) and \( \tilde{F} = -F \) to reduce it to a problem with \( \tilde{F} \) bounded from below, noticing that \( L_1 \) and \( \tilde{L}_1 \) have the same kernel and the same range. \qed
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the continuous function defined by
\begin{equation}
\varphi(u) = \sum_{m=1}^{n-1} f_m(u \varphi_m^1) \varphi_m^1.
\end{equation}

The following theorem is taken from [20].

**Theorem 12.6.1** If the functions $f_m$ $(1 \leq m \leq n-1)$ satisfy the conditions of Lemma 12.6.2 and if
\begin{equation}
\varphi(-R) \varphi(R) < 0,
\end{equation}
then, problem (12.29) has at least one solution.

**Proof.** The first assumption of Theorem 8.6.1 applied to $L_1 + F$ is satisfied for $D = B(\rho) \times [0,1]$. Now,
\[ PF(u \varphi_1^1) = \varphi(u) \varphi_1^1 \quad (u \in \mathbb{R}). \]

Hence the second assumption of Theorem 8.6.1 is satisfied, and, furthermore,
\[ |d_B[QF, B(\rho) \cap N(L), 0]| = |d_B[\varphi, -\rho, \rho], 0| = 1, \]
so that the first assumption of Theorem 8.6.1 holds as well.

**Example 12.6.1** The problem
\[ \Delta^2 x_{m-1} - \lambda_1 x_m + \exp x_m - t_m = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n, \]
has at least one solution if and only if $\sum_{m=1}^{n-1} t_m \varphi_m^1 > 0$.

The necessity follows from summing both members of the equation from 1 to $n-1$ after multiplication by $\varphi_m^1$, and the sufficiency from Theorem 12.6.1, if we observe that there exists $R > 0$ such that the function $\varphi$ defined by $\varphi(u) = \sum_{m=1}^{n-1} [\exp(u \varphi_m^1) - t_m] \varphi_m^1$ is such that $\varphi(u) > 0$ for $u \geq R$ and $\varphi(u) < 0$ for $u \leq -R$.

**Example 12.6.2** If $g : \mathbb{R} \to \mathbb{R}$ is a continuous function bounded from below or from above and $(t_1, \cdots, t_{n-1}) \in \mathbb{R}^{n-1}$ such that
\begin{equation}
-\infty < \limsup_{u \to -\infty} g(u) < \frac{\sum_{m=1}^{n-1} t_m \varphi_m^1}{\sum_{m=1}^{n-1} \varphi_m^1} < \liminf_{u \to +\infty} g(u) < +\infty,
\end{equation}
then the problem
\[ \Delta^2 x_{m-1} - \lambda_1 x_m + g(x_m) - t_m = 0 \quad (1 \leq m \leq n-1), \quad x_0 = 0 = x_n, \]
has at least one solution.

Condition (12.39) is a Landesman-Lazer-type condition for difference equations. It is easily shown to be necessary if $\limsup_{u \to -\infty} g(u) < g(v) < \liminf_{u \to +\infty} g(u)$ for all $v \in \mathbb{R}$.
Chapter 13

Bifurcation

13.1 Eigenvalues of couples of linear mappings

Let \((L, A)\) be a couple of linear mappings \(L : \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(A : \mathbb{R}^n \rightarrow \mathbb{R}^n\). The following results and concepts have been introduced in [240].

**Definition 13.1.1** \(\mu \in \mathbb{C}\) is an eigenvalue for \((L, A)\) is \(N(L - \mu A) \neq \{0\}\).

When \(A = I\), an eigenvalue for \((L, I)\) is nothing but a classical eigenvalue of \(L\). When \(N(L) \cap N(A) \neq \{0\}\), any \(\mu \in \mathbb{C}\) is an eigenvalue for \((L, 0)\). In order to be able to define the multiplicity of an eigenvalue for \((L, A)\), we make the following assumption of non-degeneracy:

\((\text{ND})\) : The set \(\sigma(L, A)\) of eigenvalues for \((L, A)\) is not \(\mathbb{C}\).

**Definition 13.1.2** Assume that \((L, A)\) satisfies condition \((\text{ND})\) and let \(\mu_0 \in \mathbb{C} \setminus \sigma(L, A)\). The spectral operator \(A_{\mu_0}\) associated to \((L, A)\) and \(\mu_0\) is the linear operator

\[ A_0 := (L - \mu_0 A)^{-1} A. \]

**Proposition 13.1.1** Assume that condition \((\text{ND})\) holds and let \(\mu_0 \in \mathbb{C} \setminus \sigma(L, A)\). Then \(\mu \in \mathbb{C}\) is an eigenvalue for \((L, A)\) if and only \(\mu - \mu_0\) is a (classical) characteristic value of \(A_0\), i.e. if and only if \(N(I - (\mu - \mu_0)A_0) \neq \{0\}\).

**Proof.** We have

\[ L - \mu A = L - \mu_0 A - (\mu - \mu_0) A = (L - \mu_0 A)[I - (\mu - \mu_0)A_0] \quad (13.1) \]

and, as \(L - \mu_0 A : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is an isomorphism (being one-to-one), \(N(L - \mu A) \neq \{0\}\) if and only if \(N(I - (\mu - \mu_0)A_0) \neq \{0\}\), i.e. if and only if \(\mu - \mu_0\) is a characteristic value of \(A_0\). \(\blacksquare\)
We now show that the (algebraic) multiplicity of a characteristic value of $A_0$ does not depend upon the choice of $\mu_0 \in \mathbb{C} \setminus \sigma(L, A)$. Recall that the algebraic multiplicity $\nu_B(\chi)$ of a characteristic value $\chi$ of a linear mapping $B : \mathbb{R}^n \to \mathbb{R}^n$ is the dimension of $N[(I - \chi B)^0]$, where $n_0$ is the smallest nonnegative integer $n$ such that

$$N[(I - \chi B)^{n+1}] = N[(I - \chi B)^n].$$

In other words,

$$\nu_B(\chi) = \dim \bigcup_{n \geq 1} N[(I - \chi B)^n].$$

**Lemma 13.1.1** If assumption (ND) holds, and if $\mu_1$ and $\mu_2$ are in $\mathbb{C} \setminus \sigma(L, A)$, then

$$[I - (\mu_1 - \mu_2)A_2][I - (\mu - \mu_1)A_1] = I - (\mu - \mu_2)A_2$$

(13.2)

**Proof.** We have

\[
\begin{align*}
I - (\mu - \mu_1)A_1 &= I - (\mu - \mu_1)(L - \mu_1)^{-1}A \\
&= I - (\mu - \mu_1)[L - \mu_2A - (\mu_1 - \mu_2)A]^{-1}A \\
&= I - (\mu - \mu_1)[I - (\mu_1 - \mu_2)(L - \mu_2A)^{-1}A]^{-1}(L - \mu_2A)^{-1}A \\
&= I - (\mu - \mu_1)[I - (\mu_1 - \mu_2)A_2]^{-1}A_2,
\end{align*}
\]

and hence

\[
\begin{align*}
(I - (\mu_1 - \mu_2)A_2)(I - (\mu - \mu_1)A_1) \\
&= (I - (\mu_1 - \mu_2)A_2)[I - (\mu - \mu_1)[I - (\mu_1 - \mu_2)A_2]^{-1}A_2] \\
&= I - (\mu_1 - \mu_2)A_2 - (\mu - \mu_1)A_2 = I - (\mu - \mu_2)A_2.
\end{align*}
\]

\[\square\]

**Proposition 13.1.2** If assumption (ND) holds, $\mu_0$ and $\mu_1$ belong to $\mathbb{C} \setminus \sigma(L, A)$, and if $\mu \in \sigma(L, A)$, then

$$\nu_{A_0}(\mu - \mu_0) = \nu_{A_1}(\mu - \mu_1).$$

**Proof.** We have, using (13.2) and the trivial commutation identity

$$[I - (\mu_1 - \mu_2)A_2]^{-1}[I - (\mu - \mu_2)A_2] = [I - (\mu - \mu_2)A_2][I - (\mu_1 - \mu_2)A_2]^{-1}$$

\[
\begin{align*}
[I - (\mu_1 - \mu_2)A_2]^{-1}[I - (\mu - \mu_2)A_2] &\quad [I - (\mu_1 - \mu_2)A_2]^{-1}[I - (\mu - \mu_2)A_2]^{-1}A_2 \\
&= [I - (\mu_1 - \mu_2)A_2]^{-1}[I - (\mu - \mu_2)A_2]^{-1}A_2 \\
&= [I - (\mu_1 - \mu_2)A_2]^{-1}A_2 = [I - (\mu - \mu_2)A_2]^{-1}A_2.
\end{align*}
\]

Hence, for any positive integer $n$,

$$N[I - (\mu_1 - \mu_2)A_2] = N[I - (\mu - \mu_2)A_2]^n,$$

and the result follows. \[\square\]
13.1. EIGENVALUES OF COUPLES OF LINEAR MAPPINGS

Proposition 13.1.2 justifies the following definition, given in [240].

**Definition 13.1.3** If Assumption (ND) holds, the multiplicity $m(\mu)$ of the eigenvalue $\mu$ of $(L, A)$ is defined by

$$m(\mu) = \nu_{A_0}(\mu - \mu_0)$$

(13.3)

where $\nu_{A_0}(\mu - \mu_0)$ is the (algebraic) multiplicity of the characteristic value $\mu - \mu_0$ of $A_0 = (L - \mu_0A)^{-1}A$ for some $\mu_0 \in \mathbb{C} \setminus \sigma(L, A)$.

When $L = I$, one can take $\mu_0 = 0$ and recover the multiplicity of the characteristic value $\mu$ of $A$.

We now give two special cases in which $\beta(\mu)$ can be easily computed. The first one can be found in [240].

**Proposition 13.1.3** If Assumption (ND) holds and if $\mu \in \sigma(L, A)$, then

$$m(\mu) = \dim N(L - \mu A)$$

(13.4)

is and only if

$$A[N(L - \mu A)] \cap R[(L - \mu A)] = \{0\}.$$  

(13.5)

**Proof.** Let $\mu_0 \in \mathbb{C} \setminus \sigma(L, A)$. We know that

$$N(L - \mu A) = N[I - (\mu - \mu_0)A_0]$$

and hence, using classical results of linear algebra,

$$m(\mu) = \dim N(L - \mu A) \iff N[I - (\mu - \mu_0)A_0] \cap R[(L - \mu_0A)] = \{0\}$$

$$\iff N(L - \mu A) \cap R[(L - \mu_0A)] = \{0\}$$

$$\iff (L - \mu A)[N(L - \mu A)] \cap R(L - \mu A) = \{0\}$$

$$\iff (\mu - \mu_0)A[N(L - \mu A)] \cap R(L - \mu A) = \{0\},$$

and the result follows. 

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The following result is essentially due to Laloux [238].

**Proposition 13.1.4** Assume that Assumption (ND) holds and that \( LA = AL \). Then, for any \( \mu \in \sigma(L, A) \), one has

\[
m(\mu) = \dim \bigcup_{n \geq 1} N(L - \mu A)^n. \tag{13.6}
\]

**Proof.** If follows immediately from \( LA = AL \) that, for any \( \mu, \nu \in \mathbb{C} \) one has

\[
(L - \mu A)(L - \nu A) = (L - \nu A)(L - \mu A),
\]

and hence, if \( \mu_0 \in \mathbb{C} \setminus \sigma(L, A) \),

\[
(L - \mu_0 A)^{-1}(L - \mu A) = (L - \mu A)(L - \mu_0 A)^{-1}.
\]

Consequently, for any positive integer \( n \),

\[
[I - (\mu - \mu_0)A_0]^n = [(L - \mu_0 A)^{-1}(L - \mu A)]^n = (L - \mu_0)^{-n}(L - \mu A)^n.
\]

Consequently,

\[
\dim N[I - (\mu - \mu_0)A_0]^n = \dim N[(L - \mu A)^n],
\]

for any integer \( n \geq 1 \), and (13.6) follows.

13.2 A Leray-Schauder formula for Brouwer index

Let \( L, A : \mathbb{R}^n \to \mathbb{R}^n \) be linear mappings such that Assumption (ND) holds, with \( \mu_0 \in \mathbb{C} \setminus \sigma(L, A) \). This implies that the set of eigenvalues \( \sigma(L, A) \) is equal, up to a translation, to the set of characteristic values of \( A_0 = (L - \mu_0 A)^{-1}A \), and hence is finite. Hence, we can assume, without loss of generality, that \( \mu_0 \in \mathbb{R} \setminus \sigma(L, A) \). In particular, the Brouwer index \( i_L(L - \mu_0 A, 0) \) is well defined.

**Lemma 13.2.1** If \( B : \mathbb{R}^n \to \mathbb{R}^n \) is linear and \( I - B \) invertible, then

\[
i_B[I - B, 0] = (-1)^m
\]

where \( m \) is the sum the multiplicities of the characteristic values of \( B \) contained in \([0, 1]\).

**Proof.** It follows from formula 5.3 that

\[
i_B[I - B, 0] = (-1)^{m'}
\]

where \( m' \) is the sum of the multiplicities of the negative characteristic values of \( I - B \). Now, as

\[
det[I - \lambda(I - B)] = det[(1 - \lambda)I_{\lambda B}] = (1 - \lambda)^n \det \left[ I - \frac{\lambda}{\lambda - 1} B \right],
\]

\( \lambda \) is an characteristic value of \( I - B \) if and only if \( \frac{\lambda}{\lambda - 1} \) is a characteristic value of \( B \) and \( \frac{\lambda}{\lambda - 1} < 0 \) if and only if \( \lambda \in [0, 1[ \). So, \( m' = m \).
The following result is essentially due to Leray and Schauder [251].

**Theorem 13.2.1** Assume that condition (ND) holds for \((L, A)\) and let \(\mu_1 < \mu_2\) be elements of \(\mathbb{R} \setminus \sigma(L, A)\). Then

\[
i_B[L - \mu_2 A, 0] = (-1)^m i_B[L - \mu_1 A, 0],
\]

where \(m\) is the sum of the multiplicities of the eigenvalues of \((L, A)\) contained in \([\mu_1, \mu_2]\).

**Proof.** From formula (13.1), we have

\[
L - \mu_2 A = (L - \mu_1 A)[I - (\mu_2 - \mu_1)A_1]
\]

where all mappings are isomorphisms, and hence

\[
i_B[L - \mu_2 A, 0] = i_B[L - \mu_1 A, 0] \cdot i_B[I - (\mu_2 - \mu_1)A_1, 0].
\]

On the other hand, using (13.7),

\[
i_B[I - (\mu_2 - \mu_1)A_1, 0] = (-1)^{m'}
\]

where \(m'\) is the sum of the multiplicities of the characteristic values of \((\mu_2 - \mu_1)A_1\) contained in \([0, 1]\) and hence the sum of the multiplicities of the characteristic values of \(A_1\) contained in \([0, \mu_2 - \mu_1]\). But the eigenvalues of \((L, A)\) are obtained by adding \(\mu_1\) to the characteristic values of \(A_1\), and their multiplicity is the multiplicity of the corresponding characteristic value of \(A_1\). Hence, \(m'\) is the sum \(m\) of the multiplicities of the eigenvalues of \((L, A)\) contained in \([\mu_1, \mu_2]\).

**Corollary 13.2.1** Assume that Assumption (ND) holds for \((L, A)\) and let \(\nu \in \sigma(L, A) \cap \mathbb{R}\). If \(i_B[L - (\nu-) A, 0]\) (resp. \(i_B[L - (\nu+) A, 0]\)) denotes the value of \(i_B[L - (\nu - \varepsilon) A, 0]\) (resp. \(i_B[L - (\nu + \varepsilon) A, 0]\)) for \(\varepsilon > 0\) such that \([\nu - \varepsilon, \nu + \varepsilon] \cap \sigma(L, A) = \{\nu\}\), then

\[
i_B[L - (\nu+) A, 0] = (-1)^{m(\nu)} i_B[L - (\nu-) A, 0]
\]

where \(m(\nu)\) denotes the multiplicity of \(\nu\).

**Proof.** Recall that \(\sigma(L, A)\) is finite and hence any eigenvalue is isolated, which insures the existence of \(\varepsilon\) above. On the other hand by Theorem 13.2.1, \(i_B[L - (\nu - \varepsilon) A, 0]\) and \(i_B[L - (\nu + \varepsilon) A, 0]\) are independent of \(\varepsilon\) as long as \([\nu - \varepsilon, \nu + \varepsilon] \cap \sigma(L, A) = \{\nu\}\), and the result follows from formula (13.8).
13.3 Bifurcation points

Let $U \subset \mathbb{R}$ be an open nonempty interval and $V \subset \mathbb{R}^n$ an open neighborhood of 0, and $F : U \times V \to \mathbb{R}^n$ be continuous and such that
\[ F(\mu, 0) = 0 \quad \text{for all } \mu \in U. \quad (13.10) \]
Consequently, equation
\[ F(\mu, x) = 0 \quad (13.11) \]
has, for any $\mu \in U$, the trivial solution $x = 0$. We are interested in finding values of $\mu$ for which a nontrivial solution exists, i.e. in the bifurcations of the solution set.

The following definition can be traced to H. Poincaré in his studies of the equilibrium of rotating fluid masses [322], where he used for the first time Kronecker’s index in bifurcation theory.

**Definition 13.3.1** The point $(\mu_0, 0) \in U \times V$ is a bifurcation point for the solutions of (13.11) with respect to $U \times \{0\}$ (briefly, a bifurcation point for (13.11)) if any neighborhood of $(\mu_0, 0)$ in $U \times V$ contains at least one zero $(\mu, x)$ of $F$ such that $x \neq 0$.

**Remark 13.3.1** If we take $F(\mu, x) = Lx - \mu Ax$ where $L, A : \mathbb{R}^n \to \mathbb{R}^n$ are linear, we see at once that any $(\mu_0, 0)$, where $\mu_0$ is an eigenvalue for $(L, A)$, is a bifurcation point of $Lx - \mu Ax = 0$. That is why bifurcation problems are sometimes also called nonlinear eigenvalue problems.

The following preliminary result is used in bifurcation theory.

**Lemma 13.3.1** Let $K \subset U$ be a nonempty compact interval such that $K \times \{0\} \neq \emptyset$ contains no bifurcation point for (13.11). Then there exists $r > 0$ such that for any $(\mu, x) \in K \times (V \cap B(r))$ satisfying (13.11), one has $x = 0$.

**Proof.** If it is not the case, there exists a sequence $(\mu_n, x_n)$ contained in $K \times (V \cap B(1/n))$ such that
\[ F(\mu_n, x_n) = 0 \quad \text{and} \quad x_n \neq 0 \quad (n = 1, 2, \ldots). \]
Going if necessary to a subsequence, we may assume that $\mu_n \to \mu_0 \in K$ and hence $(\mu_0, 0)$ is a bifurcation point for (13.11), a contradiction.

The following fundamental sufficient condition for the existence of a bifurcation point is essentially due to M.A. Krasnosel’skii [210, 212, 213]. It show how Brouwer’s index allows to detect bifurcation points.

**Theorem 13.3.1** Assume that $\mu_1 < \mu_2$ are two points of $U$ such that $i_B[F(\mu_1, \cdot), 0]$ and $i_B[F(\mu_2, \cdot), 0]$ are defined and different. Then there exists $\mu_0 \in [\mu_1, \mu_2]$ such that $(\mu_0, 0)$ is a bifurcation point for (13.11).
Proof. We prove the contrapositive form: if (13.11) has no bifurcation point on $[\mu_1, \mu_2] \times \{0\}$, then

$$i_B[F(\mu_1, \cdot), 0] = i_B[F(\mu_2, \cdot), 0].$$

From Lemma 13.3.1 with $K = [\mu_1, \mu_2]$, there exists $r > 0$ such that $\overline{B}(r) \subset V$ and $F(\mu, 0) = 0$, $(\mu, x) \in [\mu_1, \mu_2] \times \overline{B}(r) \Rightarrow x = 0$.

Define the homotopy $F : \overline{B}(r) \times [0, 1] \to \mathbb{R}^n$ by

$$F(x, \lambda) = F(x, (1 - \lambda)\mu_1 + \lambda\mu_2)$$

so that, for any $\lambda \in [0, 1]$ and any $(x\lambda) \in \partial B(r) \times [0, 1]$, one has $F(x, \lambda) \neq 0$. The homotopy invariance property 3.4.2 and the definition of the Brouwer index imply that

$$i_B[F(\mu_1, \cdot), 0] = i_B[F(\cdot, 0)] = i_B[F(\cdot, 1)] = i_B[F(\mu_2, \cdot), 0].$$

Remark 13.3.2 Although quite general and widely applied, the above sufficient condition is not necessary, in the class of continuous mappings, for the existence of a bifurcation point. In fact, if $F(\mu, x) = x - \mu x$ with $x \in \mathbb{R}^2$, $((0, 0), 1)$ is a bifurcation point. On the other hand, for any $\mu \neq 1$, $i_B[F(\mu, \cdot), 0] = \text{sign} \ (1 - \mu)^2 = 1$.

### 13.4 Bifurcation through linearization

Assume now that $F$ can be written as follows

$$F(\mu, x) = Lx - \mu Ax + R(\mu, x) \quad (13.12)$$

where $L, A : \mathbb{R}^n \to \mathbb{R}^n$ are linear and $R : U \times V \to \mathbb{R}^n$ is continuous and such that

$$\lim_{x \to 0} \frac{R(\mu, x)}{||x||} = 0 \quad (13.13)$$

uniformly on compact subintervals of $V$.

We first prove a necessary conditions for the existence of a bifurcation point $(\mu_0, 0)$ for equation

$$Lx - \mu Ax + R(\mu, x) = 0. \quad (13.14)$$

**Proposition 13.4.1** If $(\mu_0, 0)$ is a bifurcation point for equation (13.14) with $R$ satisfying (13.13), then $\mu_0$ is an eigenvalue for $(L, A)$.
Proof. We prove the contrapositive proposition. Let \( \mu_0 \in U \setminus \sigma(L, A) \) Then

\[
\|Lx - \mu Ax + R(\mu, x)\| = \|(L - \mu_0 A)x + (\mu_0 - \mu)Ax + R(\mu, x)\|
\geq \|(L - \mu_0 A)x\| - |\mu_0 - \mu|\|A\|\|x\| - |R(\mu, x)|.
\]

By our assumptions, there exists \( \alpha > 0 \) such that

\[
\|(L - \mu_0 A)x\| \geq \alpha\|x\| \quad \text{for all} \quad x \in \mathbb{R}^n.
\]

Consequently, if we first take \( \varepsilon_0 > 0 \) such that \([\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0] \subset U \) and

\[
\varepsilon_0\|A\| \leq \frac{\alpha}{3},
\]

and then \( r > 0 \) such that

\[
\|R(\mu, x)\| \leq \frac{\alpha}{3}\|x\| \quad \text{for all} \quad (\mu, x) \in [\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0] \times \mathcal{B}(r),
\]

we get, for \((\mu, x) \in [\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0] \times \mathcal{B}(r),\)

\[
\|Lx - \mu Ax + R(\mu, x)\| \geq \alpha\|x\| - \frac{\alpha}{3}\|x\| - \frac{\alpha}{3}\|x\| = \frac{\alpha}{3}\|x\|
\]

and \((\mu_0, 0)\) is not a bifurcation point for (13.14).

\[\blacksquare\]

**Remark 13.4.1** The condition of Proposition 13.4.1 is not sufficient for \((\mu_0, 0)\) to be a bifurcation point of (13.14). Indeed, taking \(L = A = I\) in \(\mathbb{R}^2\) and \(R(x_1, x_2) = (-x_2^3, x_1^3)\), we see that 1 is an eigenvalue for \((I, I)\) but the system of equations

\[
(1 - \mu)x_1 - x_2^3 = 0, \quad (1 - \mu)x_2 + x_1^3 = 0
\]

does not admit \((1, (0, 0))\) as a bifurcation point, because it has no bifurcation point. Indeed, if \((x_1, x_2, \mu)\) is a solution of (13.15), then, multiplying the first equation by \(-x_1\), the second one by \(x_2\) and adding, we obtain \(x_2^4 + x_1^4 = 0\), and hence \(x_1 = x_2 = 0\).

We prove now a sufficient condition for the existence of a bifurcation point for (13.14) essentially due to M.A. Krasnosel’skii [210, 212, 213].

**Theorem 13.4.1** If condition (ND) holds for \((L, A)\) and condition (13.13) for \(R\), and if \(\mu_0 \in U\) is an eigenvalue for \((L, A)\) having odd multiplicity \(m(\mu_0)\), then \((\mu_0, 0)\) is a bifurcation point for (13.14). Moreover, there exists \(r_0 > 0\) such that, for any \(r \in [0, r_0]\), equation (13.14) has at least one solution \((\mu_r, x_r)\) such that \(\|x_r\| = r\) and \(\mu_r \to 0\) if \(r \to 0\).

**Proof.** The eigenvalue \(\mu_0\) being isolated, there exists \(\varepsilon_0 > 0\) such that \([\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0] \subset U\) and \([\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0] \cap \sigma(L, A) = \{\mu_0\}\). Consequently,

\[
N[L - (\mu_0 - \varepsilon_0)A] = \{0\} = N[L - (\mu_0 + \varepsilon_0)A].
\]
Therefore, using Theorem 5.2.1 and Corollary 13.2.1, we obtain

\[ i_B[L - (\mu_0 + \varepsilon_0)A + R(\mu_0 + \varepsilon_0), 0] = \mu_0 [L - (\mu_0 + \varepsilon_0)A, 0] \]

\[ = (-1)^{m(\mu_0)} i_B[L - (\mu_0 - \varepsilon_0)A, 0] = -i_B[L - (\mu_0 - \varepsilon_0)A, 0] \]

\[ = -i_B[L - (\mu_0 + \varepsilon_0)A + R(\mu_0 + \varepsilon_0), 0] \]  \hspace{1cm} (13.16)

with all those indices having absolute value one. Hence all conditions of Theorem 13.3.1 hold, and \((0, \mu_0)\) is a bifurcation point for (13.14).

Now, let \(r_0 > 0\) be such that \(\overline{B}(r_0) \subset V\) and such that \(x = 0\) is the unique solution of the equations

\[ Lx - (\mu_0 \pm \varepsilon_0)Ax + R(\mu_0 \pm \varepsilon_0, x) = 0 \]

located in \(\overline{B}(r_0)\). Such a \(r_0\) exists by Lemma 13.3.1. Thus, if \(r \in ]0, r_0]\), we have, using (13.16),

\[ \text{d}_B[L - (\mu_0 - \varepsilon_0)A + R(\mu_0 - \varepsilon_0), B(r), 0] \]

\[ = \text{i}_L[L - (\mu_0 - \varepsilon_0)A + R(\mu_0 - \varepsilon_0), 0] \]

\[ = -\text{i}_L[L - (\mu_0 + \varepsilon_0)A + R(\mu_0 + \varepsilon_0), 0] \]

\[ = -\text{d}_B[L - (\mu_0 - \varepsilon_0)A + R(\mu_0 - \varepsilon_0), B(r), 0] \]

with all numbers having absolute value 1, so that the two degrees are unequal. From the contrapositive version of the homotopy invariance property 3.4.2, there must exist \(\mu_r \in [\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0]\) and \(x_r \in \partial B(r)\) such that

\[ Lx_r - \mu_r Ax_r + R(\mu_r, x_r) = 0. \]

Consequently, letting \(y_r = \frac{x_r}{\|x_r\|}\), we have

\[ Ly_r - \mu_r Ay_r + \|x_r\|^{-1}R(\mu_r, x_r) = 0. \]  \hspace{1cm} (13.17)

Hence, we can find a convergent subsequence \((\mu_{r_k}, y_{r_k})\) with limit \((\mu^*, y^*) \in [\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0] \times \partial B(1)\) when \(r_k \to 0\). From (13.17), we obtain

\[ Ly^* - \mu^* Ay^* = 0, \]

i.e. \(\mu^* \in [\mu_0 - \varepsilon_0, \mu_0 + \varepsilon_0]\) is an eigenvalue for \((L, A)\). From our construction, this implies that \(\mu^* = \mu_0^0\). Thus, any convergent subsequence of \((\mu_r)\) converges to \(\mu_0\), and the proof is complete. \(\blacksquare\)

**Remark 13.4.2** The proof of Theorem 13.4.1 also shows that every convergent subsequence of \((y_r)\) has for limit an element of \(N(L - \mu_0 A)\).

A direct consequence of Theorem 13.4.1, of Corollary 13.2.1 and of 13.1.3 is the following

**Corollary 13.4.1** If \(\mu_0\) is an eigenvalue for \((L, A)\) such that
Lemma 13.5.1

Proof. \[ (i) \ A(N(L - \mu_0 A)) \cap R(L - \mu_0 A) = \{0\} \]
\[ (i) \ dim \ N(L - \mu_0 A) \text{ is odd,} \]
then the conclusion of Theorem 13.4.1 holds.

A direct consequence of Theorem 13.4.1, of Corollary 13.2.1 and of 13.1.4 is the following

Corollary 13.4.2 If \( AL = AL \) and \( \mu_0 \) is an eigenvalue for \((L, A)\) such that
\[ \dim \bigcup_{n \geq 1} N[(L - \mu_0 A)^n] \text{ is odd,} \]
then the conclusion of Theorem 13.4.1 holds.

13.5 Global structure of bifurcation branches

The results of the previous sections give conditions to detect bifurcation points and some information about the set of solutions in the neighborhood of the corresponding bifurcation point. We now study the global structure, first obtained in 1971 by P. Rabinowitz [324, 325], following an approach introduced in 1976 by J. Ize [181] (see also [297]). We keep the notations and assumptions of the previous sections, and consider the problem (13.14) with \( \mu_0 \) be an eigenvalue for \((L, A)\) with odd multiplicity.

We need a preliminary lemma. Let \( \bar{X} = X \times \mathbb{R} \), with the norm \( \|(x, \nu)\| = (\|x\|^2 + |\nu|^2)^{1/2} \), and define, for any \( r > 0 \), the continuous mapping \( \tilde{F_r} : U \times (V - \mu_0) \to \bar{X} \) by
\[
\tilde{F_r}(x, \nu) = (F(\mu_0 + \nu, x), \|x\|^2 - r^2).
\]

Lemma 13.5.1 There exists \( \nu_0 > 0 \) and \( r_0 > 0 \) such that, for \( r \in [0, r_0] \),
\[
d_B[\tilde{F_r}, B([r^2 + \nu_0^2]^{1/2}), 0] = i_B[L - (\mu_0-)A, 0] - i_B[L - (\mu_0+)A, 0].
\]

Proof. Let \( \nu_0 > 0 \) be such that \( [\mu_0 - \nu_0, \mu_0 + \nu_0] \subset V \) and \( [\mu_0 - \nu_0, \mu_0 + \nu_0] \cap \sigma(L, A) = \{\mu_0\} \). By proceeding like in the proof of Proposition 13.4.1, there exists \( r_0 > 0 \) such that \( B(r_0) \subset U \) and such that, for any \( \lambda \in [0, 1] \), the only solution of the equations
\[
Lx - (\mu_0 \pm \nu_0) Ax + R(\mu_0, x) = 0
\]
such that \( \|x\| \leq r_0 \) is \( x = 0 \). For any \( r \in [0, r_0] \), let us define the continuous mapping \( \tilde{F_r} : V \times (v - \mu_0) \times [0, 1] \to \bar{X} \) by
\[
\tilde{F_r}(x, \nu, \lambda) = (Lx - (\mu_0 + \nu)A + \lambda R(\mu_0 + \nu, x), \lambda(\|x\|^2 - r^2) + (1 - \lambda)(\nu_0^2 - \nu^2)).
\]
If \( \tilde{F_r}(x, \nu, \lambda) = 0 \) for some \( \lambda \in [0, 1] \) and \((x, \nu) \in \partial B([r^2 + \nu_0^2]^{1/2}) \), we have
\[
\|x\|^2 - r^2 = \nu_0^2 - \nu^2, \quad \lambda(\|x\|^2 - r^2) + (1 - \lambda)(\nu_0^2 - \nu^2),
\]
and hence
\[ \|x\| = r \quad \text{and} \quad \nu = \pm \nu_0, \]
which is impossible by the choice of \( \nu_0 \) and \( r_0 \). Hence the homotopy invariance property of degree 3.4.2 implies that, for any \( r \in [0, r_0] \),
\[ d_B[\tilde{F}_r, B([r^2 + \nu_0^2]^{1/2}), 0] = d_B[\tilde{F}_r(\cdot, 1), B([r^2 + \nu_0^2]^{1/2}), 0] \]
\[ = d_B[\tilde{F}_r(\cdot, 0), B([r^2 + \nu_0^2]^{1/2}), 0]. \quad (13.18) \]
Now \( \tilde{F}_r(x, \nu, 0) = (Lx - (\mu_0 + \nu)A, \nu^2 - \nu^2) \) has only the two isolated zeros \( (0, -\nu_0) \) and \( (0, \nu_0) \), which, together with Proposition 5.1.1, implies that
\[ d_B[\tilde{F}_r(\cdot, \cdot, 0), B([r^2 + \nu_0^2]^{1/2}), 0] \]
\[ = i_B[\tilde{F}_r(\cdot, \cdot, 0), (0, -\nu_0)] + i_B[\tilde{F}_r(\cdot, \cdot, 0), (0, \nu_0)] \]
\[ = i_L[\tilde{F}_r(\cdot, \nu - \nu_0, 0), (0, 0)] + i_B[\tilde{F}_r(\cdot, \nu + \nu_0, 0), (0, 0)]. \quad (13.19) \]
Now,
\[ \tilde{F}_r(x\nu - \nu_0, 0) = (Lx - (\mu_0 - \nu_0 + \nu)Ax, 2\nu_0\nu - \nu^2) \]
and hence, by Theorem 5.2.1, we have
\[ i_B[\tilde{F}_r(\cdot, \nu_0, 0), (0, 0)] = i_B[\tilde{G}_r, (0, 0)] \]
\[ = i_B[\tilde{G}_r, (0, 0)] \quad (13.20) \]
where \( \tilde{G} : \tilde{X} \rightarrow \tilde{X} \) is defined by
\[ \tilde{G}(x, \nu) = (Lx - (\mu_0 - \nu_0)Ax, 2\nu_0(1 - \lambda) + \lambda \nu) \]
Defining now \( \tilde{G} : \tilde{X} \times [0, 1] \rightarrow \tilde{X} \) by
\[ \tilde{G}(x, \nu, \lambda) = (Lx - (\mu_0 - \nu_0)Ax, (2\nu_0(1 - \lambda) + \lambda \nu)), \]
and noticing that, for any \( \lambda \in [0, 1] \), the only zero of \( \tilde{G}(\cdot, \cdot, \lambda) \) is \( (0, 0) \), we obtain, by the homotopy invariance property 3.4.2,
\[ i_B[\tilde{G}_r, (0, 0)] = i_B[\tilde{G}(\cdot, 1), (0, 0)] \]
\[ \quad (13.21) \]
with \( \tilde{G}(x, \nu, 1) = (Lx - (\mu_0 - \nu_0)Ax, \nu) \). From the definition of degree, we obtain, where \( I \) is identity on \( \mathbb{R} \),
\[ i_B[\tilde{G}(\cdot, 1), (0, 0)] = i_B[L - (\mu_0 - \nu_0)A, 0] \quad (13.22) \]
Using relations (13.18) to (13.22), and proceeding similarly for the second term in (13.19), we finish the proof.
We can now state and prove Rabinowitz’ result on the global structure of the bifurcation branch emanating from \((0, \mu_0)\). Let us denote by \(S\) the closure of the set
\[
\{(\mu, x) \in U \times V : x \neq 0 \quad \text{and} \quad F(\mu, x) = 0\}
\]
and by \(C\) the connected component of \(S\) containing \((0, \mu_0, 0)\). The global bifurcation theorem provides information about the structure of \(C\).

**Theorem 13.5.1** Let \(\mu_0\) be an eigenvalue for \((L, A)\) with odd multiplicity. Then either

(i) \(C\) is not compact in \(U \times V\) (in case \(U \times V = \mathbb{R} \times \mathbb{R}^n\), this means that \(C\) is unbounded in \(\mathbb{R} \times \mathbb{R}^n\)), or

(ii) \(C\) contains a finite number of points \((\mu_j, 0)\) with \(\mu_j \in U\) an eigenvalue for \((L, A)\) \((j = 1, 2, \ldots, m)\). Furthermore, the number of those eigenvalues \(\mu_j\) having odd multiplicity is even.

**Proof.** Suppose \(C\) is compact in \(U \times V\). The set of eigenvalues for \((L, A)\) being finite, \(C\) contains a finite number of points of the form \((\mu_j, 0)\) \((j = 0, 1, \ldots, m)\) with \(\mu_j\) an eigenvalue for \((L, A)\). \(C\) being a connected compact subset of \(U \times V\), Corollary 7.2.1 implies the existence of a bounded set \(D\), open in \(U \times V\) (and hence in \(\mathbb{R} \times \mathbb{R}^n\)) such that, on \(\partial D\), \(F\) has no zero \((\mu, x)\) with \(x \neq 0\), and such that \(D\) contains no other point of the form \((\mu, 0)\), with \(\mu\) an eigenvalue for \((L, A)\), than the \((\mu_j, 0)\). For any \(r > 0\), define the continuous mapping \(F_r : \bar{D} \rightarrow \mathbb{R}^n \times \mathbb{R}\) by
\[
F_r(x, \mu) = (F(\mu, x), \|x\|^2 - r^2),
\]
Since \(F_r(x, \mu) = 0\) implies \(\|x\| = r\), \(d_B[F_r, D, 0]\) is well defined. Furthermore, given \(0 < r_1 < r_2\), let us introduce the homotopy \(\mathcal{F} : \mathcal{D} \times [0, 1] \rightarrow \mathbb{R}^n \times \mathbb{R}\) by
\[
\mathcal{F}(x, \mu, \lambda) = (F(\mu, x), \|x\|^2 - (1 - \lambda)r_1^2 - \lambda r_2^2).
\]
Again, \(\mathcal{F}(x, \mu, \lambda) \neq 0\) for any \((x, \mu, \lambda) \in \partial \mathcal{D} \times [0, 1]\), and, using the homotopy invariance property, we obtain
\[
d_B[F_r, D, 0] = d_B[F(0, \cdot, \cdot), D, 0] = d_B[F(0, \cdot, \cdot), D, 1] = d_B[F_{r_2}, D, 0],
\]
i.e. \(d_B[F_r, D, 0]\) is independent of \(r\). But, if we take \(r > 0\) such that \(D \subset B(r) \times \mathbb{R}\), then \(F_r\) has no zero in \(\bar{D}\), which implies, by the existence property 3.4.1 that
\[
d_B[F_r, D, 0] = 0 \quad \text{for any} \quad r > 0. \tag{13.23}
\]
Now, by Proposition 13.4.1 and Lemma 13.3.1, for any \(\varepsilon > 0\), there exists \(r_{\varepsilon} > 0\) such that \(\mu \in U \setminus \bigcup_{j=0}^{m} [\mu_j - \varepsilon, \mu_j + \varepsilon], x \in \bar{D} \cap \overline{B(r)}\) and \(F(\mu, x) = 0\) imply \(x = 0\). Consequently, if \((x, \mu)\) is a zero of \(F_r\) such that \(0 < r \leq r_{\varepsilon}\), then \(\mu \in [\mu_j - \varepsilon, \mu_j + \varepsilon]\) for some \(j \in \{0, 1, \ldots, m\}\), so that, using the additivity-excision property of degree,
13.6. AN APPLICATION TO DIFFERENCE EQUATIONS

the definition of Brouwer’s index, and Lemma 13.5.1, we obtain, by taking $\varepsilon$ and $r$ sufficiently small

$$0 = d_B[F_r, D, 0] = \sum_{j=0}^{m} (i_B[L - (\mu_j -)A, 0] - i_B[L - (\mu_j +)A, 0]).$$

Since, by Corollary 13.2.1, one has

$$i_B[L - (\mu_j +)A, 0] = (-1)^{m(\mu_j)}i_B[L - (\mu_j -)A, 0],$$

the only contributions in the sum above come from the $\mu_j$ having an odd multiplicity, there must be an even number of them. □

**Example 13.5.1** The example of $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$F(\mu, x) = x - \mu x + x^2$$

demonstrates that the first alternative of Theorem 13.5.1 may occur, with the unbounded branch $x = \mu - 1$ emanating from the simple eigenvalue $\mu = 1$. The example, given in [149], of $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(\mu, x_1, x_2) = (x_1 - \mu x_1^3 + 2x_2 - \mu x_2^3 - x_1^3),$$

for which $\mu_1 = 1$ and $\mu_2 = 2$ are simple eigenvalues and which has a pair of non-trivial solutions given by

$$(x_1, x_2) = \left[ \pm(\mu - 1)^{1/8}(2 - \mu)^{3/8}, \pm(2 - \mu)^{1/8}(\mu - 1)^{3/8} \right] \quad (\mu \in [1, 2])$$

shows that the alternative (ii) may occur. An appropriate combination of the examples shows that (i) and (ii) can occur simultaneously.

13.6 An application to difference equations

Keeping the notations of Chapter 12, let us consider the two-point boundary value problem

$$\Delta^2 x_{m-1} - \mu x_m + f_m(\mu, x_m) = 0 \quad (m = 1, \ldots, n - 1), \quad x_0 = x_n = 0,$$  \hspace{1cm} (13.24)

where $n \geq 2$ is a fixed integer and each $f_m : \mathbb{R}^2 \to \mathbb{R}$ is continuous $(m = 1, \ldots, n - 1)$ and such that

$$\lim_{y \to 0} \frac{f_m(\mu, y)}{|y|} = 0$$  \hspace{1cm} (13.25)

uniformly on compact $\mu$-intervals. As shown in Chapter 12, this problem can be written as the equation in $V^{n-1} \simeq \mathbb{R}^{n-1}$

$$Lx - \mu x + f(\mu, x) = 0$$  \hspace{1cm} (13.26)
where $L$ is the Jacobi matrix given in (12.4), $x = (x_1, \ldots, x_{n-1})$ and $f(\mu, x) = (f_1(\mu, x_1), \ldots, f_{n-1}(\mu, x_{n-1}))$.

As shown in Lemma 12.2.1, the linear problem

$$L x - \mu x = 0 \quad (13.27)$$

has nontrivial solutions if and only if

$$\mu = \mu_k := 2 \cos \frac{k\pi}{n} - 2 \quad (k = 1, \ldots, n - 1)$$

and the corresponding eigenvectors $v^k$ are multiple of

$$\varphi^k := \left( \sin \frac{k\pi}{n}, \sin \frac{2k\pi}{n}, \ldots, \sin \frac{(n-1)k\pi}{n} \right).$$

and the sequence of components of each $\varphi^k$ changes sign exactly $k - 1$ times. So define

$$S_k^+ = \{ v = (v_1, \ldots, v_k) \in V^{n-1} : v_1 > 0 \text{ and } (v_1, \ldots, v_k) \text{ changes sign } k - 1 \text{ times} \} \quad (13.28)$$

$$S_k^- = -S_k^+, \quad S_k = S_k^+ \cup S_k^- \quad (k = 1, \ldots, n - 1) \quad (13.29)$$

It is clear that $S_k^+, S_k^-$ and $S_k$ are open in $V^{n-1}$, that $\alpha S_k = S_k$ for any $\alpha \in \mathbb{R} \setminus \{0\}$, and that any $v \in S_k$, provides a unique $w \in S_k^+$ such that $\|w\| = 1$.

Let $S$ denote the closure of the set of $(\mu, x)$ such that $Lx - \mu x + f(\mu, x) = 0$ and $x \neq 0$.

**Lemma 13.6.1** If $\mu_k \in \sigma(L)$, there exists a neighborhood $N_k$ of $(\mu_k, 0)$ in $\mathbb{R} \times V^{n-1}$ such that if $(\mu, x) \in S \cap N_k$, then $x \in S_k$.

**Proof.** If it were not the case, there would exist a sequence $(\alpha_j, x_j)$ in $S$ converging to $(\mu_k, 0)$ and such that $x_j \notin S_k$. Then,

$$L \left( \frac{x_j}{\|x_j\|} \right) - \alpha_j \frac{x_j}{\|x_j\|} + \frac{f(\alpha_j, x_j)}{\|x_j\|} = 0 \quad (j \in \mathbb{N}), \quad (13.30)$$

and, going if necessary to a subsequence, we can assume that $(\frac{x_j}{\|x_j\|})$ converges to some $v$ with $\|v\| = 1$. Consequently, going to the limit in (13.30) and using (13.25), we obtain

$$Lv - \mu_k v = 0, \quad \|v\| = 1,$$

and hence $v \in S_k$. Consequently, there exists $j_0$ such that $\frac{x_j}{\|x_j\|} \in S_k$ for $j \geq j_0$, which implies that $x_j \in S_k$ for $j \geq j_0$, a contradiction. $\blacksquare$

We can now describe the structure of the bifurcation branches of (13.26).

**Theorem 13.6.1** If condition (13.25) holds, then, for each $k \in \{1, \ldots, n - 1\}$, equation (13.26) possesses a connected component of solutions $C_k$ in $\mathbb{R} \times V^{n-1}$ with $C_k \subset (\mathbb{R} \times S_k) \cup \{(\mu_k, 0)\}$, and $C_k$ is unbounded in $\mathbb{R} \times V^{n-1}$.
Proof. First notice that each eigenvalue $\mu_k$ has multiplicity one. Hence, for each $k = 1, 2, \ldots, n-1$, Theorem 13.5.1 implies the existence of a connected component $C_k$ of $S$ satisfying the alternative of the conclusion of this theorem. By Lemma 13.6.1, there exists a neighborhood $N_k$ of $(\mu_k, 0)$ such that

$$C_k \cap N_k \subset (\mathbb{R} \times S_k) \cup \{ (\mu_k, 0) \}. \quad (13.31)$$

If

$$C_k \subset (\mathbb{R} \times S_k) \cup \{ (\mu_k, 0) \}, \quad (13.32)$$

then, by Theorem 13.5.1, it is either unbounded in this set, or it contains $(\mu_j, 0)$ for some $j \neq k$. But, by Lemma 13.6.1 again, any nontrivial solution of (13.26) near $(\mu_j, 0)$ lies in $\mathbb{R} \times S_j$. Since $S_j \cap S_k = \emptyset$ when $j \neq k$, the validity of (13.32) shows that $C_k$ cannot contain $(\mu_j, 0)$ for $j \neq k$, and hence must be unbounded in $(\mathbb{R} \times S_k) \cup \{ (\mu_k, 0) \}$.

To verify (13.32), notice that (13.31) shows that (13.32) holds for that part of $C_k$ in $N_k$. If we assume it does not hold globally, there is some $(\mu, x) \in \partial (\mathbb{R} \times S_k) \setminus C_k$ such that $(\mu, x) \neq (\mu_k, 0)$. Hence $x \in \partial S_k$. Now, $x \in \partial S_k$ if and only if there exists at least one $j \in \{2, \ldots, n-2\}$.
14.1 Initial value and periodic problems

Let $T > 0$ be fixed, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, (t, y) \mapsto f(t, y)$ be locally Lipschitzian in $y$ and continuous. Then it is well known that for each $(s, x) \in \mathbb{R} \times \mathbb{R}^n$, there exists a unique solution $y(t; s, x)$ of the Cauchy problem

$$ y'(t) = f(t, y(t)), \quad y(s) = x, $$

which is defined on a maximal interval $[\tau_-(s, x), \tau_+(s, x)]$, for some

$$ -\infty \leq \tau_-(s, x) < s < \tau_+(s, x) \leq +\infty. $$

Moreover, $y$ is continuous on the set

$$ G = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : \tau_-(s, x) < t < \tau_+(s, x)\} $$
on which it is defined, and $G$ is open.

A T-periodic solution of the differential equation

$$ y'(t) = f(t, y(t)) \tag{14.1} $$
is a solution of (14.1) defined over $[0, T]$ and such that

$$ y(0) = y(T). $$

If we assume in addition that the function $f$ is T-periodic with respect to $t$, i.e. such that

$$ f(t, y) = f(t + T, y) $$
for all $t \in \mathbb{R}$, then a T-periodic solution of (14.1) can be continued as a solution defined over $\mathbb{R}$ and such that

$$ y(t) = y(t + T) $$
for all $t \in \mathbb{R}$. As already observed by Poincaré [323] at the end of the XIXth century, $y(t; 0, x)$ will be a $T$-periodic solution of (14.1) if and only if $x \in \mathbb{R}^n$ is such that $\tau_+(0, x) > T$ and

$$x = y(T; 0, x),$$

i.e. $x$ is a fixed point of the Poincaré operator $P_T$ defined by

$$P_T(x) = y(T; 0, x).$$

### 14.2 Bounded perturbations of some linear systems

Let $A : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $h : \mathbb{R} \to \mathbb{R}^n$ be continuous and $T$-periodic and consider the linear system

$$y' = A(t)y + h(t).$$

(14.2)

By the variation of constants formula, the solution of eq. (14.2) such that $y(0) = x$ is given by

$$y(t) = Y(t)x + \int_0^t Y(t)Y^{-1}(s)h(s)\, ds,$$

(14.3)

where $Y(t)$ is the fundamental matrix solution of equation $y' = A(t)y$, i.e. the solution of the problem

$$Y' = A(t)Y, \quad Y(0) = I.$$

The homogeneous equation

$$y' = A(t)y$$

(14.4)

has a non-trivial $T$-periodic solution if and only if one can find $x \neq 0$ such that

$$[Y(T) - I]x = 0,$$

i.e. if and only if $\det [I - Y(T)] = 0$. On the other hand, the initial conditions $x$ of the possible $T$-periodic solutions of eq. (14.2) are the solutions of the linear system in $\mathbb{R}^n$

$$[I - Y(T)]x = \int_0^T Y(T)Y^{-1}(s)h(s)\, ds,$$

(14.5)

and such a non-homogeneous linear system will have a solution for any continuous $T$-periodic $h$ if and only if $\det [I - Y(T)] \neq 0$. In other words, we have proved the following result.
Proposition 14.2.1 The linear differential system (14.2) has a T-periodic solution for each continuous T-periodic h if and only if the homogeneous system (14.4) only has the trivial T-periodic solution, in which case the T-periodic solution of (14.2) is unique.

Let now \( g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be continuous, T-periodic in \( t \), locally Lipschitzian in \( x \), and consider the nonlinear differential system
\[
y' = Ay + g(t, y).
\] (14.6)

The following result is a nonlinear extension of Proposition 14.2.1.

Theorem 14.2.1 If system (14.4) only has the trivial T-periodic solution and if there exists \( M > 0 \) such that
\[
\|g(t, x)\| \leq M \quad \text{for all} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,
\] (14.7)
then system (14.6) has at least one T-periodic solution.

Proof. It follows from condition (14.7) and classical results on the Cauchy problem that all the solutions of eq. (14.6) exist over \( \mathbb{R} \). Using the variation of constants formula (14.3), we see that the solution \( y(t; 0, x) \) of eq. (14.6) such that \( y(0) = x \) satisfies the integral equation
\[
y(t; 0, x) = Y(t)x + \int_0^t Y(t)Y^{-1}(s)g(s, y(s; 0, x)) \, ds, \quad (t \in \mathbb{R}).
\]
Hence, \( y(t; 0, x) \) is a T-periodic solution of eq. (14.6) if and only if \( x \) verifies the nonlinear equation in \( \mathbb{R}^n \)
\[
x = \left[ I - Y(T) \right]^{-1} \int_0^T Y(T)Y^{-1}(s)g(s, y(s; 0, x)) \, ds := G(x).
\] (14.8)
\( G : \mathbb{R}^n \to \mathbb{R}^n \) is continuous and such that, for all \( x \in \mathbb{R}^n \),
\[
\|G(x)\| \leq \left\| \left[ I - Y(T) \right]^{-1} Y(T) \right\| M \int_0^T \|Y^{-1}(s)\| \, ds := R,
\]
and hence \( G \) continuously maps \( \mathbb{R}^n \) into the closed ball \( \overline{B}(r) \subset \mathbb{R}^n \). By Brouwer’s fixed point theorem 8.1.3, \( G \) has a fixed point in \( \overline{B}(r) \).

Remark 14.2.1 The conclusion of Theorem 14.2.1 may not hold if the linear system (14.4) has a non-trivial solution, as show by the elementary example
\[
y' = 1,
\]
which corresponds to \( n = 1 \), \( A(t) \equiv 0 \) and \( g(t, y) \equiv 1 \), and which obviously has no T-periodic solution for any \( T > 0 \).
14.3 Stampacchia’s method

Guido Stampacchia has introduced in 1947 [374, 375] an interesting approach to study boundary value problems for ordinary differential systems, based upon finite-dimensional techniques. Like in the previous section, let \( A : [0, T] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) and \( g : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be continuous, and let \( B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), \) and \( d \in \mathbb{R}^n \). Consider the differential system

\[
y'(t) = A(t)y(t) + g(t, y(t)) \quad (t \in [0, T])
\]

with the two-point boundary condition

\[
By(0) + Cy(T) = d.
\]  

Notice that, for \( B = -C = I, \ d = 0 \) (14.10) reduces to the periodic boundary conditions, and for \( B = C = I, \ d = 0 \) to the anti-periodic boundary condition.

Using the variation of constants formula, if \( Y \) denotes the fundamental matrix solution of the linear homogeneous system

\[
y'(t) = A(t)y(t),
\]

the solution of system (14.9) such that \( y(0) = a \in \mathbb{R}^n \) satisfies the nonlinear integral equation

\[
y(t) = Y(t)a + \int_0^t Y(t)Y^{-1}(s)g(s, y(s))\,ds,
\]

for all \( t \geq 0 \) for which it exists. Stampacchia’s idea, which is an adaptation of an earlier idea of L. Tonelli [388] for proving the existence of a solution to Cauchy’s problem, consists in defining the sequence of mappings \( y_k : [-\frac{T}{k}, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) \((k \geq 1)\) as follows

\[
y_k(t, a) = \begin{cases} 
  a & \text{if } -\frac{T}{k} \leq t \leq 0 \\
  Y(t)a + \int_0^t Y(t)Y^{-1}(s)g(s, a)\,ds & \text{if } 0 < t \leq k^{-1}T \\
  Y(t)a + \int_0^t Y(t)Y^{-1}(s)g(s, y_k(s - k^{-1}T, a))\,ds & \text{if } (j-1)k^{-1}T < t \leq jk^{-1}T,
\end{cases}
\]

where \( j = 2, \ldots, k \). By construction, each \( y_k \) is continuous on \([-k^{-1}T, T] \times \mathbb{R}^n\), and satisfies the identity

\[
y_k(t, a) = Y(t)a + Y(t) \int_0^t Y^{-1}(s)g(s, y_k(s - k^{-1}T, a))\,ds \quad (t \in [0, T]).
\]

Furthermore,

\[
y_k'(t, a) = Y'(t)a + Y'(t) \int_0^t Y^{-1}(s)g(s, y_k(s - k^{-1}T, a))\,ds + Y(t)Y^{-1}(t)g(t, y_k(t - k^{-1}T, a)) = A(t)y_k(t, a) + g(t, y_k(t - k^{-1}T, a)).
\]

We can now prove the following existence theorem.
14.3. STAMPACCHIA’S METHOD

Theorem 14.3.1 Assume that the following conditions hold.

(i) \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) is bounded, namely there exists \( M > 0 \) such that \( \| f(t, y) \| \leq M \) for all \( (t, y) \in [0, T] \times \mathbb{R}^n \).

(ii) \( B + CY(T) \) is nonsingular.

Then problem (14.9)-(14.10) has at least one solution.

Proof. Let \( (y_k) \) be the sequence of functions defined by Stampacchia’s algorithm (14.12). We first show that for each \( k \geq 1 \), there exists at least one \( a_k \in \mathbb{R}^n \) such that

\[
By_k(0, a_k) + Cy_k(T, a_k) = d. \quad (14.15)
\]

Using formula (14.12), condition (14.15) can be written

\[
Ba_k + C \left[ Y(T)a_k + \int_0^T Y(T)Y^{-1}(s)g(s, y_k(s - k^{-1}T, a_k)) \right] = d,
\]

or, using Assumption (ii),

\[
a_k = (B + CY(T))^{-1} \left[ d - \int_0^T Y(T)Y^{-1}(s)g(s, y_k(s - k^{-1}T, a_k)) \right]. \quad (14.16)
\]

The mapping \( F_k : \mathbb{R}^n \to \mathbb{R}^n \) defined by the right-hand member of (14.16) is continuous, and it follows easily from Assumption (i) that it is bounded, by a constant \( R > 0 \) independent of \( k \). So, Brouwer fixed point theorem 8.1.3 implies the existence of at least one fixed point \( a_k^* \in \overline{B}(R) \).

Next, we prove that the family \( \{y_k(\cdot, a_k^*) : k = 1, 2, \ldots \} \) is relatively compact in \( C([0, T], \mathbb{R}^n) \), or equivalently, using Ascoli-Arzelà’s theorem, is bounded and equicontinuous. The boundedness is a direct consequence of Stampacchia’s construction of \( y_k \) and of Assumption (i). For the equicontinuity, it follows from (14.14) and Assumption (i) that there exists \( S > 0 \) such that \( \|y_k(t, a_k^*)\| \leq S \) for all \( k = 1, 2, \ldots \). Thus, going ifbju necessary to a subsequence, we can assume that \( (y_k(\cdot, a_k^*)) \) converges uniformly on \([0, T]\) to some \( y^* \in C([0, T], \mathbb{R}^n) \). In particular, \( a_k^* = y_k(0, a_k^*) \to y^*(0) \) if \( k \to \infty \). From relations (14.15), we deduce that

\[
By^*(0) + Cy^*(T) = d,
\]

i.e. that \( y^* \) satisfies the boundary condition (14.10). On the other hand, using identity (14.13) and Lebesgue dominated convergence theorem, we obtain

\[
y^*(t) = Y(t)y^*(0) + \int_0^t Y^{-1}(s)g(s, y^*(s)) \, ds \quad (t \in [0, T]),
\]

which means that \( y^* \) is of class \( C^1 \) and is solution of (14.9). \( \blacksquare \)
CHAPTER 14. POINCARÉ’S OPERATOR

Remark 14.3.1 Elementary considerations of linear algebra and the variation of constant formula immediately imply that assumption (ii) holds if and only if the linear boundary value problem

\[ y'(t) = A(t)y(t), \quad By(0) + Cy(T) = 0 \]  

only has the trivial solution.

Remark 14.3.2 When \( B = -C = I \) and \( d = 0 \), Theorem 14.3.1 generalizes Theorem 14.2.1 by suppression the assumption of Lipschitz continuity upon \( g \).

Corollary 14.3.1 For any continuous and bounded \( g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \), and any \( d \in \mathbb{R}^n \), the problem

\[ y'(t) = g(t, y(t)), \quad y(0) + y(T) = d \]  

has at least one solution. This is in particular the case for the anti-periodic problem \( y(0) + y(T) = 0 \).

Corollary 14.3.2 For any continuous and bounded \( g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \), and any \( a \neq 0 \), the problem

\[ y'(t) = ay(t) + g(t, y(t)), \quad y(0) = y(T) \]  

has at least one solution.

14.4 The method of lower and upper solutions

Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be continuous, \( c \in \mathbb{R} \) and consider the periodic boundary value problem

\[ x'' + cx' = f(t, x), \quad x(0) = x(T), \quad x'(0) = x'(T). \]  

(14.18)

The following concept, which can be traced to Oskar Perron, was introduced in the present setting by G. Scorza-Dragoni [354] and by M. Nagumo [291]. See [86] for the development of many aspect of lower and upper solutions.

Definition 14.4.1 We say that \( \alpha \in C^2([0, T], \mathbb{R}) \) (resp. \( \beta \in C^2([0, T], \mathbb{R}) \)) is a lower solution (resp. upper solution) for problem (14.18) if

\[ \alpha''(t) + c\alpha'(t) \geq f(t, \alpha(t)) \quad (t \in [0, T]), \quad \alpha(0) = \alpha(T), \quad \alpha'(0) \geq \alpha'(T). \]  

(14.19)

(resp.

\[ \beta''(t) + c\beta'(t) \leq f(t, \beta(t)) \quad (t \in [0, T]), \quad \beta(0) = \beta(T), \quad \beta'(0) \leq \beta'(T). \]

(14.20)

The lower (resp. upper) solution is called strict if strict inequalities hold in the differential inequality part of the definition.
The existence of an ordered couple of lower and upper solutions implies the existence and localization of a solution. For periodic problems, this result was first proved by H.W. Knobloch [204] in 1963 when \( f \) is locally Lipschitzian in \( x \). The proof given here is taken from [267].

**Theorem 14.4.1** If problem (14.18) admits a lower solution \( \alpha \) and an upper solution \( \beta \) such that \( \alpha(t) \leq \beta(t) \) for all \( t \in [0, T] \), then it has at least one solution \( x \) such that

\[
\alpha(t) \leq x(t) \leq \beta(t) \quad (t \in [0, T]).
\]

Furthermore, if \( \alpha \) and \( \beta \) are strict, then \( \alpha(t) < x(t) < \beta(t) \) for all \( t \in [0, T] \).

**Proof.** We first introduce a modified problem which coincides with (14.18) when \( \alpha(t) \leq x(t) \leq \beta(t) \) \( (t \in [0, T]) \). Define \( \gamma \in C([0, T] \times \mathbb{R}) \) by

\[
\gamma(t, x) = \begin{cases} 
\alpha(t) & \text{if } x < \alpha(t) \\
x & \text{if } \alpha(t) \leq x \leq \beta(t) \\
\beta(t) & \text{if } x > \beta(t).
\end{cases}
\]

It is clear that \( \gamma \) is bounded. Consider the modified problem

\[
x'' + cx' - x = f(t, \gamma(t, x)) - \gamma(t, x), \quad x(0) = x(T), \quad x'(0) = x'(T). \tag{14.22}
\]

We show that any possible solution \( x \) of (14.22) satisfies (14.21), and hence is a solution of (14.18). We show the first inequality, the second one being similar. It the left inequality does not hold, then \( \max_{[0, T]}(\alpha - x) > 0 \), and hence, if it is reached at \( t^* \in [0, T] \), one has

\[
\alpha(t^*) > x(t^*), \quad \alpha'(t^*) = x'(t^*),
\]

and

\[
\alpha''(t^*) \leq x''(t^*) = -cx'(t^*) + x(t^*) + f(t, \gamma(t^*, x(t^*))) - \gamma(t^*, x(t^*)) = -c\alpha'(t^*) + x(t^*) + f(t^*, \alpha(t^*)) - \alpha(t^*) < -c\alpha'(t^*) + f(t, \alpha(t^*)),
\]

a contradiction with the definition of lower solution. If \( \max_{[0, T]}(\alpha - x) \) is reached at 0 or \( T \), then, by periodicity, it is reached at 0 and \( T \), so that

\[
\alpha'(0) - x'(0) \leq 0 \leq \alpha'(T) - x'(T) \leq \alpha'(0) - x'(0),
\]

i.e.

\[
\alpha'(0) - x'(0) = 0 = \alpha'(T) - x'(T).
\]

Hence,

\[
\alpha''(0) \leq x''(0) = -cx'(0) + x(0) + f(t, \gamma(0, x(0))) - \gamma(0, x(0)) = -c\alpha'(0) + x(0) + f(0, \alpha(0)) - \alpha(0) < -c\alpha'(0) + f(t - 0, \alpha(0)),
\]
a contradiction with the definition of lower solution. The case of strict lower and upper solutions is treated similarly.

Hence, it remains to prove that the modified problem (14.22) has at least one solution. If we write it in system form by letting \( y_1 = x, y_2 = x' \), we get

\[
y_1' = y_2, \quad y_2' = -cy_2 + y_1 + f(t, \gamma(t, y_1)) - \gamma(t, y_1),
\]

\[
y_1(0) - y_1(T) = 0, \quad y_1'(0) - y_1'(T) = 0.
\]

(14.23)

which is of the type (14.9)-(14.10) with \( n = 2 \),

\[
A(t) = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}, \quad g(t, y_1, y_2) = \begin{pmatrix} 0 \\ f(t, \gamma(t, y_1)) - \gamma(t, y_1) \end{pmatrix},
\]

\[
B = -C = I, \quad d = 0.
\]

Furthermore, \( g \) is bounded over \([0, T] \times \mathbb{R}^2\), and the linear problem

\[
y_1' = y_2, \quad y_2' = -cy_2, \quad y_1(0) - y_1(T) = 0, \quad y_1'(0) - y_1'(T) = 0
\]

has only the trivial solution, as matrix

\[
\begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}
\]

has real nonzero eigenvalues. Hence, the existence of a solution follows from Stampacchia’s theorem 14.3.1.

\[ \blacksquare \]

### 14.5 Strict lower and upper solutions

Consider problem (14.18) and assume that \( f \) is locally Lipschitzian with respect to \( x \). The following result can be traced, in different versions, to [266] and [305].

**Theorem 14.5.1** Assume that (14.18) admits an ordered couple of strict lower and upper solutions \( \alpha \) and \( \beta \), let \( M > 0 \) be defined by

\[
M = \max_{(t,u) \in [0,T] \times \mathbb{R}} |f(t, \gamma(t, u)) - \gamma(t, u)|.
\]

(14.24)

and let \( P_T \) be Poincaré’s operator associated to problem (14.18). Then,

\[
d_B[I - P_T, \alpha(0), \beta(0)[\times] - TM, TM[, 0] = -1.
\]

(14.25)

**Proof.** Necessarily, \( \alpha(t) < \beta(t) \) for all \( t \in [0, T] \) and the solutions of the modified problem (14.22) are all contained in the set of \( T \)-periodic functions such that

\[
\alpha(t) < x(t) < \beta(t) \quad (t \in [0, T]).
\]
14.5. STRICT LOWER AND UPPER SOLUTIONS

In particular, their initial positions are such that \( \alpha(0) < x(0) < \beta(0) \). If \( x \) is constant, that is all we need. If \( x \) is not constant, multiplying both members of (14.22) by \( x'' \) and integrating the result over \([0,T]\) gives, using T-periodicity

\[
\int_0^T |x''(t)|^2 \, dt + \int_0^T |x'(t)|^2 \, dt = \int_0^T [f(t, \gamma(t, x(t)))] x''(t) \, dt
\]

and hence, using Cauchy-Schwarz inequality,

\[
\left( \int_0^T |x''(t)|^2 \, dt \right)^{1/2} < \sqrt{T} M
\]

where \( M \) is given by (14.24). Consequently, using the fact that \( x' \) must vanish at some \( \tau \in [0,T] \),

\[
|x'(t)| \leq \sqrt{T} \left( \int_0^T |x''(t)|^2 \, dt \right)^{1/2} < T M
\]

for all \( t \in [0,T] \). In particular, the initial velocities are such that \( |x'(0)| < TM \). Therefore, if we consider any bounded open set \( D \) in \( \mathbb{R}^2 \) containing \( \alpha(0), \beta(0) \times -TM, TM \) the Brouwer degree \( d_B[I - P_T, D, 0] \), where \( P_T \) is Poincaré’s operator associated to (14.22), is well defined. To compute it, let us consider the family of T-periodic problems

\[
x'' + cx' - x = \lambda[f(t, \gamma(t, x)) - \gamma(t, x)], \quad x(0) = x(T), \quad x'(0) = x'(T), \quad (14.26)
\]

indexed by \( \lambda \in [0, 1] \), which reduce to (14.22) for \( \lambda = 1 \). It can be written, equivalently, as

\[
x(t) = \lambda \int_0^T G(t, s)[f(s, \gamma(s, x(s))) - \gamma(s, x(s))] \, ds,
\]

where \( G(t, s) \) is the Green matrix of the associated linear problem

\[
x'' + cx' - x = h(t), \quad x(0) = x(T), \quad x'(0) = x'(T),
\]

so that, we also have

\[
x'(t) = \lambda \int_0^T G'_t(t, s)[f(s, \gamma(s, x(s))) - \gamma(s, x(s))] \, ds,
\]

and consequently,

\[
|x(t)| < \Gamma_1 M, \quad |x'(t)| < \Gamma_2 M,
\]

where the \( \Gamma_j \) only depend upon the Green matrix \( G \). In other words, the initial conditions \((x(0), x'(0))\) of any possible solutions of (14.26) are contained in the
open bounded set \( | - \Gamma_1 M, \Gamma_1 M [\times | - \Gamma_2 M | \times | \Gamma_2 M | . \) If \( P_T (\cdot, \lambda) \) denotes Poincaré’s operator associated to (14.26), the homotopy invariance property 3.4.2 of Brouwer degree implies that, for any bounded open set \( D \supset | - \Gamma_1 M, \Gamma_1 M [\times | - \Gamma_2 M | \times | \Gamma_2 M | \) one has

\[
d_B [I - P_T (\cdot, 1), D, 0] = d_B [I - P_T (\cdot, 0), D, 0].
\]

Now, for \( \lambda = 0 \), (14.26) reduces to the linear problem

\[
x'' + cx' - x = 0, \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{14.27}
\]

and hence \( P_T (\cdot, 0) \) is linear. Consequently, if \( \lambda_1 = -\frac{c}{2} - \sqrt{\frac{c^2}{4} + 1} \) and \( \lambda_2 = -\frac{c}{2} + \sqrt{\frac{c^2}{4} + 1} \) are the characteristic values of (14.27), elementary computations show that the solution of Cauchy’s problem

\[
x'' + cx' - x = 0, \quad x(0) = y_1, \quad x'(0) = y_2
\]

is given by

\[
x(t) = (\lambda_1 - \lambda_2)^{-1} \left[ (-\lambda_2 y_1 + y_2) e^{\lambda_1 t} + (\lambda_1 y_1 - y_2) e^{\lambda_2 t} \right],
\]

so that

\[
x'(t) = (\lambda_1 - \lambda_2)^{-1} \left[ \lambda_1 (-\lambda_2 y_1 + y_2) e^{\lambda_1 t} + \lambda_2 (\lambda_1 y_1 - y_2) e^{\lambda_2 t} \right],
\]

and

\[
[I - P_T (\cdot, 0)](y_1, y_2) = (\lambda_1 - \lambda_2)^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} \lambda_1 - \lambda_2 + \lambda_2 e^{\lambda_1 T} - \lambda_1 e^{\lambda_2 T} \\ -\lambda_1 \lambda_2 (-e^{\lambda_1 T} + e^{\lambda_2 T}) \end{pmatrix} \begin{pmatrix} e^{\lambda_1 T} \\ e^{\lambda_2 T} \end{pmatrix}
\]

Hence

\[
det [I - P_T (\cdot, 0)] = 1 - e^{\lambda_1 T} - e^{\lambda_2 T} + e^{(\lambda_1 + \lambda_2) T} = (1 - e^{\lambda_1 T})(1 - e^{\lambda_2 T}),
\]

and, as \( \lambda_1 < 0 < \lambda_2 \), we have

\[
d_B [I - P_T (\cdot, 0), D, 0] = i_B [I - P_T (\cdot, 0)] = sign (1 - e^{\lambda_1 T})(1 - e^{\lambda_2 T}) = -1.
\]

Now, as (14.18) and (14.22) coincide on the set of \( C^1 \) functions \( u \) such that \( \alpha(t) \leq u(t) \leq \beta(t) \), Poincaré’s operator \( P_T \) of (14.18) and \( P_T (\cdot, 1) \) coincide on a neighborhood of the set of initial conditions of solutions \( x \) of (14.18) such that \( \alpha(t) \leq x(t) \leq \beta(t) \) for all \( t \in [0, T] \), and hence

\[
d_B [I - P_T, \{ \alpha(0), \beta(0) \}] \times TM, TM, [0] = -1.
\]
14.6 Krasnosel’skii-Perov’s existence theorem

One obstacle in applying the Poincaré method lies in the fact that the maximal interval of existence of \( y(t; 0, x) \) can be difficult to estimate and that \( y(t; 0, x) \) is rarely explicitly known, so that the same is true for \( P_T \). To overcome the second difficulty, one can use the homotopy invariance of degree. One way of doing this leads to an interesting existence theorem whose special cases can be traced to I. Berstein and A. Halanay [24] and which is due, in its general form, to M.A. Krasnosel’skii and A.I. Perov [216], [217] (see also [214]).

**Theorem 14.6.1** Assume that there exists an open bounded set \( G \subset \mathbb{R}^n \) such that the following conditions hold:

1. For each \( x \in \overline{G} \), the solution \( y(t; 0, x) \) of (14.1) exists at least over \([0, T]\).

2. For each \( \lambda \in [0, 1] \) and each \( x \in \partial G \), one has \( y(\lambda T; 0, x) \neq x \).

3. For each \( x \in \partial G \), one has \( f(0, x) \neq 0 \).

Then

\[
\partial B[P_T, G, 0] = (-1)^n \partial B[f(0, \cdot), G, 0].
\]

If we assume moreover that

4. \( \partial B[f(0, \cdot), G, 0] \neq 0 \),

then equation (14.1) has at least one \( T \)-periodic solution \( y \) such that \( y(0) \in \overline{G} \).

**Proof.** It follows from Assumption 3 that

\[
\alpha := \min_{x \in \partial G} \| f(0, x) \| > 0,
\]

and hence, by continuity of \( f \), there exists \( \lambda_0 \in [0, 1] \) such that

\[
\left\| \frac{1}{T} \int_0^T f(\lambda s, y(\lambda s, 0, x)) \, ds - f(0, x) \right\|
\leq \alpha
\]

whenever \( \lambda \in [0, \lambda_0] \). Hence, by Rouché’s theorem 3.4.2,

\[
d_B[(1/T) \int_0^T f(\lambda s, y(\lambda s, 0, x)) \, ds, G, 0] = d_B[f(0, \cdot), G, 0] \quad (14.28)
\]

for \( \lambda \in [0, \lambda_0] \). On the other hand, for all \( \lambda \in [0, 1] \), we have

\[
(1/\lambda T)[x - y(\lambda T, 0, x)] = -(1/\lambda T) \int_0^{\lambda T} f(t, y(t, 0, x)) \, dt
= -(1/T) \int_0^T f(\lambda s, y(\lambda s, 0, x)) \, ds,
\]
and hence, for \( \lambda \in [\lambda_0, 1] \),
\[
d_B[(1/T)[I - y(\lambda T, 0, \cdot)], G, 0] = d_B[-(1/T) \int_0^T f(\lambda s, y(\lambda s, 0, x)), G, 0]. \tag{14.29}
\]
Therefore, it follows from Assumption 2, the homotopy invariance property 3.4.2, (14.28) and (14.29) that
\[
d_B[I - P_T, G, 0] = d_B[I - y(\lambda_0 T, 0, \cdot), G, 0] = d_B[-(1/T) \int_0^T f(\lambda_0 s, y(\lambda_0 s, 0, x)), G, 0]
\]
\[
= (-1)^n d_B[(1/T) \int_0^T f(\lambda_0 s, y(\lambda_0 s, 0, x)), G, 0]
\]
\[
= (-1)^n d_B[f(0, \cdot), G, 0].
\]
If Assumption 4 also holds, then, by the existence theorem 3.4.1, \( I - P_T \) has at least one zero in \( G \).

14.7 Degree of gradient mappings

The following consequence of Theorem 14.6.1 is an extension of a result of M.A. Krasnosel’skii [214] due to H. Amann [11]. For a function \( V : \Omega \subset \mathbb{R}^n \to \mathbb{R} \), and \( c \in \mathbb{R} \), we denote by \( V^c \) and \( \overline{V}^c \) respectively the sets \( \{ x \in \Omega : V(x) < c \} \) and \( \{ x \in \Omega : V(x) \leq c \} \).

**Theorem 14.7.1** Let \( \Omega \subset \mathbb{R}^n \) be open and \( V \in C^1(\Omega, \mathbb{R}) \) with gradient \( \nabla V \) locally Lipschitzian. Assume that there exists \( \beta \in \mathbb{R} \) such that \( \overline{V}^\beta \subset \Omega \) is compact, and \( \alpha < \beta \), \( r > 0 \) and \( x_0 \in \Omega \) such that
\[
\overline{V}^\alpha \subset B_{x_0}(r) \subset V^\beta,
\]
and \( \nabla V(x) \neq 0 \) for every \( x \in \overline{V}^\beta \setminus \overline{V}^\alpha \). Then
\[
d_B[\nabla V, V^\beta, 0] = 1.
\]

**Proof.** We consider the associated gradient system
\[
y'(t) = -\nabla V(y(t)), \tag{14.30}
\]
and observe that for each \( c \leq \beta \), the set \( \overline{V}^c = \{ x \in \mathbb{R}^n : V(x) \leq c \} \) is bounded (as contained in \( \overline{V}^\beta \)) and is positively invariant for (14.30). Indeed, if \( y(t; 0, x) \) is the solution of (14.30) such that \( y(0; 0, x) = x \), then, denoting by \( \langle \cdot, \cdot \rangle \) the inner product in \( \mathbb{R}^n \), we have
\[
\frac{d}{dt} \langle y(t; 0, x) \rangle = \langle \nabla V(y(t; 0, x)), y'(t; 0, x) \rangle = -\|\nabla V(y(t; 0, x))\|^2 \leq 0, \tag{14.31}
\]
so that if $x \in \overline{V^c}$, then $V[y(t;0,x)] \leq V(x) \leq c$, i.e. $y(t;0,x) \in \overline{V^c}$ for $t \in [0,\tau_+(0,x)]$. As $\overline{V^c}$ is bounded, this implies that $\tau_+(0,x) = +\infty$ (see e.g. [71]). Notice also that if $x \in \partial V^\beta$ i.e. if $V(x) = \beta$, then, by (14.31),

$$\frac{d}{dt} V[y(t;0,x)]_{t=0} = -\|\nabla V(x)\|^2 < 0,$$

and hence $V[y(t;0,x)] < V(x)$ for all $t > 0$. Consequently, $y(t;0,x) \neq x$ for all $t > 0$. It then follows from Theorem 14.6.1 with $G = V^\beta$ that, for all $t > 0$, one has

$$d_B[I - g(t;0,\cdot), V^\beta, 0] = (-1)^n d_B[\nabla V, V^\beta, 0] = d_B[\nabla V, V^\beta, 0].$$

We shall now show that $d_B[I - g(t;0,\cdot), V^\beta, 0] = 1$ for $t$ sufficiently large. To this effect, if

$$\gamma = \min_{\overline{V^\beta} \setminus V^\alpha} \|\nabla V(x)\|^2,$$

then $\gamma > 0$. If $x \in \partial V^\beta$, then, by the invariance of $\overline{V^\beta}$, $y(t;0,x) \in \overline{V^\beta}$ for all $t \geq 0$. If $y(t;0,x) \in \overline{V^\beta} \setminus V^\alpha$ for $t \in [0,\tau]$, then

$$\beta - \alpha \geq V(x) - V[y(\tau;0,x)] = \int_0^\tau \frac{d}{dt} V[y(t;0,x)] dt = \int_0^\tau \|\nabla V[y(t;0,x)]\|^2 dt \geq \gamma \tau.$$

Consequently, for $\tau > \frac{\beta - \alpha}{\gamma}$, we necessarily have $y(\tau;0,x) \in V^\alpha$, and hence $y(\tau;0,x) \in B(x_0;r)$. Thus $y(\tau;0,\cdot)$ maps $\partial V^\beta$ into $B_{x_0}(r)$. Consequently, for each $\lambda \in [0,1]$ and each $x \in \partial V^\beta$, we have

$$\|(1 - \lambda)(x - x_0) + \lambda(x - y(\tau;0,x))\| = \|x - x_0 - \lambda(y(\tau;0,x) - x_0)\|$$

$$\geq \|x - x_0\| - \|y(\tau;0,x) - x_0\| > 0,$$

as $\|x - x_0\| > r$ for $x \in \partial V^\beta$. The homotopy invariance theorem 3.4.2 then implies that

$$d_B[I - g(\tau;0,\cdot), V^\beta, 0] = d_B[I - x_0, V^\beta, 0] = 1.$$

The following consequence of Theorem 14.7.1 is a result of M.A. Krasnosel’skii [214] for the Brouwer degree of the gradient of a coercive real function.

**Corollary 14.7.1** Let $V \in C^1(\mathbb{R}^n, \mathbb{R})$, with gradient $\nabla V$ locally Lipschitzian, be such that $\nabla V(x) \neq 0$ for some $r_0 > 0$ and all $x \in \mathbb{R}^n$ with $\|x\| \geq r_0$. If $V$ is coercive, i.e. if

$$V(x) \to +\infty \text{ as } \|x\| \to \infty,$$

then $d_B[\nabla V, B(r), 0] = 1$ for all $r \geq r_0$.

**Proof.** Let $\alpha = \max_{\|x\| \leq r_0} V(x)$. By the coercivity, $V^\alpha$ is bounded and hence there exists $r > 0$ such that $\overline{V^\alpha} \subset B(r)$. One then takes $\beta > \max_{\|x\| \leq r_0} V(x)$ and all the assumptions of Theorem 14.7.1 with $x_0 = 0$ are satisfied. Thus, $d_B[\nabla V, V^\beta, 0] = 1$, and the result follows from the excision theorem 3.4.1.
Another consequence of Theorem 14.7.1 was first derived by E. Rothe [334].

**Corollary 14.7.2** Let $U$ be an open neighborhood of $x_0 \in \mathbb{R}^n$ and $V \in C^1(U, \mathbb{R})$, with gradient $\nabla V$ locally Lipschitzian. If $x_0$ is an isolated critical point of $V$ at which $V$ has a local minimum, then for all sufficiently small $\rho > 0$ one has

$$d_B[\nabla V, B_\rho(x), 0] = 1.$$  

**Proof.** Without loss of generality, we can take $U = B(\rho)$, $x_0 = 0$, $V(0) = 0$, with $\rho > 0$ such that $V(x) > 0$ for all $x \in U$. Let us fix $0 < r_1 < r_2 < \rho$ and let $\beta = \min_{r_1 \leq \|x\| \leq r_2} V(x)$, so that $\beta > 0$. As $V^\beta$ is an open neighbourhood of 0, there exists $r > 0$ such that $B(r) \subset V^\beta$. If we take $\alpha = 1/2 \min_{r_1 \leq \|x\| \leq r_2} V(x)$, then $\alpha < \beta$ and all conditions of Theorem 14.7.1 are satisfied. The result follows from this theorem and from the excision theorem 3.4.1. 

**Remark 14.7.1** All the above results hold when $\nabla V$ is only assumed to be continuous. This can be shown using a partition of unity argument in the proofs above. We shall use freely this fact in the sequel.

A further consequence of Theorem 14.6.1 concerns an autonomous system of the form

$$x'(t) = f(x(t)), \quad (14.33)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitzian and such that $f(0) = 0$. Recall that the equilibrium 0 is said to be **stable** if for each $\epsilon > 0$ we can find $\delta > 0$ such that $\|y(t; 0, z)\| \leq \epsilon$ for each $t \geq 0$ and $z \in \mathbb{R}^n$ such that $\|z\| \leq \delta$, and (uniformly) **asymptotically stable** if it is stable and **attractive**, i.e. there exists $\beta > 0$ such that for each $\eta > 0$ one can find $T > 0$ such that $\|y(t; 0, z)\| \leq \eta$ whenever $t \geq T$ and $\|z\| \leq \beta$. See [337] for details.

**Theorem 14.7.2** If 0 is an isolated zero of $f$ and is asymptotically stable, then there exists $\rho > 0$ such that

$$d_B[-f, B(\rho), 0] = 1.$$  

**Proof.** We first notice that the stability of 0 implies that all solutions of sufficiently small initial condition exist on $[0, +\infty]$ and we claim that there exists some $\rho_0 \in [0, \beta]$ such that, for each $\rho \in [0, \rho_0]$, each $z$ with $\|z\| = \rho$ and each $t > 0$ we have $y(t; 0, z) \neq z$. If it is not the case, that, for each integer $k \geq 1$, such that $1/k \leq \beta$, we can find some $\rho_k \in [0, 1/k]$, some $z_k$ with $\|z_k\| = \rho_k$ and some $t_k > 0$ such that $y(t_k; 0, z_k) = z_k$, i.e. such that $y(t; 0, z_k)$ is $t_k$-periodic. But, this implies that $y(t; 0, z_k) \not\to 0$ for $t \to +\infty$ and contradicts the attractivity of 0. Consequently, for all $z \in B(\rho_0) \setminus \{0\}$, all $\lambda \in [0, 1]$, and for all $T > 0$, we have $y(\lambda T; 0, z) \neq z$. Hence, by Theorem 14.6.1,

$$d_B[-f, B(\rho), 0] = d_B[I - y(T; 0, \cdot), B(\rho), 0]$$
for all $T > 0$ and all $\rho \in [0, \rho_0]$. For $T > 0$ associated to $\eta = \rho/2$ in the definition of attractivity, we have $\|y(T; 0, z)\| \leq \rho/2$ for all $z \in \overline{B}(\rho)$, and hence
\[
\|z - \lambda y(T; 0, z)\| \geq \rho/2
\]
whenever $|z| = \rho$ and $\lambda \in [0, 1]$, so that, by Rouché’s theorem 3.4.2
\[
d_B[I - y(T; 0, \cdot), B(\rho), 0] = d_B[I, B(\rho), 0] = 1,
\]
and the result follows.

For more delicate results on the degree when 0 is only assumed to be stable, the reader may consult [385, 386].
CHAPTER 14. POINCARÉ’S OPERATOR
Chapter 15

Stability and index of periodic solutions

15.1 Introduction

Topological degree techniques have been widely applied in the study of the existence and the multiplicity of solutions of nonlinear boundary value problems, and in particular of periodic solutions of nonlinear differential equations. Much less has been done using those techniques in the study of the stability of those solutions. Concentrating on the periodic case, we can quote some pioneering studies for arbitrary systems ([214],[276]) and for second order equations ([290],[205],[263]). Among the more recent results, we may quote [269] (for periodic solutions of first order scalar equations), [303],[304],[305],[306],[307],[243],[244]. We shall analyze briefly here some of the results of R. Ortega taken from [303] and [304]. The reader can consult the original papers, as well as the survey [308] for more details and further developments.

15.2 Planar periodic systems

We consider the following two-dimensional system

\[ x' = f(t, x), \]  

where the continuous mapping \( f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) is T-periodic with respect to \( t \) for fixed \( x \) and of class \( C^1 \) with respect to \( x \) for fixed \( t \). We denote by \( x(t; x_0) \) the solution of (15.1) such that \( x(0; x_0) = x_0 \) and by \( P_T \) Poincaré’s mapping \( P_T : x_0 \mapsto x(T; x_0) \). This mapping is defined for the (open) subset of \( \mathbb{R}^2 \) made of the \( x_0 \) such that \( x(\cdot; x_0) \) is defined over \([0,T]\). Recall that \( x(\cdot; x_0) \) is a T-periodic solution of (15.1) if and only if \( x_0 \) is a fixed point of \( P_T \). If \( x \) is an isolated T-periodic solution of (15.1), then \( x(0) \) is an isolated fixed point of \( P_T \) and hence the Brouwer index

\[ \text{ind}[P_T, x(0)] := d_B[I - P_T, B_{x_0}(r), 0], \]

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where \( r > 0 \) is sufficiently small, is well defined, called the **index of period** \( T \) of \( x \), and denoted by \( \gamma_T(x) \). Similarly, we need to consider the mapping \( P_{2T} : x_0 \mapsto x(2T; x_0) \) and if \( x \) is an isolated 2T-periodic solution of (15.1), we define the **index of period** \( 2T \) of \( x \) by

\[
\gamma_{2T}(x) = \text{ind}[P_{2T}, x(0)] = d_B[I - P_{2T}, B(x_0)(r), 0].
\]

Notice that every \( T \)-periodic solution \( x \) of (15.1) is also a \( 2T \)-periodic solution, and if isolated as a \( 2T \)-periodic solution, is also isolated as a \( T \)-periodic solutions. It admits both indices \( \gamma_T \) and \( \gamma_{2T} \).

**Definition 15.2.1** A \( T \)-periodic solution \( x \) of (15.1) is called **nondegenerate** of period \( T \) (resp. of period \( 2T \)) if the linearized equation

\[
y' = f_x(t, x(t))y
\]

has no nontrivial \( T \)-periodic (resp. \( 2T \)-periodic) solution.

If \( x \) is \( T \)-periodic and nondegenerate of period \( 2T \), an implicit function argument shows that \( x \) is isolated for periods \( T \) and \( 2T \) and the linearization formula for degree together with the Leray-Schauder formula for linear mappings show that

\[
|\gamma_T(x)| = |\gamma_{2T}(x)| = 1.
\]

More explicitly, if \( x \) is a solution of (15.1) with initial value \( x_0 \), then from the identity

\[
x'(t; x_0) = f(t, x(t; x_0))
\]

we deduce

\[
[x'_{x_0}(t; x_0)]' = f_x'(t, x(t; x_0))x'_{x_0}(t; x_0),
\]

\[
x'_0(x_0) = I,
\]

and hence \( x'_{x_0}(t, x_0) \) is the fundamental matrix \( Y(t) \) of the variational system (15.2) associated to the solution \( x(t; x_0) \). In particular, \( Y(T) = x'_{x_0}(T; x_0) = P'_T(x_0) \) is the **monodromy matrix** of equation (15.2) when \( x(t; x_0) \) is \( T \)-periodic. Now, if \( \mu_1 \) and \( \mu_2 \) are the **characteristic multipliers** of (15.2), i.e. the eigenvalues of a monodromy matrix, we have, by linear algebra

\[
\det [I - P'_T](x_0) = (1 - \mu_1)(1 - \mu_2).
\]

Consequently, if \( x(t, x_0) \) is isolated (period \( T \)) and if \( \mu_j \neq 1 \) \((j = 1, 2)\), Theorem 5.2.1 implies that, for sufficiently small \( r > 0 \),

\[
\gamma_T(x_0) = \text{ind}[I - P_{2T}, B_{x_0}(r), 0] = \text{sign } J_{I - P_{2T}}(x_0) = \text{sign } \det [I - P'_T](x_0) = \text{sign } (1 - \mu_1)(1 - \mu_2).
\]  

(15.3)
15.2. PLANAR PERIODIC SYSTEMS

Now, for each $x_0 \in \mathbb{R}^2$, we have

$$P_{2T}(x_0) = x(2T; x_0) = x[T; x(T; x_0)] = P_T(P_T(x_0)),$$

and hence

$$P'_{2T}(x_0) = P'_T(P_T(x_0)) \circ P'_T(x_0),$$

which implies in particular, as $x(t; x_0)$ is $T$-periodic, that

$$P'_{2T}[x_0] = P'_T[x_0] \circ P'_T[x_0] \quad (15.4)$$

As $P'_T(x_0) = Y(T)$, we get

$$\det[I - P''_{2T}(x_0)] = \det\{[I - P'_T[x_0] \circ P'_T[x_0]] = \det[I - Y(T) \circ Y(T)]$$

$$= (1 - \mu_1^2)(1 - \mu_2^2).$$

Consequently, if $x$ is a $T$-periodic solution which is isolated of period $2T$ and if $\mu_j \neq \pm 1 (j = 1, 2)$, then

$$\gamma_{2T}(x) = \text{sign}[(1 - \mu_1^2)(1 - \mu_2^2)]. \quad (15.5)$$

**Theorem 15.2.1** Assume that condition

$$\text{div } f(t, x) < 0, \quad (15.6)$$

holds for all $(t, x)$, and let $x$ be a $T$-periodic solution of (15.1) which is nondegenerate of period $2T$. Then $\gamma_{2T}(x) = 1$ (resp. $-1$) if and only if $x$ is uniformly asymptotically stable (resp. is unstable).

**Proof.** Let $\mu_1$ and $\mu_2$ be the characteristic multipliers of the variational equation (15.2) considered as an equation with $T$-periodic coefficients. Recall that if $Y(t)$ is the fundamental matrix of (15.2), then $\mu_1$ and $\mu_2$ are the eigenvalues of the matrix $Y(T)$, which implies that the identity

$$\mu^2 - (\mu_1 + \mu_2)\mu + \mu_1 \mu_2 = \det[Y(T) - \mu I],$$

is valid for all $\mu \in \mathbb{C}$, and that $\mu_2 = \overline{\mu_1}$ whenever $\mu_1$ is complex. In particular, using Liouville’s formula and assumption (15.6), we have

$$\mu_1 \mu_2 = \det Y(T) = \exp \left( \int_0^T \text{div } f_x(s, x(s)) \, ds \right) \in ]0, 1[. \quad (15.7)$$

The nondegeneracy condition implies that $\mu_i \notin \{-1, 1\}$, $(i = 1, 2)$. Condition (15.7) implies that if say $\mu_1$ is not real, then $\mu_2 = \overline{\mu_1}$ and $|\mu_1|^2 = |\mu_2|^2 = \mu_1 \mu_2 < 1$. Thus, in any case,

$$|\mu_i| \neq 1, (i = 1, 2). \quad (15.8)$$
Thus, if one of the $\mu_i$ (and hence the other one too) is not real, we have $|\mu_i| < 1$, $i = 1, 2$, and

$$\gamma_{2T}(x) = \text{sign} |(1 - \mu^2_1)|^2 = 1.$$ 

Now, $x$ is uniformly asymptotically stable if and only if $|\mu_i| < 1$, ($i = 1, 2$), and hence, in the case of non real multipliers, the statement about the uniform stability is proved. If the multipliers are real, then $\gamma_{2T} = 1$ if and only if $\mu_1^2$ and $\mu_2^2$ are both smaller or both larger than one, and the second case is excluded by relation (15.7), so that the statement about the uniform stability is proved also in this case. Finally, $x$ is unstable if and only if one multiplier, say $\mu_1$ has absolute value greater than one, which can only arrive if they are real. Thus, this is equivalent to the fact that $\mu_1^2 > 1$ so that necessarily, by relation (15.7), $\mu_2^2 < 1$ and, by (15.5), those two conditions are equivalent to $\gamma_{2T} = -1$.

### 15.3 The case of a second order differential equation

We consider in this section the second order differential equation

$$u'' + cu' + g(t, u) = 0,$$  

(15.9)

with $c > 0$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function, $T$-periodic with respect to the first variable and having a continuous partial derivative with respect to the second variable. Such an equation is equivalent to the planar system

$$x'_1 = x_2, \quad x'_2 = -cx_2 - g(t, x_1),$$  

(15.10)

This system satisfies condition (15.6), because

$$\text{div} (x_2, -cx_2 - g(t, x_1)) = -c < 0$$

for all $(x_1, x_2) \in \mathbb{R}^2$. The aim of this section is to characterize the stability of a $T$-periodic solution $u$ of (15.9) (i.e. of the associated $T$-periodic solution $x = (u, u')$ of (15.10)) in terms of its index $\gamma_T(u) := \gamma_T(x)$ only. If $\alpha$ and $\beta$ are real functions defined over $[0, T]$, we shall write $\alpha \ll \beta$ if $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$ and the strict inequality holds on a subset of positive Lebesgue measure.

The following lemma is proved in [277].

**Lemma 15.3.1** Assume that the real function $\beta$ is continuous over $[t_0, t_0 + T]$ and such that

$$\beta \ll \frac{\pi^2}{T^2}.$$ 

Then the two-point boundary value problem

$$w''(t) + \beta(t)w = 0, \quad w(t_0) = 0 = w(t_0 + \pi^2/T^2)$$

only has the trivial solution.
Lemma 15.3.2 Assume that $c > 0$ and that the real function $\alpha$ is continuous, $T$-periodic and such that

$$\alpha < \frac{\pi^2}{T^2} + \frac{c^2}{4}.$$  

Then the linear equation

$$v'' + cv' + \alpha(t)v = 0 \quad (15.11)$$

does not admit negative characteristic multipliers.

Proof. Setting

$$v(t) = \exp \left( -\frac{c}{2} t \right) w(t),$$

equation (15.11) becomes

$$w'' + \left[ \alpha(t) - \frac{c^2}{4} \right] w = 0 \quad (15.12)$$

and the corresponding solutions $v$ and $w$ obviously have the same zeros. If (15.11) admits a negative characteristic multiplier $\mu$, then it has a nontrivial solution $v$ such that

$$v(t + T) = \mu v(t)$$

for all $t \in \mathbb{R}$. If $t$ is such that $v(t) \neq 0$, then $v$ has opposite signs at $t$ and $t + T$ and therefore must vanish at some $t_0 \in [t, t + T]$, and hence also at $t_0 + T$. Thus $w$ is a solution of equation (15.12) satisfying the two-point boundary conditions

$$y(t_0) = y(t_0 + T) = 0.$$  

But $\frac{\pi^2}{T^2}$ is the first eigenvalue of the two-point boundary value problem on an interval of length $T$ for the operator $-w''$ and Lemma 15.3.1 together with the assumption over $\alpha$ imply that $w$ must be identically zero, a contradiction. \hfill \blacksquare

Theorem 15.3.1. Assume that $u$ is an isolated $T$-periodic solution of (15.9) such that condition

$$g_u'(t, u(t)) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4} \quad (15.13)$$

holds for all $t \in \mathbb{R}$. Then $u$ is uniformly asymptotically stable (resp. unstable) if and only if $\gamma_T(u) = 1$ (resp. $\gamma_T(u) = -1$).

Proof. If $g_u'(t, u(t)) = \frac{\pi^2}{T^2} + \frac{c^2}{4}$ for all $t \in \mathbb{R}$, the variational equation

$$v'' + cv' + g_u'(t, u(t))v = 0, \quad (15.14)$$
has constant coefficients and the characteristic exponents have negative real parts, which immediately implies that $u$ is uniformly asymptotically stable and $\gamma_T(u) = \gamma_2T(u) = 1$. We can therefore assume that

$$g_u(\cdot, u(\cdot)) \leq \frac{\pi^2}{T^2} + \frac{\epsilon^2}{4}.$$  

Lemma 15.3.2 implies that the variational equation (15.14) has no negative characteristic multipliers. We already know from the proof of Theorem 15.2.1 that $\gamma_T(u) = \gamma_2T(u) = 1$ and $u$ is uniformly asymptotically stable if the characteristic multipliers are not real. If the $\mu_i$ are real, they are both positive and

$$\text{sign}[(1 - \mu_1^2)(1 - \mu_2^2)] = \text{sign}[(1 - \mu_1)(1 + \mu_1)(1 + \mu_2)] = \text{sign}[(1 - \mu_1)(1 - \mu_2)],$$

so that $\gamma_T(u) = \gamma_2T(u)$ and the result follows from Theorem 15.2.1.

Remark 15.3.1 The following example from [303] shows that the assumption in Theorem 15.3.1 is optimal. Let $\beta$ be a $T$-periodic function defined by

$$\beta(t) = \begin{cases} -\omega_1^2 + \frac{\epsilon^2}{4} & \text{if } t \in [0, a[, \\ \omega_2^2 + \frac{\epsilon^2}{4} & \text{if } t \in ]a, T[, \\ \end{cases}$$

for some $0 < a < T$ and $\omega_1, \omega_2 > 0$. We consider the equation

$$u'' + cu' + \omega(t)u = 0.$$  

If the assumption of Theorem 15.3.1 does not hold for this equation, then $\omega_2^2 > \frac{\pi^2}{T^2}$ and we can fix $a$ so that $\omega_2(T - a) = \pi$. Direct integration of the equation shows that one can select $\omega_1$ in such a way that the trivial solution is unstable and has index 1. Hence, $\frac{\pi^2}{T^2} + \frac{\epsilon^2}{4}$ cannot be replaced by any greater constant. One could object that in this example the function $\omega$ is not continuous but this can be of course easily overcome by an approximation argument.

Remark 15.3.2 Using a class of systems introduced by Smith [369] and for which Massera’s convergence theorem holds, Ortega [305] has obtained the following partial improvement of Theorem 15.3.1:

Proposition 15.3.1 Let $u$ be a $T$-periodic solution of (15.9) such that

$$g_u'(t, u(t)) < \frac{\epsilon^2}{4}$$

for all $t \in \mathbb{R}$. Then the conclusion of Theorem 15.3.1 holds.

Remark 15.3.3 Combining degree techniques with the theory of center manifolds, Ortega [306] has obtained the following improvement of Theorem 15.3.1:

Proposition 15.3.2 Let $x$ be a $T$-periodic solution of (15.1) which is isolated of period $2T$. Then $\gamma_{2T}(x) = 1$ (resp. $\neq 1$) if and only if $x$ is uniformly asymptotically stable (resp. is unstable).

He has also obtained a version of this result in $\mathbb{R}^n$. 
15.4 Second order equations with convex nonlinearity

We consider in this section the second order parametric equation

\[ u'' + cu' + g(t, u) = s, \]  

(15.15)

where \( c > 0, \) \( s \) is a real parameter and the continuous function \( g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is \( T \)-periodic with respect to the first variable and has a continuous partial derivative with respect to the second variable. Assume moreover that \( g \) is strictly convex, namely that

\[ [g_u'(t, u) - g_u'(t, v)](u - v) > 0, \]  

(15.16)

for all \( t \in \mathbb{R} \) and all \( u \neq v \) in \( \mathbb{R} \). To insure that condition (15.13) holds, we assume also that

\[ \lim_{u \to +\infty} g_u'(t, u) \leq \frac{\pi^2}{T^2} + \frac{c^2}{4}. \]  

(15.17)

Finally, let us assume that the coercitivity condition

\[ \lim_{|u| \to \infty} g(t, u) = +\infty \]  

(15.18)

introduced in [109] holds uniformly in \( t \in [0, T] \). Following [303], we complete the existence discussion given in [109] by results on the stability of the corresponding \( T \)-periodic solutions. The following lemma is used in the proof.

**Lemma 15.4.1** Consider the differential operator \( L_\alpha \) defined by

\[ L_\alpha(u) = u'' + cu' + \alpha(.)u, \]

acting on \( T \)-periodic functions, where \( \alpha \in L^1(\mathbb{R}/T\mathbb{Z}) \) satisfies condition

\[ \alpha \ll \frac{\pi^2}{T^2} + \frac{c^2}{4}. \]  

(15.19)

Then the following conclusions are true.

(a) For each real \( \mu \) each possible \( T \)-periodic solution \( u \) of equation \( L_\alpha(u) = \mu \) is either trivial or different from zero for each \( t \in [0, T] \).

(b) Let \( \alpha_1 \) and \( \alpha_2 \) be functions in \( L^1(\mathbb{R}/T\mathbb{Z}) \) satisfying (15.19) and such that \( \alpha_1 \ll \alpha_2 \). Then equations

\[ L_{\alpha_i}(u) = 0, \quad (i = 1, 2) \]

cannot admit nontrivial \( T \)-periodic solutions simultaneously.
Proof. If a nontrivial T-periodic solution \( u \) of equation \( L_\alpha(u) = \mu \) vanishes at some \( t_0 \), then it vanishes also at \( t_0 + T \), and, by the same reasoning as in the proof of Lemma 15.3.2, the function \( w \) defined by \( w(t) = \exp(-\frac{c^2}{4}t)u(t) \) is a solution of the differential equation

\[
 w'' + [\alpha(t) - \frac{c^2}{4}]w = 0
\]

which vanishes at \( t_0 \) and \( t_0 + T \), a contradiction with Lemma 15.3.1. To prove (b), we first notice that conclusion (a) obviously holds also for the adjoint operator \( L^*_\alpha \) defined by \( L^*_\alpha(u) = u'' - cu' + \alpha(.)u \). If both equations \( L_\alpha(u) = 0 \) admit nontrivial T-periodic solutions \( u_1 \) and \( u_2 \) respectively, then the adjoint equation \( L^*_\alpha(u) = 0 \) admits a nontrivial T-periodic solution \( \varphi \). By the part (a) of the Lemma, we can assume without loss of generality that \( u_2 > 0 \) and \( \varphi > 0 \) on \([0,T]\). Moreover,

\[
 L_\alpha_1(u_2) = L_\alpha_2(u_2) + (\alpha_1 - \alpha_2)u_2 = (\alpha_1 - \alpha_2)u_2,
\]

so that, using the Fredholm alternative, we get

\[
 0 = \int_0^T [\alpha_1(t) - \alpha_2(t)]u_2(t)\varphi(t) \, dt,
\]

which is a contradiction with the assumption \( \alpha_1 \ll \alpha_2 \).

The following lemma is essentially Lemma 2 of [109].

**Lemma 15.4.2** Assume that condition (15.18) holds. Then there exists an increasing function \( M(s) \) such that each possible T-periodic solution \( u \) of (15.15) satisfies the inequality

\[
 |u(t)| + |u'(t)| < M(|s|), \quad (t \in \mathbb{R}).
\]

**Proof.** By assumption (15.18), there exists \( \gamma \in \mathbb{R} \) such that

\[
 g(t, u) \geq \gamma \tag{15.20}
\]

for all \((t, u) \in [0, T] \times \mathbb{R} \). Let \( u \) be a possible T-periodic solution of (15.15). Then, integrating both members of (15.15) over \([0, T] \), we get

\[
 s = \overline{g(\cdot, u(\cdot))} := \frac{1}{T} \int_0^T g(t, u(t)) \, dt \geq \gamma. \tag{15.21}
\]

Furthermore,

\[
 \int_0^T |u''(t) + cu'(t)| \, dt = \int_0^T |g(t, u(t)) - s| \, dt \tag{15.22}
\]

\[
 \leq \int_0^T |g(t, u(t)) - \gamma| \, dt + T|\gamma - s| \tag{15.23}
\]

\[
 = \int_0^T |g(t, u(t)) - \gamma| \, dt + T|\gamma - s| \tag{15.24}
\]

\[
 = T(s - \gamma) + T|\gamma - s| = 2T(s - \gamma)^+. \tag{15.25}
\]
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Consequently, if \( \tau \in [0, T] \) is such that \( u'(\tau) = 0 \), we deduce from (15.22) that, for all \( t \in [0, T] \),

\[
|u'(t)| = \left| \int_\tau^t e^{-c(t-\sigma)} [u''(\sigma) + cu'(\sigma)] d\sigma \right| 
\]

(15.26)

\[
\leq \int_0^T |u''(\sigma) + cu'(\sigma)| d\sigma \leq 2T(s - \gamma)^+, 
\]

(15.27)

which immediately implies that, for all \( t \in [0, T] \),

\[
|u(t) - u(0)| \leq 2T^2(s - \gamma)^+. 
\]

(15.28)

Now, it follows from condition (15.18) that there exists an increasing function \( R(s) \) of \( s \) such that

\[
f(t, u) > s \text{ whenever } t \in [0, T] \text{ and } |u| \geq R(s). \] 

(15.29)

Consequently, if \( |u(0)| \geq R(s) + 2T^2(s - \gamma)^+ \), then, for all \( t \in [0, T] \),

\[
|u(t)| = |u(0) + u(t) - u(0)| \geq |u(0)| - |u(t) - u(0)| \geq R(s),
\]

so that

\[
\overline{g(\cdot, u(\cdot))} > s,
\]

a contradiction with (15.21). Hence, \( |u(0)| < R(s) + 2T^2(s - \gamma)^+ \), and, consequently, we have, for all \( t \in [0, T] \),

\[
|u(t)| \leq |u(0)| + |u(t) - u(0)| < R(s) + 4T^2(s - \gamma)^+. 
\]

(15.30)

and it suffices to take \( M(s) = R(s) + (4T^2 + 2T)(s - \gamma)^+ \).

We also need the following preliminary result.

**Lemma 15.4.3** Assume that the function \( g \) satisfies conditions (15.17) and (15.18). Then, given \( s_1, s_2 \in \mathbb{R} \) and \( u_1, u_2 \) be \( T \)-periodic solutions of (15.15) for \( s = s_1, s = s_2 \) respectively such that \( u_1 \neq u_2 \). Then either \( u_1 > u_2 \) or \( u_1 < u_2 \) on \([0, T]\).

**Proof.** The difference \( v = u_1 - u_2 \) satisfies an equation of the form

\[
L_\alpha(v) = s_1 - s_2,
\]

for some \( T \)-periodic function \( \alpha \) which, because of conditions (15.16) and (15.17) satisfies the conditions of Lemma 15.4.1. The conclusion follows from this lemma.

We can now state and prove the main result of this section.

**Theorem 15.4.1** Assume that the function \( g \) satisfies conditions (15.16), (15.17) and (15.18). Then there exists \( s_0 \in \mathbb{R} \) such that the following conclusions hold.
(i) If \( s > s_0 \), equation (15.15) has exactly two T-periodic solutions, one uniformly asymptotically stable and another unstable.

(ii) If \( s = s_0 \), equation (15.15) has a unique T-periodic solution which is not asymptotically stable.

(iii) If \( s < s_0 \), every solution of equation (15.15) is unbounded.

**Proof.** We first follow the arguments of [109]. If \( s^* \) is such that \( g(t,0) \leq s^* \) for all \( t \in [0,T] \), then 0 is an upper solution for (15.15) with T-periodic boundary conditions and \( s = s^* \). From condition (15.18), there exists \( R > 0 \) such that \( g(t,-R) \geq s^* \) for all \( t \in \mathbb{R} \), and hence \( -R \) is a lower solution for (15.15) with T-periodic boundary conditions and \( s = s^* \). Hence, by Theorem 14.4.1, equation (15.15) with \( s = s^* \) has at least one T-periodic solution taking values in \([-R,0]\).

In other words, the set \( S \) of \( s \in \mathbb{R} \) such that (15.15) has at least one T-periodic solution is non-empty. On the other hand, if \( u \) is a T-periodic solution of (15.15), then \( \overline{f}(s,u(s)) \geq \gamma \), which shows that \( S \) is bounded below by \( \gamma \). Now, \( S \) is an interval unbounded from above, because if \( s_1 \) belong to \( S \), \( u_1 \) is a corresponding T-periodic solution of (15.15) with \( s = s_1 \), and if \( s_1 < s \), then

\[
u''(t) + cu'(t) + g(t,u(t)) = s_1 < s,\]

so that \( u_1 \) is an upper solution. Now it follows again from assumption (15.18) that one can find \( R \geq -\min_{0 \leq t \leq T} u_1 \) so that \( f(t,-R) \geq s \) for all \( t \in [0,T] \), (15.15) with T-periodic boundary conditions. Using again Theorem (14.4.1), we obtain the existence of a T-periodic solution lying between \( -R \) and \( u_1(t) \). Let \( s_0 := \inf S \), so that \( s_0 \geq \gamma \). A limiting and compactness argument using the conclusion (a) of Lemma 15.4.2 implies that \( s_0 \in S \). Thus, \( s \geq s_0 \) equation (15.15) has at least one T-periodic solution, and for \( s < s_0 \), equation (15.15) has no T-periodic solution.

We now show that equation (15.15) has at most two T-periodic solutions, so that they are necessarily isolated. Otherwise, let \( u_i \), \( (i = 1,2,3) \) be three different T-periodic solutions. By Lemma 14.4.3, they can be ordered, say \( u_1 < u_2 < u_3 \). Setting \( v_1 = u_2 - u_1 \), \( v_2 = u_3 - u_2 \), we see immediately that \( v_i \) satisfies equation

\[L_{o_i}(v_i) = 0 \text{ with } o_i = [g(t,u_{i+1}) - g(t,u_i)]/v_i, \quad (i = 1,2),\]

The strict convexity of \( g \) implies that \( o_1 < o_2 \) on \([0,T]\). Using Lemma 14.4.1, we obtain that either \( v_1 \) or \( v_2 \) must be zero, a contradiction.

Let now \( u_0 \) be a T-periodic solution of equation (15.15) for \( s = s_0 \). It follows from the continuity of the index that \( u_0 \) is degenerate and that \( \gamma_T(u_0) = 0 \), because, if it was not the case, equation (15.15) would have a T-periodic solutions for all \( s \in [s_0 - \epsilon,s_0] \) for some \( \epsilon > 0 \). Notice now that if \( u_1 \) is another T-periodic solution of equation (15.15) for some \( s_1 \geq s_0 \), then, by Lemma 15.4.3, either

\[u_1 < u_0 \text{ and hence } g'(u_1) < g'(u_0) \text{ on } [0,T],\]

or

\[u_1 > u_0 \text{ and hence } g'(u_1) > g'(u_0) \text{ on } [0,T].\]
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By the second part of Lemma 15.4.1, we conclude that $u_1$ is nondegenerate, and hence that $s_1 > s_0$. So we have proved that, for $s = s_0$ equation (15.15) has a unique T-periodic solution $u_0$ which satisfies $\gamma_T(u_0) = 0$, and hence cannot be uniformly asymptotically stable by Theorem 15.3.1.

A consequence of the above reasoning is the existence, for $s > s_0$, of exactly two T-periodic solutions of (15.15) and they are nondegenerate. Given $s_1 > s_0$, By Lemma 15.4.1, the Poincaré operator $P^*_{T,s_1}$ associated to equation (15.15) (written as a planar system) has no fixed point on the boundary of the nonempty subset $\Omega = \{(x_1, x_2) \in \text{dom} P_T : |x_1| + |x_2| < M_0\}$, with $M_0 = M(\max(|s_0|, |s_1|)$, for each $s \in [s_0, s_1]$. Hence, by the homotopy invariance property 3.4.2,

$$d_B I - P^*_{T,s_1},\Omega,0] = d_B[I - P^*_{T,s_0},\Omega,0] = \gamma_T(u_0) = 0.$$  \hspace{1cm} (15.31)

Now, if $u_1$ denotes the T-periodic solution whose existence has been proved using the method of lower and upper solutions, $u_1$ is non-degenerate and hence $\gamma_T(u_1) = \pm 1$.

Therefore, by (15.31) and Proposition 5.1.1, there must exist a second T-periodic solution $u^*_1$ such that

$$\gamma_T(u^*_1) = -\gamma_T(u_1).$$

Thus, by Theorem 15.3.1, equation (15.15) has exactly one uniformly asymptotically stable and one unstable T-periodic solution for each $s > s_0$. Finally, let $u$ be a bounded solution of equation (15.15) for some $s < s_0$. The equation easily implies that $u'$ is also bounded and given $u_+ > \sup u(t) \geq \inf u(t) > u_-$ with $|u_\pm|$ large enough, let us consider the associated truncated equation

$$u'' + cu' + g^*(t, u) = s.$$ \hspace{1cm} (15.32)

where the function (bounded and Lipschitzian in $u$) $g^*$ is defined by

$$g^*(t, u) = \begin{cases} g(t, u) & \text{if } u_- < u < u_+, \\ g(t, u_-) & \text{if } u \leq u_-, \\ g(t, u_+) & \text{if } u \geq u_+.
\end{cases}$$

By construction, $u$ is also a solution of this equation (for which the Cauchy problem is globally uniquely solvable) with $|u| + |u'|$ bounded. A theorem of Massera [259] implies then that equation (15.32) should have a T-periodic solution, a contradiction with Lemma 15.4.2 and the nonexistence of T-periodic solution for equation (15.15) when $s < s_0$. Thus, for those values of $s$, equation (15.15) has no bounded solution.

Remark 15.4.1 Theorem 15.4.1 can be considered as a partial extension to a second order differential equation of the result of [271] which gives a complete qualitative picture of the set of solutions for first order scalar equations with convex and coercive nonlinearities.

Remark 15.4.2 More results on equation 15.15, including the existence of some subharmonics of order 2 can be found in [304].
15.5 Periodic solutions of pendulum-type equations

In this section, we study periodic problems of the form

\[ u'' + cu' + f(t, u) = p(t) + s, \quad u(0) = u(T), \quad u'(0) = u'(T), \quad (15.33) \]

where \( c \geq 0, s \in \mathbb{R}, f : [0, T] \times \mathbb{R} \to \mathbb{R} \) is continuous, \( 2\pi \)-periodic with respect to \( u \), such that \( f_u \) exists and is continuous for any \( u \in \mathbb{R} \), \( p : [0, T] \to \mathbb{R} \) is continuous and has mean value

\[ \overline{p} := \frac{1}{T} \int_0^T p(t) \, dt \]

equal to zero. A special case is the forced pendulum equation, for which \( g(t, u) = a \sin u \).

T-periodic solutions of (15.33) which differ by a multiple of \( 2\pi \) will be identified.

Because of the boundedness and regularity of \( f \), solutions of the Cauchy problem for the differential equation in (15.33) exist over \( \mathbb{R} \) and are unique, so that Poincaré’s operator \( P_T \) is well defined everywhere. On the other hand, it follows from the invariance of (15.33) under the transformation \( u \to u + 2\pi \) and the uniqueness of Cauchy problem that

\[ P_T(u_0 + 2\pi, v_0) = P_T(u_0, v_0) \quad (15.34) \]

for all \((u_0, v_0) \in \mathbb{R}^2\). Again, let \( S \) denote the set of \( s \in \mathbb{R} \) such that equation (15.33) has at least one T-periodic solution. If (15.33) has a T-periodic solution \( u \), then, integrating both members of (15.33) over \([0, T]\) we obtain, if we set

\[ f_L(t) := \min_{u \in \mathbb{R}} f(t, u), \quad f_M(t) := \max_{u \in \mathbb{R}} f(t, u), \quad (15.35) \]

\[ \overline{f_L} \leq s \leq \overline{f_M} \]

and hence \( S \subset [\overline{f_L}, \overline{f_M}] \) is bounded.

**Proposition 15.5.1** \( S \) is not empty.

**Proof.** Let \( a = u(0) \) and \( v(t) = u(t) - a \), so that \( v(0) = 0 \). As \( u' = v' \), the search of T-periodic solutions of (15.33) is equivalent to the search of \( a, v(t) \) such that

\[ u'' + cu' + f(t, a + v) = p(t) + s, \quad v(0) = v(T), \quad v'(0) = v'(T). \quad (15.36) \]

If we let \( u'(0) = v'(0) = v_0 \), it is easy to see that the solution \( v(t; v_0, s) \) of the differential equation in (15.36) such that \( v(0) = 0, v'(0) = v_0 \) verifies the identities

\[ v'(t) = v_0 + \int_0^t e^{-(t-\tau)} [p(\tau) + s - f(\tau, a + v(\tau, v_0, s)) \, d\tau] \quad (15.37) \]

\[ v(t) = v_0 t + \int_0^t \int_0^\sigma e^{-(\sigma-\tau)} [p(\tau) + s - f(\tau, a + v(\tau, v_0, s)) \, d\tau \, d\sigma. \]
Consequently, for a given $a, v$ will be a solution of problem \((15.36)\) if $v_0$ and $s$ are such that

$$
\int_{0}^{T} e^{-c(T-\tau)} [p(\tau) + s - f(\tau, a + v(\tau, v_0, s))] \, d\tau = 0 \quad (15.38)
$$

Let $s_0$ be such that

$$
v_0 T + \int_{0}^{s_0} e^{-c(s-\tau)} [p(\tau) + s - f(\tau, a + v(\tau, v_0, s))] \, d\tau \, ds = 0.
$$

It follows from the boundedness of $f$ that there exists $s^* > 0$ such that the left-hand member of the first equation in \((15.38)\) is positive when $s \geq s^*$ and all $v_0 \in \mathbb{R}$, and is negative when $s \leq s^*$ and all $v_0 \in \mathbb{R}$. On the other hand, for the same reason, there exists $v^* > 0$ such that the left-hand member of the second equation in \((15.38)\) is positive when $v_0 \geq v^*$ and $s \in [-s^*, s^*]$, and negative when $v_0 \leq -v^*$ and $s \in [-s^*, s^*]$. Consequently, Poincaré-Miranda’s theorem implies the existence of $(\tilde{s}, \tilde{v}_0) \in [-s^*, s^*] \times [-v^*, v^*]$ such that conditions \((15.38)\) hold, i.e. such that $v(t; \tilde{v}, \tilde{s})$ solves problem \((15.36)\) with $s = \tilde{s}$. In other words, equation \((15.33)\) with $s = \tilde{s}$ has at least one $T$-periodic solution, or, equivalently, $\tilde{s} \in S$.

Let us now prove that $S$ is a closed interval.

**Proposition 15.5.2** $S$ is a closed (possibly degenerate) interval.

**Proof.** If $S$ contains only one point, there is nothing to prove. If it is not the case, let $s_1 < s_2$ be elements of $S$, and $u_1, u_2$ be corresponding $T$-periodic solutions of \((15.33)\) with $s = s_1$ and $s = s_2$ respectively. If $s_1 \leq s \leq s_2$, then

$$
u_1'' + cu_1' + f(t, u_1) = p(t) + s_1 \leq p(t) + s,
$$

$$
u_2'' + cu_2' + f(t, u_2) = p(t) + s_2 \geq p(t) + s,
$$

which shows that $u_1$ is an upper solution and $u_2$ is a lower solution for equation \((15.33)\) with $T$-periodic boundary conditions. The same is true for $u_2 + 2k\pi$ ($k \in \mathbb{Z}$), and hence, by a suitable choice of $k$, we can assume that $u_0(t) = u_2(t) + 2k\pi \leq u_1(t)$ for all $t \in [0, T]$. Consequently, Theorem 14.4.1 implies the existence of at least one $T$-periodic solution between $u_0$ and $u_1$, i.e. $s \in S$.

To show that $S$ is closed, we first need some a priori estimates on the possible $T$-periodic solutions of \((15.33)\). If $u$ is such a solution and $\tau \in [0, T]$ is such that $u'(\tau) = 0$, integrating the equation implies that, for all $t \in [0, T]$,

$$
u'(t) = \int_{\tau}^{t} e^{-c(t-\sigma)} [p(\sigma) + s - f(\sigma, u(\sigma))] \, d\sigma
$$

and hence

$$|u'(t)| \leq M_1$$

for some $M_1 > 0$ depending only upon $c, p, s, T$ and a bound upon $|f|$. Consequently, for all $t \in [0, T]$,

$$|u(t)| \leq |u(0)| + \int_{0}^{t} |u'(\sigma)| \, d\sigma \leq |u(0)| + T M_1,$$
and, as we identify the solutions differing from a multiple of $2\pi$, we can restrict $u(0)$ in such a way that $|u(0)| < 2\pi$, so that, for all $t \in [0, T]$,

$$|u(t)| < 2\pi + TM_1 := M_2.$$ 

From those a priori estimates, a limiting and compactness argument easily implies that $S$ is closed.

The following lemma [303] is useful in computing the Brouwer degree of $I - P_T$ for pendulum-like equations.

**Lemma 15.5.1** Let $B = [\alpha_-, \alpha_+ ] \times [\beta_-, \beta_+ ]$ be an open rectangle and $h : \overline{B} \to \mathbb{R}^2$ be continuous and such that

$$h(\alpha_-, x_2) = h(\alpha_+, x_2) \quad \text{for all} \quad x_2 \in [\beta_-, \beta_+] ,$$  

$$h_1(x_1, \beta_+) < 0 < h_1(x_1, \beta_-) \quad \text{for all} \quad x_1 \in [\alpha_-, \alpha_+] .$$  

Then $d_B[h, B, 0] = 0$.

**Proof.** Define the mapping $g : [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ by

$$g(y_1, y_2) = h(\{(1/2)[\alpha_+ - \alpha_-]y_1 + \alpha_+ + \alpha_-\}, (1/2)[(\beta_+ - \beta_-)]y_2 + \beta_+ + \beta_-\} ,$$

so that, as easily checked,

$$d_B[h, B, 0] = d_B[g, ] - 1, 1[2, 0] ,$$

and define the mapping $g^* : [-1, 1]^2 \to \mathbb{R}^2$ by

$$g^*(y_1, y_2) = g(-y_1, y_2) .$$

By definition of degree,


On the other hand, it is easy to see that, by assumptions,

$$(1 - \lambda)g(y_1, y_2) + \lambda g^*(y_1, y_2) \neq 0$$

whenever $(y_1, y_2) \in \partial[-1, 1]^2$, and hence, by the homotopy invariance property 3.4.2, we have


Hence $d_B[g, ] - 1, 1[2, 0] = 0$.  


15.5. PERIODIC SOLUTIONS OF PENDULUM-TYPE EQUATIONS

Some simple estimates on Poincaré’s operator are necessary to conclude.

**Lemma 15.5.2** There exist $\rho > 0$ such that, for any $(x_0, y_0) \in \mathbb{R}^2$ verifying $|y_0| > \rho$, one has

$$y_0[x_0 - x(T; x_0, y_0)] > 0.$$  

**Proof.** We know that the solution $x(t; x_0, y_0)$ of (15.33) such that $x(0) = x_0, x'(0) = y_0$ satisfies the identities

$$x'(t; x_0, y_0) = y_0 + \int_0^t e^{-c(t-\tau)}[p(\tau) + s - f(\tau, x(\tau; x_0, y_0))] \, d\tau,$$

and hence

$$x(T; x_0, y_0) - x_0 = y_0 T + \int_0^T \int_0^t e^{-c(t-\tau)}[p(\tau) + s - f(\tau, x(\tau; x_0, y_0))] \, d\tau \, dt.$$  

Letting

$$M = \max_{(t, x, s) \in [0,T] \times \mathbb{R} \times [J_L, J_M]} |p(t) + s - f(t, x)|,$$

we obtain

$$y_0^2 T - |y_0|M \leq y_0 x(T; x_0, y_0) - x_0$$

and the result follows.  

**Proposition 15.5.3** Assume that the set of solutions of (15.33) is finite and given by $u_1, \ldots, u_m$. Then

$$\sum_{j=1}^m \gamma_T(u_j) = 0.$$  

**Proof.** Let $\alpha \neq u_j(0)$ ($j = 1,2,\ldots,m$), and consider the rectangle $R = [\alpha, \alpha + 2\pi[\times] - \rho, \rho[, where $\rho > 0$ is given by Lemma 15.5.2. It follows from Lemma 15.5.1 that $d_B[I - P_T, R, 0] = 0$, and the result follows from Proposition 5.1.1.

This result provides some information about the values of the indices of solutions.

Let $S = [s_1, s_2]$.

**Corollary 15.5.1** Assume that the set of solutions of (15.33) is finite and given by $u_1, \ldots, u_m$.

(i) If $s = s_1$ or $s = s_2$, then $\gamma_T(u_j) = 0$ ($j = 1,2,\ldots,m$).

(ii) If $s_1 < s_2$ and $s \in [s_1, s_2[$, there exist $k,l \in \{1,2,\ldots,m\}$ such that $\gamma_T(u_k) < 0$ and $\gamma_T(u_l) = 1$. 

Proof. (i) If some index were not zero, the stability of the index under small perturbations would imply the existence of solutions of (15.33) for \( s_0 - \varepsilon \) or for \( s_1 + \varepsilon \) and \( \varepsilon > 0 \) sufficiently small, a contradiction.

(ii) As observed in the previous section, if \( s_1 < s < s_2 \), a \( T \)-periodic solution \( u_1 \) of (15.33) with \( s = s_1 \) is a strict upper solution for (15.33) with \( T \)-periodic boundary conditions, and a \( T \)-periodic solution \( u_2 \) of (15.33) with \( s = s_2 \) is a strict lower solution for (15.33) with \( T \)-periodic boundary conditions. By a suitable addition of some \( 2k\pi \) to one of them, one can assume that they are ordered, and hence Theorem 14.5.1 implies that the index of at least one of the solution must be negative. On the other hand, Proposition 15.5.3 implies then that one of them must be positive.
Chapter 16

Guiding functions

16.1 Definition and preliminaries

We have already considered, in Chapter 14, the concept of a gradient system

\[ y'(t) = -\nabla V(y(t)), \]

where \( V \in C^1(\mathbb{R}^n, \mathbb{R}) \) has a locally Lipschitzian gradient \( \nabla V \). The T-periodic solutions of this gradient system are particularly simple because, if \( y \) is such a T-periodic solution, then, we have

\[
\int_0^T \|y'(t)\|^2 \, dt = -\int_0^T \langle \nabla V(y(t)), y'(t) \rangle \, dt = -\int_0^T \frac{d}{dt} V(y(t)) \, dt = V(y(T)) - V(y(0)) = 0.
\]

Consequently, each possible T-periodic solution of a gradient system is a constant, and hence a zero of \( \nabla V \). This observation and the analogy with Lyapunov’s second method in stability theory (see e.g. [337]), has led M.A. Krasnosel’skii and his school to introduce the important concept of guiding or directing function in the study of periodic solutions of ordinary differential equations (see e.g. [214], [219], [265], [337] for references).

Definition 16.1.1 We say that \( V \in C^1(\mathbb{R}^n, \mathbb{R}) \) is a guiding function for (14.1) if there exists \( \rho_0 > 0 \) such that

\[
\langle \nabla V(x), f(t, x) \rangle > 0 \quad (16.1)
\]

for all \( t \in \mathbb{R} \) and all \( x \in \mathbb{R}^n \) such that \( \|x\| \geq \rho_0 \).
So, for large \( x \), the qualitative behavior of the vector field \( f \) is similar to that of \( \nabla V \). By (16.1), we have necessarily

\[
\nabla V(x) \neq 0
\]

for all \( x \in \mathbb{R}^n \) with \( \|x\| \geq \rho_0 \), and hence the Brouwer degree \( d_B[\nabla V, B(r), 0] \) is defined and constant for all \( r \geq \rho_0 \). It is called the index at infinity of \( V \) and denoted by \( i_B[V, \infty] \). It follows from Corollary 14.7.1 that if \( V \) is a coercive guiding function, then

\[
i_B[V, \infty] = 1.
\]

By (16.1) and (16.2), we have also, for all \( \lambda \in [0, 1], t \in \mathbb{R} \) and all \( x \in \mathbb{R}^n \) with \( \|x\| \geq \rho_0 \),

\[
(\nabla V(x), (1 - \lambda)\nabla V(x) + \lambda f(t, x)) > 0,
\]

and hence, by the homotopy invariance theorem 3.4.2, we get

\[
i_B[V, \infty] = d_B[\nabla V, B(r), 0] = d_B[f(t, \cdot), B(r), 0]
\]

for all \( t \in \mathbb{R} \) and all \( r \geq \rho_0 \).

The existence of a guiding function implies some a priori localization of the possible T-periodic solutions of eq. (14.1).

**Lemma 16.1.1** If eq. (14.1) admits a guiding function \( V \), and if we set

\[
R_0 = \max_{\|x\| \leq \rho_0} V(x),
\]

then any possible T-periodic solution \( y \) of eq. (14.1) is such that

\[
V(y(t)) \leq R_0 \quad \text{for all} \quad t \in \mathbb{R}.
\]

**Proof.** Let \( y \) be a possible T-periodic solution of eq. (14.1). If the conclusion of the lemma does not hold, there exists \( t_0 \in \mathbb{R} \) such that \( V(y(t_0)) > R_0 \). Consequently,

\[
V(y(\tau)) = \max_{t \in \mathbb{R}} V(y(t)) > R_0,
\]

which implies that

\[
\|y(\tau)\| > \rho_0, \quad 0 = \frac{d}{dt}[V(y(\tau))] = (\nabla V(y(\tau)), y'(\tau)) = (\nabla V(y(\tau)), f(\tau, y(\tau))),
\]

a contradiction the the definition of a guiding function.

The following lemma can be found in [214] and [219].

**Lemma 16.1.2** Assume that \( f \) is locally Lipschitzian with respect to \( y \), that \( \tau_+(s, x) = +\infty \) for all the solutions of (14.1) with initial conditions \( y(s) = x \), and that (14.1) admits a guiding function \( V \). Then, for each \( t, s \in \mathbb{R} \) with \( t > s \), there exists some \( \rho_1 = \rho_1(t, s) \geq \rho_0 \) such that for each \( x \in \mathbb{R}^n \) with \( \|x\| > \rho_1(t, s) \) and each \( \tau \in ]s, t] \), one has \( x \neq y(\tau; s, x) \).
Proof. For \( t > s \), let
\[
\rho_1 = \rho_1(t, s) = \max\{\|y(\tau; \sigma, x)\| : s \leq \sigma < t, \|x\| \leq \rho_0\},
\]
so that \( \rho_1 \geq \rho_0 \). We claim that this \( \rho_1 \) satisfies the conclusion of Lemma 16.1.2. If it is not the case, there will exist \( x_0 \in \mathbb{R}^n \) with \( \|x_0\| > \rho_1 \), and \( \tau_0 \in ]s, t] \) such that
\[
y(\tau_0; s, x_0) = x_0.
\] (16.7)
By the definition of \( \rho_1 \), we have necessarily
\[
\|y(\tau; s, x_0)\| > \rho_0
\]
for all \( \tau \in [s, \tau_0] \). Hence, letting \( v(\tau) = V(y(\tau; s, x_0)) \), we get
\[
v'(\tau) = \langle \nabla V(y(\tau; s, x_0)), y'(\tau; s, x_0) \rangle = \langle \nabla V(y(\tau; s, x_0)), f(\tau, y(\tau; s, x_0)) \rangle > 0,
\]
whenever \( \tau \in [s, \tau_0] \). Therefore,
\[
V(y(\tau_0; s, x_0)) = v(\tau_0) > v(s) = V(x_0),
\]
a contradiction with (16.7).
\[\blacksquare\]
The following Corollary is an easy consequence of Lemma 16.1.2.

**Corollary 16.1.1** Assume that \( f \) is locally Lipschitzian with respect to \( y \), \( T \)-periodic with respect to \( t \), that \( \tau_+(s, x) = +\infty \) for all the solutions of (14.1) with initial conditions \( y(s) = x \) and that (14.1) admits a guiding function \( V \). Then, there exists some \( \rho_1 = \rho_1(T, 0) \geq \rho_0 \) such that for each \( x \in \mathbb{R}^n \) with \( \|x\| > \rho_1 \) and each \( \tau \in [0, T] \), one has \( x \neq y(\tau; 0, x) \).

### 16.2 Systems with continuable solutions

We can now state and prove an existence theorem for \( T \)-periodic solutions due to M.A. Krasnosel’skii and V.V. Strygin [218] (see also [214]-[219]).

**Theorem 16.2.1** Assume that \( f \) is locally Lipschitzian with respect to \( y \), \( T \)-periodic with respect to \( t \), and that the following conditions hold:

1. (14.1) admits a guiding function \( V \).
2. \( i_B[V, \infty] \neq 0 \).
3. \( \tau_+(s, x) = +\infty \) for each solution of (14.1) with initial conditions \( y(s) = x \), \( s \in \mathbb{R}, \ x \in \mathbb{R}^n \).

Then (14.1) has at least one \( T \)-periodic solution \( y \) such that \( V(y(t)) \leq R_0 \) for all \( t \in \mathbb{R} \) and \( R_0 \) defined in (16.5).
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Proof. Let $\rho_1 = \rho_1(T, 0)$ be given by Corollary 16.1.1, $\rho > \rho_1$, and $G = B(\rho) \subset \mathbb{R}^n$. By Corollary 16.1.1, for each $x \in \partial G$, we have $y(\lambda T; 0, x) \neq x$ for all $\lambda \in [0, 1]$. By the existence of the guiding function, we have $f(t, x) \neq 0$ for all $t \in \mathbb{R}$ and all $x \in \partial G$, and moreover

$$d_B[f(0, \cdot), G, 0] = d_B[\nabla V, G, 0] = i_B[V, \infty] \neq 0,$$

by Assumption 3. The result then follows from Theorem 14.6.1 and Lemma 16.1.1.

The following consequence of Theorem 16.2.1, due to M.A. Krasnosel’skii and A.I. Perov [216], has assumptions which are more easy to verify in practice.

**Corollary 16.2.1** Assume that $f$ is locally Lipschitzian with respect to $y$, $T$-periodic with respect to $t$. If (14.1) has a guiding function $V$ such that $V$ or $-V$ is coercive, then (14.1) has at least one $T$-periodic solution $y$ such that $V(y(t)) \leq R_0$ for all $t \in \mathbb{R}$ and $R_0$ defined in (16.5).

Proof. Assume first that $-V$ is coercive. Then, letting $W = -V$, we see that

$$\langle \nabla W(x), f(t, x) \rangle < 0,$$

for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ with $\|x\| \geq \rho_0$. Hence, by a reasoning made in the proof of Theorem 14.7.1, we see that for each $c \geq \rho_0$, the sets $W^{-1} = \{x \in \mathbb{R}^n : W(x) \leq c\}$ are bounded (by the coercivity of $W$) and positively invariant for the solutions of (14.1). Consequently, Assumption 3 of Theorem 16.2.1 holds. Moreover,

$$i_B[V, \infty] = d_B[\nabla V, B(\rho_0), 0] = (-1)^n d_B[\nabla W, B(\rho_0), 0] = (-1)^n \neq 0.$$

Thus all the assumptions of Theorem 16.2.1 are satisfied and (14.1) has a $T$-periodic solution. If $V$ is coercive, then letting $u(t) = y(-t)$, we see that $y$ is a $T$-periodic solution of (14.1) if and only if $u$ is a $T$-periodic solution of

$$u'(t) = -f(-t, u(t)). \quad (16.8)$$

For this equation, $W = -V$ is a guiding function such that $-W$ is coercive, and the first part of the proof implies the existence of a $T$-periodic solution $u$.

**Example 16.2.1** Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of odd degree and $h : \mathbb{R} \to \mathbb{R}$ a continuous $T$-periodic function. Then equation

$$y'(t) = p(y(t)) + h(t)$$

has at least one $T$-periodic solution.

Indeed, if

$$p(x) = \sum_{j=0}^{2m+1} a_j x^j,$$
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with \( a_{2m+1} \neq 0 \), let us take

\[
V(x) = \frac{a_{2m+1}}{2}x^2.
\]

Then \( V \) or \( -V \) is coercive, and

\[
a_{2m+1}x[p(x) + h(t)]
\]

\[
= a_{2m+1}x^{2m+2}[1 + \sum_{j=0}^{2m} a_{2m+1}^{-2}a_j x^{j-2m-1} + a_{2m+1}^{-2}h(t)x^{-2m-1}] > 0
\]

for \( |x| \) sufficiently large. Thus \( V \) is guiding function for the above equation and the existence follows.

As another application, we can prove the following existence theorem due to G. Gustafson and K. Schmitt [158].

**Theorem 16.2.2** Assume that \( f \) is continuous, locally Lipschitzian in \( x \), \( T \)-periodic in \( t \), and that there exists \( R > 0 \) such that, for all \( t \in \mathbb{R} \) and all \( x \in \mathbb{R}^n \) with \( \|x\| = R \), one has

\[
\langle x, f(t, x) \rangle \geq 0. \tag{16.9}
\]

Then eq. (14.1) has at least one \( T \)-periodic solution \( y \) such that \( \|y(t)\| \leq R \) for all \( t \in \mathbb{R} \).

**Proof.** Let us define \( g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) by

\[
g(t, x) = f(t, x) \quad \text{if} \quad \|x\| \leq R,
\]

\[
g(t, x) = \left(1 - \frac{R}{\|x\|}\right)x + f \left(t, \frac{R}{\|x\|}x\right) \quad \text{if} \quad \|x\| > R.
\]

Then \( g \) is continuous, \( T \)-periodic in \( t \) and locally Lipschitzian in \( x \). Furthermore, for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \) with \( \|x\| > R \), we have

\[
\langle x, g(t, x) \rangle = \left(1 - \frac{R}{\|x\|}\right)\|x\|^2 + \frac{\|x\|}{R} \left(\frac{R}{\|x\|}x, f \left(t, \frac{R}{\|x\|}x\right)\right) > 0,
\]

so that \( V(x) = \frac{\|x\|^2}{2} \) is a coercive guiding function for equation

\[
y' = g(t, y) \tag{16.10}
\]

if we choose any \( \rho_0 > R \). By Corollary 16.2.1, eq. (16.10) has at least one \( T \)-periodic solution \( y \) such that \( \|y(t)\| \leq \rho_0 \) for all \( t \in \mathbb{R} \) and all \( \rho_0 > R \), and hence such that \( \|y(t)\| \leq R \) for all \( t \in \mathbb{R} \). Thus, \( y \) is a \( T \)-periodic solution of eq. (14.1).
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Remark 16.2.1 The conclusion of Theorem 16.2.2 also holds if condition (16.9) is replaced by

\[ \langle x, f(t, x) \rangle \leq 0 \quad \text{whenever} \quad t \in \mathbb{R}, \quad \|x\| = R. \] (16.11)

An interesting consequence of Theorem 16.2.2 for the scalar equation

\[ y' = g(t, y) \] (16.12)

with \( g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) continuous, T-periodic in \( t \), locally Lipschitzian in \( y \), is the following result.

Corollary 16.2.2 Assume that there exists \( A < B \) such that, for all \( t \in \mathbb{R} \), one has

\[ g(t, A) \leq 0 \leq g(t, B) \quad \text{or} \quad g(t, A) \geq 0 \geq g(t, B). \] (16.13)

Then eq. (16.12) has at least one T-periodic solution such that, for all \( t \in \mathbb{R} \),

\[ A \leq y(t) \leq B. \]

Proof. Letting

\[ u(t) = y(t) - \frac{A + B}{2}, \quad f(t, u) = g \left( t, u + \frac{A + B}{2} \right), \]

we see that \( y \) is a T-periodic of eq. (16.12) if and only if \( u \) is a T-periodic solution of equation

\[ u' = f(t, u). \] (16.14)

Now, conditions (16.13) become

\[ f \left( t, \frac{A - B}{2} \right) \leq 0 \leq f \left( t, \frac{B - A}{2} \right) \quad \text{or} \quad f \left( t, \frac{A - B}{2} \right) \geq 0 \geq f \left( t, \frac{B - A}{2} \right), \]

i.e. the one-dimensional version of (16.9) or of (16.11) with \( R = \frac{B - A}{2} \). \( \blacksquare \)

As a special case which is useful in mathematical demography, let us consider the Kolmogorov equation

\[ u' = uh(t, u), \] (16.15)

where \( h : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, T-periodic in \( t \), locally Lipschitzian in \( u \). We are interested in the positive T-periodic solutions of eq. (16.15), and hence we set

\[ u(t) = \exp y(t), \]

which transforms eq. (16.15) into

\[ y' = h(t, \exp y). \] (16.16)
Corollary 16.2.3 Assume that the following conditions hold.

1. \( h(t,0) > 0 \) for all \( t \in \mathbb{R} \).
2. \( h(t,C) \leq 0 \) for some \( C > 0 \).

Then eq. (16.15) has at least one positive \( T \)-periodic solution.

Proof. It suffices to prove that eq. (16.16) has a \( T \)-periodic solution. By assumption 2, we have \( h(t, \exp B) \leq 0 \) for \( B = \log C \) and all \( t \in \mathbb{R} \). By assumption 1 and \( T \)-periodicity, we have \( h(t,0) \geq \alpha > 0 \) for all \( t \in \mathbb{R} \) and some \( \alpha > 0 \), and hence there exist \( A < B \) such that \( h(t, \exp A) \geq \alpha / 2 > 0 \) for all \( t \in \mathbb{R} \). The result follows from Corollary 16.2.2.

In particular, the conditions of Corollary 16.2.3 are satisfied for Verhulst equation with \( T \)-periodic continuous positive coefficients \( a \) and \( b \)

\[
    u' = u[a(t) - b(t)u].
\]

Notice that, in this special case, the transformation \( u = \frac{1}{v} \) transforms the problem into the first order linear differential equation

\[
    v' = -a(t)v + b(t),
\]

for which the (unique) positive \( T \)-periodic can be computed explicitly.

Further applications of the method of guiding functions can be found in [214, 219].

16.3 Systems with non-continuable solutions

We now describe a generalization of Theorem 16.2.1, which can be found in [215], and does not require the global extendability of the solutions of the Cauchy problem. Its proof requires the introduction of the following concepts. Consider the value

\[
    \alpha(\rho_0) = \inf_{0 \leq t_0 \leq T; \|x_0\| \leq \rho_0} [\tau_+(t_0, x_0) - \tau_-(t_0, x_0)], \quad (16.17)
\]

By a classical result (see e.g. [231]), \( \tau_+ - \tau_- \) is a lower semicontinuous function, and hence the infimum is reached. Therefore, this value is equal to either \( +\infty \) or some positive number. Without special notice we shall use the equalities

\[
    \alpha(\rho_0) = \inf_{\tau \leq t_0 \leq T + \tau; \|x_0\| \leq \rho_0} [\tau_+(t_0, x_0) - \tau_-(t_0, x_0)], \quad \tau \in \mathbb{R},
\]

and

\[
    \alpha(\rho_0) = \inf_{t_0 \in \mathbb{R}; \|x_0\| \leq \rho_0} [\tau_+(t_0, x_0) - \tau_-(t_0, x_0)],
\]

which follow from the \( T \)-periodicity of \( f \) with respect to \( t \).
Definition 16.3.1 A solution \( y(t) \) of (14.1) is called a \((T, \rho_0, \rho_1)\)-solution of (14.1) \((\rho_1 > \rho_0)\) if it is defined on some closed bounded interval \([t_1, t_2] \) and

(i) \[ \| y(t_1) \| = \| y(t_2) \| = \max_{t_1 \leq t \leq t_2} \| y(t) \| = \rho_1, \]

(ii) \[ \| y(t_0) \| \leq \rho_0 \text{ for some } t_0 \in (t_1, t_2), \]

(iii) \[ t_2 - t_1 \leq T. \]

Remark 16.3.1 According to the periodicity of the function \( f(t, x) \) we can assume without loss of generality that the values \( t_0, t_1, t_2 \) satisfy the inequalities

\[ 0 \leq t_1 \leq T, \quad 0 \leq t_1 < t_0 < t_2 \leq 2T. \]

Remark 16.3.2 If (14.1) has some \((T, \rho_0, \rho_1)\)-solution \( y(t) \) \((t_1 \leq t \leq t_2) \) then (14.1) has a \((T, \rho_0, \rho)\)-solution for any \( \rho \in [\rho_0, \rho_1] \). Indeed, consider the values

\[ t_{1, \rho} = \max \{ t \in [t_1, t_0] \mid y(t) = \rho \}, \quad t_{2, \rho} = \min \{ t \in [t_0, t_2] \mid y(t) = \rho \}. \]

The function \( y(t) \) considered on the interval \([t_{1, \rho}, t_{2, \rho}] \) is the \((T, \rho_0, \rho)\)-solution of (14.1).

Lemma 16.3.1 Assume that \( f \) is \( T \)-periodic in \( t \), locally Lipschitzian in \( y \) and continuous. If \( T < \alpha(\rho_0) \), then there exist \( \rho_* > \rho_0 \) such that equation (14.1) has no \((T, \rho_0, \rho_1)\)-solution for \( \rho_1 \geq \rho_* \).

Proof. Due to Remark 16.3.2, it is sufficient to prove the existence of \( \rho_* > \rho_0 \) such that the system (14.1) has no \((T, \rho_0, \rho_1)\)-solution. If it is not the case, we can find a sequence \((\rho_n)\) with \( \rho_n > \rho_0 \) and \( \rho_n \to \infty \) such that, for every \( n \) the system (14.1) has a \((T, \rho_0, \rho_n)\)-solution \( y_n(t) \) \((t^n_1 \leq t \leq t^n_2) \). It means that

\[ \| y_n(t^n_1) \| = \| y_n(t^n_2) \| = \max_{t^n_1 \leq t \leq t^n_2} \| y_n(t) \| = \rho_n, \]

and \( \| y_n(t^n_0) \| \leq \rho_0 \) for some \( t^n_0 \in ]t^n_1, t^n_2[. \) Due to Remark 16.3.1 we can assume that

\[ 0 \leq t^n_0 \leq T, \quad 0 \leq t^n_1 < t^n_0 < t^n_2 \leq 2T, \quad t^n_2 - t^n_1 \leq T. \]

Without loss of generality (consider if necessary some subsequences), we can suppose that all the sequences \((t^n_1), (t^n_0), (t^n_2)\) and \((y_n(t^n_0))\) converge to some limits \( t_1^*, t_0^*, t_2^*, y_0^* \). Obviously, \( 0 \leq t_1^* \leq T, \quad t_0^* \leq t_2^* \leq 2T, \quad t_2^* - t_1^* \leq T, \quad \| y_0^* \| \leq \rho_0. \)

Consider the solution \( y^*(t) = y(t; t_0^*, y_0^*) \) of (14.1). Since by assumption the inequality

\[ T < \tau_+ (t_0^*, y_0^*) - \tau_- (t_0^*, y_0^*) \]

holds, then the solution \( y^*(t) \) is defined at least on one of the closed intervals \([t_1^*, t_0^*], [t_0^*, t_2^*]\), say the interval \([t_1^*, t_0^*]\). Now,

\[ y_n(t_1^*) = y(t_1^*; t_0^*, y_n(t_0^*)), \quad y^*(t_1^*) = y(t_1^*; t_0^*, y_0^*). \]
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and the classical theorem on the continuity of solutions with respect to initial values (see e.g. [71]) implies that for sufficiently large \( n \) the estimate

\[
\| y_n(t_1^n) - y^\ast(t_1^n) \| < 1
\]

is valid; therefore

\[
\rho_n = \| y_n(t_1^n) \| < 1 + \| y^\ast(t_1^n) \| < \infty,
\]

so that

\[
\limsup_{n \to \infty} \rho_n \leq 1 + \| x^\ast(t_1^\ast) \| < \infty.
\]

This contradiction proves the lemma.

We now prove a result on the existence of a priori estimates for the periodic solutions of (14.1).

**Lemma 16.3.2** Assume that the conditions of Lemma 16.3.1 hold and that (14.1) has a guiding function \( V \). Then each possible \( T \)-periodic solution \( y(t) \) of system (14.1) satisfies the a priori estimate

\[
\| y(t) \| < \rho_\ast, \quad t \in \mathbb{R}.
\]

**Proof.** Let the statement of the lemma be false. Then (14.1) has a \( T \)-periodic solution \( y(t) \) such that

\[
\max_{0 \leq t \leq T} \| y(t) \| = r \geq \rho_\ast. \tag{16.18}
\]

Consequently, either

\[
\min_{0 \leq t \leq T} \| y(t) \| \geq \rho_0, \quad \text{or for some } t_0 \in [0,T] \tag{16.19}
\]

\[
\| y(t_0) \| < \rho_0. \tag{16.20}
\]

Let (16.19) be valid. Put

\[
v(t) = V[y(t)], \quad t \in \mathbb{R}. \tag{16.21}
\]

Due to the \( T \)-periodicity of \( y(t) \), the real function (16.21) is also \( T \)-periodic. On the other hand

\[
v'(t) = (f[t,y(t)],V'[y(t)]) > 0,
\]

which contradicts the periodicity of \( v(t) \). Let (16.20) be valid for some \( t_0 \in [0,T] \). Let

\[
t_1 = \max \{ t < t_0 : \| y(t) \| = \rho_\ast \}, \quad t_2 = \min \{ t > t_0 : \| y(t) \| = \rho_\ast \}
\]

Since \( y(t) \) is \( T \)-periodic, \( t_2 - t_1 \leq T \). Therefore \( y(t) \) is a \((T,\rho_0,\rho_\ast)\)-solution of (14.1). But Lemma 16.3.1 implies that (14.1) has no \((T,\rho_0,\rho_\ast)\)-solutions. This contradiction proves the lemma. ■
CHAPTER 16. GUIDING FUNCTIONS

We can now state and prove the following existence theorem ([215]).

**Theorem 16.3.1** Assume that $f$ is $T$-periodic with respect to $T$, locally Lipschitzian with respect to $y$ and continuous. Assume that (14.1) has a guiding function $V$ such that $i_B[V, \infty] \neq 0$ and that $T < \alpha(\rho_0)$. Then equation (14.1) has at least one $T$-periodic solution $y$ with $\max_{t \in \mathbb{R}} \|y(t)\| < \rho_*$. 

**Proof.** Let

$$
\beta(r) = \max_{0 \leq t \leq T, |y| \leq r} \|f(t, y)\|, \quad 0 \leq r < \infty.
$$

This function is non-decreasing and since $\langle f(t, y), \nabla V(y) \rangle > 0$, $\|y\| \geq \rho_0$ its values are positive for $r \geq \rho_0$. Put

$$
\varphi(y) = \begin{cases} 
1 & \text{if } 0 \leq \|y\| \leq \rho_* \\
\text{linear} & \text{if } \rho_* \leq \|y\| \leq 1 + \rho_* \\
(1 + \beta(\|y\|))^{-1} & \text{if } 1 + \rho_* \leq \|y\|.
\end{cases}
$$

Consider the auxiliary equation

$$
\frac{dy}{dt} = \varphi(y)f(t, y). \quad (16.22)
$$

Since the right-hand side of (16.22) is uniformly bounded we see that all the solutions of (16.22) are infinitely continuable both to the left and to the right. Since, $\varphi(y) > 0$ for all $y \in \mathbb{R}^n$, the function $V(y)$ is a guiding function for system (16.22), with the same $\rho_0$, and the index at infinity of the guiding function $V(y)$ is different from 0. Therefore Theorem 16.2.1 implies the existence of at least one $T$-periodic solution of (16.22). Let us show that every $T$-periodic solution $y(t)$ of (16.22) is a $T$-periodic solution of (14.1), i.e. let us prove the estimate

$$
\|y(t)\| \leq \rho_*, \quad t \in \mathbb{R}. \quad (16.23)
$$

We shall use constructions which has been already used in the proof of Lemma 16.3.2. Assume (16.23) is not valid. If

$$
\min_{0 \leq t \leq T} \|y(t)\| \geq \rho_0
$$

then the $T$-periodic scalar-valued function $w(t) = V[y(t)]$ ($t \in \mathbb{R}$) satisfies the estimate

$$
\frac{dw(t)}{dt} = \varphi[y(t)] \langle f(t, y(t)), \nabla V[y(t)] \rangle > 0,
$$

which is impossible. Now, if $\|y(t_0)\| < \rho_0$, for some $t_0 \in [0, T]$ then, for some $t_1 < t_1$ with $t_2 - t_1 < T$, $y(t_1 \leq t \leq t_2)$ is a $(T, \rho_0, \rho_*)$-solution of (16.22). Therefore $\|y(t_0)\| \leq \rho_*$ for $t \in [t_1, t_2]$ and $y(t)$ is a $(T, \rho_0, \rho_*)$-solution of system (14.1). This contradicts Lemma 16.3.1. Hence (16.23) is proved, which implies that every $T$-periodic solution of (16.22) is a $T$-periodic solution of (14.1).
16.4 Generalized guiding functions

In this section, we extend Theorem 16.3.1 by weakening the definition of a guiding function. The following concept was introduced in [265].

**Definition 16.4.1** We say that $V \in C^1(\mathbb{R}^n, \mathbb{R})$ is a generalized guiding function for (14.1) if there exists some $\rho_0 > 0$ such that

$$\langle f(t, x), \nabla V(x) \rangle \geq 0,$$

for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$ such that $\|x\| \geq \rho_0$.

We want to replace in Theorem 16.3.1 the assumption of the existence of a guiding function by that of a generalized guiding function $V$ such that $\nabla V(x) \neq 0$ for $\|x\| \geq \rho_0$. A natural idea to prove such a result would be to consider the perturbed problems

$$y' = f(t, y) + \epsilon \nabla V(y),$$

for which $V$ is a guiding function, as immediately checked. But $\nabla V$ is not necessarily locally Lipschitzian, and hence it is necessary to replace it by another locally Lipschitzian function having the same qualitative behavior.

**Lemma 16.4.1** Let $V \in C^1(\mathbb{R}^n, \mathbb{R})$ be a generalized guiding function for (14.1) such that $\nabla V(x) \neq 0$ whenever $\|x\| \geq \rho_0$. Then there exists a locally Lipschitzian function $g(x) : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\langle \nabla V(x), g(x) \rangle > 0, \quad \|x\| \geq \rho_0.$$  \hfill (16.24)

**Proof.** The continuity of $\nabla V$ implies that for every $x$, $\|x\| \geq \rho_0$ there is an $r(x) \in [0, 1]$ such that

$$\langle \nabla V(y), \nabla V(x) \rangle > 0, \quad y \in B(x; r(x)).$$  \hfill (16.25)

The open balls $B_x(r(x))$ form a cover of the closed set $F = \{x : \|x\| \geq \rho_0\} \subset \mathbb{R}^n$. Let us choose from this cover a countable subcover

$$B_{x_i}(r(x_i)), \quad i = 1, 2, \ldots$$  \hfill (16.26)

such that every point $x$ belongs to a finite number only of balls (16.26). Define for every $i = 1, 2, \ldots$ the function $\beta_i : \mathbb{R}^n \to \mathbb{R}, x \mapsto \text{dist}(x, \mathbb{R}^n \setminus B_{x_i}(r(x_i)))$. Each $\beta_i$ is locally Lipschitzian. Put, for $x \in \mathbb{R}^n$,

$$g(x) = \sum_{i=1}^{\infty} \beta_i(x) \nabla V(x_i), \quad x \in \mathbb{R}^n.$$  \hfill (16.27)

In this sum, for every $x$, a finite number of values $\beta_i(x)$ only differ from 0. The function (16.27) is obviously locally Lipschitzian on $\mathbb{R}^n$, and

$$\langle \nabla V(x), g(x) \rangle = \sum_{i=1}^{\infty} \beta_i(x) \langle \nabla V(x), \nabla V(x_i) \rangle$$

$$= \sum_{\{i : x \in B_{x_i}(r(x_i))\}} \beta_i(x) \langle \nabla V(x), \nabla V(x_i) \rangle > 0.$$
We can now state and prove the main result of this section, which is taken from [215].

**Theorem 16.4.1** Assume that $f$ is $T$-periodic with respect to $T$, locally Lipschitzian with respect to $y$ and continuous. Assume that (14.1) has a generalized guiding function $V$ such that $\nabla V(x) \neq 0$ whenever $\|x\| \geq \rho_0$ and $i_B[V, \infty] \neq 0$. If $T < \alpha(\rho_0)$, then equation (14.1) has at least one $T$-periodic solution $x$ with $\max_{t \in \mathbb{R}} \|x(t)\| \leq \rho_*$.

**Proof.** Let $g$ be given by Lemma 16.4.1 and consider the family of equations

$$y' = f(t, y) + \epsilon g(y), \quad (16.28)$$

with $\epsilon \geq 0$. Let $[\tau_-(s, \epsilon), \tau_+(s, \epsilon)]$ denote the maximal existence interval of the solution of the Cauchy problem $y(s) = x$, for equation (16.28). We know (see e.g. [231]) that $\tau_+ - \tau_-$ is lower semicontinuous. In particular, $\tau_+ - \tau_-$ reaches its infimum on each compact set. We first claim that there exist $\epsilon^* > 0$ such that

$$T < \inf_{0 \leq s \leq T; \|x\| \leq \rho_0, 0 \leq \epsilon \leq \epsilon^*} [\tau_+(s, x, \epsilon) - \tau_-(s, x, \epsilon)].$$

If it is not the case, then, for each positive integer $k$, we have

$$T \geq \inf_{0 \leq s \leq T; \|x\| \leq \rho_0, 0 \leq \epsilon \leq 1/k} [\tau_+(s, x, \epsilon) - \tau_-(s, x, \epsilon)] = \tau_+(s_k, x_k, \epsilon_k) - \tau_-(s_k, x_k, \epsilon_k),$$

for some $s_k \in [0, T]$, $\|x_k\| \leq \rho_0$, $\epsilon_k \in [0, 1/k]$. Going if necessary to a subsequence, we can assume that $s_k \to s^*$ and $x_k \to x^*$ for some $s^* \in [0, T]$ and $\|x^*\| \leq \rho_0$. Therefore, by lower semicontinuity,

$$T \geq \liminf_{k \to \infty} [\tau_+(s_k, x_k, \epsilon_k) - \tau_-(s_k, x_k, \epsilon_k)] = \tau_+(s^*, x^*, 0) - \tau_-(s^*, x^*, 0) > T,$$

a contradiction. Let $(\epsilon_n)$ be a sequence in $[0, \epsilon^*]$ which converges to zero. For each positive integer $n$, equation (16.28) with $\epsilon = \epsilon_n$ satisfies the conditions of Theorem 16.3.1 and hence has at least one $T$-periodic solution $y_n(t)$, such that

$$\|y_n(t)\| \leq \rho_*, \quad t \in \mathbb{R}. $$

Since

$$\|y_n(t)\| \leq \max_{0 \leq t \leq T; \|y\| \leq \rho_*} \|f(t, y)\| + \epsilon_n \max_{\|y\| \leq \rho_*} \|g(y)\|, \quad t \in \mathbb{R}; \quad n = 1, 2, \ldots,$$

the sequence $(y_n)$ is bounded and hence the sequence $(y_n)$ is compact in $C = C[0, T]$. Therefore one can choose a subsequence $(y_{n_k})$ ($k = 1, 2, \ldots$) converging in $C$ to some function $y$. Every function $y_{n_k}$ is $T$-periodic and satisfies the integral equation

$$y_{n_k}(t) = y_{n_k}(0) + \int_0^t f[s, y_{n_k}(s)] \, ds + \epsilon_{n_k} \int_0^t g[y_{n_k}(s)] \, ds.$$

Consequently, letting $k$ going to infinity, we see that $y$ is $T$-periodic and is a solution of

$$y(t) = y(0) + \int_0^t f[s, y(s)] \, ds,$$

and hence a $T$-periodic solution of the differential equation (14.1).
16.4. GENERALIZED GUIDING FUNCTIONS

Remark 16.4.1 The above proof implies the estimate \( \|y(t)\| \leq \rho_\ast (t \in \mathbb{R}) \). This estimate does not mean that there is an a priori estimate of all the \( T \)-periodic solutions of equation (14.1). Under assumptions of Theorem 16.3.1, \( T \)-periodic solutions of (14.1) can have an arbitrary large norm in \( C \). In those cases they cannot be constructed by the procedure used above in the proof of Theorem 16.4.1. A very simple example is given by the equation \( y' = 0 \).

The application of Theorem 16.4.1 requires precise estimates of the value \( \alpha(\rho_0) \). An answer to this problem makes use of theorems on differential inequalities (see e.g. [214],[236]). For example, let \( M(u) (u \geq 0) \) be a positive and continuous function. Natural examples of \( M(u) \) are the functions

\[
a + bu^p (p \geq 1), \quad a + bu \ln (1 + u), \quad a + be^u, \quad a + be^{u^2}, \tag{16.29}
\]

etc. Denote by \( u(t) (t \geq 0) \) a solution of Cauchy problem

\[
\frac{du}{dt} = M(u), \quad u(0) = \rho_0^2.
\]

The solution \( u(t) \) increases (since \( M(u) > 0 \)). Let \([0, m(\rho_0)]\) be the maximal interval where \( u(t) \) is defined. Assume that

\[
\langle f(t, x), x \rangle \leq \frac{1}{2} M(\|x\|^2), \quad \|x\| \geq \rho_0. \tag{16.30}
\]

Consider some solution \( y(t) \) of (14.1) satisfying

\[
y(t_0) = x_0, \quad \|x_0\| = \rho_0, \quad \|y(t)\| \geq \rho_0, \quad (t \geq t_0).
\]

Assumption (16.30) (for \( t > t_\ast \)) implies

\[
\frac{d}{dt}\|y(t)\|^2 = 2 \left\langle \frac{dy}{dt}(t), y(t) \right\rangle = 2 \langle f[t, y(t)], y(t) \rangle \leq M(\|y(t)\|^2).
\]

Therefore the estimate

\[\|y(t)\|^2 \leq u(t - t_0)\]

holds for \( t \in [t_0, t_0 + m(\rho_0)]\). We have proved the following result.

**Proposition 16.4.1** The estimate (16.30) guarantees an estimate

\[\tau_+(t_0, x_0) \geq t_0 + m(\rho_0)\]

for every \( t_0 \in \mathbb{R} \) and \( \|x_0\| \geq \rho_0 \).

The next statement is dual to Proposition 16.4.1 and can be proved analogously. Consider one function \( M_1 \) of type (16.29). Let the function \( M_1(u) (u \geq 0) \) be positive and continuous. Denote by \( u_1(t) (t \geq 0) \) a solution of Cauchy problem

\[
\frac{du_1}{dt} = M_1(u), \quad u_1(0) = \rho_0^2.
\]

Let \([0, m_1(\rho_0)]\) be the maximal interval where \( u_1(t) \) is defined.
Proposition 16.4.2 The estimate
\[
\langle f(t, x), x \rangle \geq -\frac{1}{2} M_1 \|x\|^2, \quad \|x\| \geq \rho_0
\]
guarantees an estimate
\[
\tau_-(t_0, x_0) \leq t_0 - m_1(\rho_0)
\]
for every \( t_0 \in \mathbb{R} \) and \( \|x_0\| \geq \rho_0 \).

Condition (16.30) implies \( \alpha(\rho_0) \geq m(\rho_0) \), condition (16.31) implies \( \alpha(\rho_0) \geq m_1(\rho_0) \). If both conditions hold then
\[
\alpha(\rho_0) \geq m(\rho_0) + m_1(\rho_0).
\]
In this case the assumption over \( T \) of Theorem 16.4.1 can be rewritten as
\[
T < m(\rho_0) + m_1(\rho_0).
\]
A suitable choice of functions \( M(u) \) and \( M_1(u) \) makes possible the obtention of sharp estimates for the period \( T \).

If we cannot estimate the value of \( \alpha(\rho_0) \), then Theorem 16.4.1 has the following meaning: the existence of a generalized guiding function of non-zero index at infinity guarantees the existence of \( T \)-periodic solutions for sufficiently small \( T > 0 \), i.e. of oscillations with sufficiently high frequency.

As a special case of Theorem 16.4.1, we can obtain the following improvement of Corollary 16.2.1, which was first proved in [265]. Its proof is similar to that of Corollary 16.2.1.

Corollary 16.4.1 Assume that \( f \) is locally Lipschitzian with respect to \( y \), \( T \)-periodic with respect to \( t \). If (14.1) has a generalized guiding function \( V \) such that \( \nabla V(x) \neq 0 \) whenever \( \|x\| \geq \rho_0 \) and \( V \) or \( -V \) is coercive, then (14.1) has at least one \( T \)-periodic solution.
Chapter 17

Computing degree in dimension two

17.1 The Kronecker index on a closed simple curve

Let $D \subset \mathbb{R}$ be an open bounded set such that $\partial D$ is a closed Jordan curve of class $C^2$, and let $f \in C^1(\overline{D}, \mathbb{R})$ be such that $0 \notin f(\partial D)$. Recall that the Kronecker index of $f$ on $\partial D$ is defined by

$$i_K[f, \partial D] = \int_{\partial D} \|f\|^{-2}[f_1 df_2 - f_2 df_1].$$

So, if $\partial D$ has the parametric representation $\varphi : [0, 2\pi] \to \mathbb{R}^2$, with $\varphi(0) = \varphi(2\pi)$ and $\varphi(s) \neq \varphi(s')$ for $s \neq s'$ in $[0, 2\pi]$, we have

$$i_K[f, \partial D] = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{F_1(s) F_2'(s) - F_2(s) F_1'(s)}{F_1^2(s) + F_2^2(s)} \right] ds,$$

if we write

$$F_j(s) = f_j(\varphi(s)), \quad (j = 1, 2).$$

**Theorem 17.1.1** Assume that $F_1$ has a finite number of zeros

$$0 \leq s_1 < s_2 < \ldots < s_N < 2\pi,$$

and that

$$F_1'(s_k) \neq 0, \quad (1 \leq k \leq N).$$
Then

\[ i_K[f, \partial D] = -\frac{1}{2} \sum_{k=1}^{N} \text{sign } F_2(s_k) \text{sign } F_1'(s_k). \tag{17.2} \]

**Proof.** For \( \epsilon > 0 \) sufficiently small and \( 1 \leq k \leq N \), we have

\[ \int_{s_k+\epsilon}^{s_{k+1}-\epsilon} \frac{F_1(s)F_2'(s) - F_2(s)F_1'(s)}{F_1^2(s) + F_2^2(s)} \, ds = \int_{s_k+\epsilon}^{s_{k+1}-\epsilon} \left[ \arctan \frac{F_2(s)}{F_1(s)} \right]' \, ds \]

\[ = \arctan \frac{F_2(s_{k+1} - \epsilon)}{F_1(s_{k+1} - \epsilon)} - \arctan \frac{F_2(s_k + \epsilon)}{F_1(s_k + \epsilon)}. \tag{17.3} \]

Now,

\[ \lim_{\epsilon \to 0} \arctan \frac{F_2(s_{k+1} - \epsilon)}{F_1(s_{k+1} - \epsilon)} = \lim_{\epsilon \to 0} \frac{F_2(s_{k+1} - \epsilon)}{F_1(s_{k+1} - \epsilon) - F_1(s_{k+1})} = -\frac{\pi}{2} \text{sign } F_2(s_{k+1}) \text{sign } F_1'(s_{k+1}), \tag{17.4} \]

\[ \lim_{\epsilon \to 0} \arctan \frac{F_2(s_k + \epsilon)}{F_1(s_k + \epsilon)} = \lim_{\epsilon \to 0} \frac{F_2(s_k + \epsilon)}{F_1(s_k + \epsilon) - F_1(s_k)} = \frac{\pi}{2} \text{sign } F_2(s_k) \text{sign } F_1'(s_k). \]

Consequently, using (17.1) and (17.4), we obtain, in the case where \( 0 < s_1 \) (the case \( s_1 = 0 \) is similar)

\[ 2\pi i_K(f, \partial D) = \lim_{\epsilon \to 0} \left[ \int_{0}^{s_1-\epsilon} \frac{F_1(s)F_2'(s) - F_2(s)F_1'(s)}{F_1^2(s) + F_2^2(s)} \, ds \right. \]

\[ + \sum_{k=1}^{N-1} \int_{s_k+\epsilon}^{s_{k+1}-\epsilon} \frac{F_1(s)F_2'(s) - F_2(s)F_1'(s)}{F_1^2(s) + F_2^2(s)} \, ds \]

\[ + \left. \int_{s_N+\epsilon}^{2\pi} \frac{F_1(s)F_2'(s) - F_2(s)F_1'(s)}{F_1^2(s) + F_2^2(s)} \, ds \right] \]

\[ = \lim_{\epsilon \to 0} \left[ \arctan \frac{F_2(s_1 - \epsilon)}{F_1(s_1 - \epsilon)} - \arctan \frac{F_2(0)}{F_1(0)} \right. \]

\[ + \sum_{k=1}^{N-1} \left[ \arctan \frac{F_2(s_{k+1} - \epsilon)}{F_1(s_{k+1} - \epsilon)} - \arctan \frac{F_2(s_k + \epsilon)}{F_1(s_k + \epsilon)} \right] \]

\[ + \arctan \frac{F_2(2\pi)}{F_1(2\pi)} - \arctan \frac{F_2(s_N + \epsilon)}{F_1(s_N + \epsilon)} \]

\[ = -\frac{\pi}{2} \sum_{k=0}^{N-1} \text{sign } F_2(s_{k+1}) \text{sign } F_1'(s_{k+1}) \]

\[ - \frac{\pi}{2} \sum_{k=1}^{N} \text{sign } F_2(s_k) \text{sign } F_1'(s_k). \]
Therefore, by adding both members of those two formulas, we get

\[ i_K[f, \partial D] = -\frac{1}{2} \sum_{k=1}^{N} \text{sign } F_2(s_k) \text{ sign } F'_1(s_k). \]

\[ \text{Example 17.1.1} \]

Let

\[ f : \mathbb{R}^2 \to \mathbb{R}^2, \ (x_1, x_2) \mapsto (x_1^2 - x_2^2, 2x_1 x_2), \]

and \( \partial D := \partial B(1) \), with parametrization \( \varphi(s) = (\cos s, \sin s) \). Then

\[ F_1(s) = \cos^2 s - \sin^2 s = \cos 2s, \quad F_2(s) = 2 \cos s \sin s = \sin 2s, \]

and hence \( i_K[f, \partial B(1)] = -\frac{1}{2}(-4) = 2. \)

\[ \text{Remark 17.1.1} \]

Formula (17.2) can be interpreted geometrically, if we remember that \( i_K[f, \partial D] \) is the total number of anticlockwise rotations of \( f \circ \varphi \) around zero. By continuity, this algebraic number of rotations of \( f \circ \varphi(s) \) when \( s \) varies from 0 to \( 2\pi \) is equal to the difference between the number of zeros of \( F_1 \) at which \( F_2 > 0 \) and \( F'_1 < 0 \) and the number of zeros of \( F_1 \) at which \( F_2 > 0 \) and \( F'_1 > 0 \). In other words,

\[ i_K[f, \partial D] = - \sum_{k:F_2(s_k)>0} \text{sign } F'_1(s_k) = - \sum_{k:F_2(s_k)>0} \text{sign } [F'_1(s_k)F_2(s_k)] \]

\[ = - \sum_{k:F_2(s_k)>0} \text{sign } F'_1(s_k) \text{ sign } F_2(s_k). \]  

But this algebraic number of rotations is as well equal to the difference between the number of zeros of \( F_1 \) at which \( F_2 < 0 \) and \( F'_1 > 0 \) and the number of zeros of \( F_1 \) at which \( F_2 < 0 \) and \( F'_1 < 0 \). In other words

\[ i_K[f, \partial D] = \sum_{k:F_2(s_k)<0} \text{sign } F'_1(s_k) = - \sum_{s_k:F_2(s_k)<0} \text{sign } [F'_1(s_k)F_2(s_k)] \]

\[ = - \sum_{k:F_2(s_k)<0} \text{sign } F'_1(s_k) \text{ sign } F_2(s_k). \]  

Therefore, by adding both members of those two formulas, we get

\[ i_K[f, \partial D] = -\frac{1}{2} \sum_{k=1}^{N} \text{sign } [F_2(s_k)F'_1(s_k)] = -\frac{1}{2} \sum_{k=1}^{N} \text{sign } F_2(s_k) \text{ sign } F'_1(s_k), \]

which is (17.2).
CHAPTER 17. COMPUTING DEGREE IN DIMENSION TWO

Remark 17.1.2 If we write \( F_1(s_k - 0) \) (resp. \( F_1(s_k + 0) \)) for \( F_1(s_k - \varepsilon) \) (resp. \( F_1(s_k + \varepsilon) \)) for sufficiently small \( \varepsilon > 0 \), formula (17.2) can be written as well

\[
i_K[f, \partial D] = \frac{1}{2} \sum_{k=1}^{N} \text{sign } F_2(s_k) \text{sign } [F_1(s_k - 0) - F_1(s_k + 0)]
\]

as \( F_1(s_k) = 0 \ (1 \leq k \leq N) \).

Remark 17.1.3 Let \( 0 \leq r_1 \) be the smallest zero of \( F_2 \) in \([0, 2\pi]\). Then \( \partial D \) can be parametrized as well on \([r_1, r_1 + 2\pi]\) and \( F_2^+ \) and \( F_2^- \) are open subsets contained in \([r_1, r_1 + 2\pi]\). Hence, formulas (17.6) and (17.7) can be respectively rewritten as

\[
i_K[f, \partial D] = -d_B[F_1, F_2^+, 0], \quad i_K[f, \partial D] = d_B[F_1, F_2^-, 0],
\]

so that

\[
i_K[f, \partial D] = \frac{1}{2} \{ d_B[F_1, F_2^+, 0] - d_B[F_1, F_2^-, 0] \}
\] (17.9)

Example 17.1.2 For the mapping (17.5) and \( D = B(1) \), we have

\[ F_2^+ = ]0, \pi/2[ \cup ]\pi, 3\pi/2[, \quad F_2^- = ]\pi/2, \pi[ \cup ]3\pi/2, 2\pi[,
\]

and hence

\[
i_K[f, \partial B(1)] = -\frac{1}{2} \{ d_B[\cos 2\cdot, ]0, \pi/2[, 0] + d_B[\cos 2\cdot, ]\pi, 3\pi/2[0] \}
+ \frac{1}{2} \{ d_B[\cos 2\cdot, ]\pi/2, \pi[, 0] + d_B[\cos 2\cdot, ]3\pi/2, 2\pi[0] \}
= -\frac{1}{2} (-1 - 1) + \frac{1}{2} (1 + 1) = 2.
\]

17.2 Factorized multilinear mappings

Definition 17.2.1 A multilinear mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is an application of the form

\[
f_1(x_1, x_2) = \sum_{i=0}^{m} a_i x_1^{m-i} x_2^i, \quad f_2(x_1, x_2) = \sum_{j=1}^{n} b_j x_1^{n-j} x_2^j,
\] (17.10)

where the \( a_i \) and \( b_j \) are real numbers. \( f \) is said to be nondegenerate if \( f^{-1}(\{0\}) = \{0\} \), and degenerate in the opposite case.
17.2. FACTORIZED MULTILINEAR MAPPINGS

The zeros of \( f_1 \) are located on the straight lines through the origin whose angular coefficients are the real solutions of the equation

\[
f_1(1, k) := \sum_{i=0}^{m} a_i k^i = 0,
\]

(17.11)

(including the solution \( k = +\infty \) if the degree of \( f_1(1, k) \) is smaller than \( m \), i.e. if \( a_m = 0 \)), and the zeros of \( f_2 \) are located on the straight lines through the origin whose angular coefficients are the solutions of the equation

\[
f_2(1, k) := \sum_{j=0}^{n} b_j k^j = 0,
\]

(17.12)

(including the solution \( k = +\infty \) if the degree of \( f_2(1, k) \) is smaller than \( n \), i.e. if \( b_n = 0 \)). \( f \) is therefore nondegenerate if the two equations (17.11) and (17.12) have no common root (including \( k = +\infty \)). For a nondegenerate multilinear mapping \( f \), the Brouwer degree \( d_B[f, B(r), 0] \) exists for each \( r > 0 \), and is independent of \( r \).

Let us assume \( f \) nondegenerate and write it in a factorized form

\[
f_1(x_1, x_2) = K_1 \prod_{i=1}^{\tilde{m}} (x_2 - k_i x_1)^{p_i}, \quad f_2(x_1, x_2) = K_2 \prod_{j=1}^{\tilde{n}} (x_2 - l_j x_1)^{q_j},
\]

(17.13)

where the \( k_i (1 \leq i \leq \tilde{m}) \) (resp. \( l_j (1 \leq j \leq \tilde{n}) \)) are the distinct roots of equation (17.11) (resp. (17.12)), with respective multiplicities \( p_i \) and \( q_j \). If some of the roots are equal to \( +\infty \), the corresponding factor will be taken equal to \( -x_1 \). If we define \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
g_1(x_1, x_2) := \prod_{i=1}^{\tilde{m}} (x_2 - k_i x_1)^{p_i},
\]

\[
g_2(x_1, x_2) := \prod_{j=1}^{\tilde{n}} (x_2 - l_j x_1)^{q_j},
\]

then

\[d_B[f, B(r), 0] = \text{sign } K_1 \text{ sign } K_2 d_B[g, B(r), 0],\]

and it suffices to compute the last degree.

One can neglect several factors of \( g \) without changing its degree. The following lemmas as well as the final theorem have been proved in 1954 by J. Cronin [79] (seen also [81]).

**Lemma 17.2.1** One can suppress the pairs of factors \((x_2 - k_i x_1)(x_2 - k_i x_1)\) in which \( k_{i_1} = k_{i_2} \) without changing \( d_B[g, B(r), 0] \).
Proof. In this case, 
\[(x_2 - k, x_1)(x_2 - k_2, x_1) = x_2^2 + |k_1|^2 x_1^2,\]
and \(p_1 = p_2\). Using the homotopy \(G\) defined by
\[G_1(x_1, x_2, \lambda) := \prod_{i \neq i_1, i_2} (x_2 - k_i, x_1)^{p_i}[\lambda + (1 - \lambda)(x_2 - k_{i_1}, x_1)^{p_{i_1}}(x_2 - k_{i_2}, x_1)^{p_{i_2}}],\]
\[G_2(x_1, x_2, \lambda) := g_2(x_1, x_2), \quad \lambda \in [0, 1],\]
one easily shows that \(G(\cdot, \lambda)^{-1}(\{0\}) = \{0\}\) for each \(\lambda \in [0, 1]\). Theorem 3.4.2 implies that
\[d_B[g, B(r), 0] = d_B[G(\cdot, 0), B(r), 0] = d_B[G(\cdot, 1), B(r), 0].\]

**Lemma 17.2.2** One can suppress the factors \((x_2 - k, x_1)^{p_i}\), where \(k_i\) is real and \(p_i\) is even, without changing \(d_B[g, B(r), 0]\).

Proof. One uses in this case the homotopy \(G\) defined by
\[G_1(x_1, x_2, \lambda) := \prod_{i \neq i_1} (x_2 - k_i, x_1)^{p_i}[\lambda + (1 - \lambda)(x_2 - k_{i_1}, x_1)^{p_{i_1}}],\]
\[G_2(x_1, x_2, \lambda) := g_2(x_1, x_2), \quad \lambda \in [0, 1].\]

**Lemma 17.2.3** One can suppress the pairs of factors \((x_2 - k_1, x_1), (x_2 - k_2, x_1)\) such that \(k_1 < k_2\) and there exists no \(l_j\) or \(k_i\) verifying the inequalities
\[k_1 < l_j < k_2 \quad \text{or} \quad k_1 < k_i < k_2\]
without changing \(d_B[g, B(r), 0]\).

Proof. To show it, one uses the homotopy \(G\) defined by
\[G_1(x_1, x_2, \lambda) := \prod_{i \neq i_1, i_2} (x_2 - k_i, x_1)^{p_i}(x_2 - k_{i_1})[x_2 - [(1 - \lambda)k_{i_2} + \lambda k_{i_1}]x_1],\]
\[G_2(x_1, x_2, \lambda) := g_2(x_1, x_2), \quad \lambda \in [0, 1],\]
which, for \(\lambda = 1\), reduces the problem to the case of Lemma 17.2.2.

**Lemma 17.2.4** One can suppress the pairs of factors \((x_2 - k_1, x_1), (x_2 - k_m, x_1)\), where \(k_1\) and \(k_m\) are the smallest and the largest among the numbers
\[k_1, \ldots, k_m, l_1, \ldots, l_n\]
(uniquely in the case where both belong to the \(k_i\)) without changing \(d_B[g, B(r), 0]\).
Proof. One uses in this case the homotopy $G$ defined by

$$G_1(x_1, x_2, \lambda) := \left[ \prod_{i=1}^{\tilde{m}-1} (x_2 - k_i x_1)^{p_i} \right] H(x_1, x_2, \lambda)[(1 - \lambda x_2 - k_{\tilde{m}} x_1],$$

$$G_2(x_1, x_2, \lambda) := g_2(x_1, x_2), \quad \lambda \in [0, 1],$$

where

a. if $k_1 < 0$, $H(x_1, x_2, \lambda) = (1 - \lambda)x_2 - k_1 x_1$, which reduces the problem, for $\lambda = 1$, to the case of Lemma 17.2.2;

b. if $k_1 = 0$, $H(x_1, x_2, \lambda) = x_2 - \lambda(-\varepsilon)x_1$, with $\varepsilon > 0$, which reduces the problem to case a when $\lambda = 1$;

c. if $k_1 > 0$, $H(x_1, x_2, \lambda) = x_2 - (1 - \lambda)k_1 x_1$, which reduces the problem to case b when $\lambda = 1$.

Lemma 17.2.5 One can suppress the factors listed above with the $k_i$ replaced by the $l_j$, without changing $d_B[g, B(r), 0]$.

If all the factors of $g_1$ or of $g_2$ are included in the five types listed above, then $d_B[g, B(r), 0] = 0$, as $g$ is homotopic to a mapping having one of its components constant. If it is not the case, one is reduced to compute the degree of a mapping $h$ whose both components have the same number $r$ of factors $(x_2 - k_i x_1), (x_2 - l_j x_1)$, where the $k_i$ and $l_j$ are such that either

$$k_1 < l_1 < k_2 < l_2 < \ldots < k_r < l_r \quad \text{or} \quad l_1 < k_1 < l_2 < k_2 < \ldots < l_r < k_r, \quad (17.14)$$

and all the exponents $p_i$ and $q_j$ equal to one.

Theorem 17.2.1 If the nondegenerate mapping $f$ given by formula (17.13) is such that

$$\tilde{m} - \tilde{n} = 1 \pmod{2}, \quad (17.15)$$

then $d_B[f, B(r), 0] = 0$.

Proof. The operations given by the lemmas above always consist in suppressing an even number of linear factors in one or another component of $g$. Hence, if condition (17.15) holds, one always obtains by reduction a component which is constant.

It remains to consider the case of a multilinear mapping $h : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$h_1(x_1, x_2) = \prod_{i=1}^{r} (x_2 - k_i x_1), \quad h_2(x_1, x_2) = \prod_{j=1}^{r} (x_2 - l_j x_1), \quad (17.16)$$

where the $k_i$ and $l_j$ verify condition (17.14).
Chapter 17. Computing Degree in Dimension Two

Theorem 17.2.2 If the nondegenerate mapping $f$ given by formula (17.13) is reduced to the form (17.16), then

$$d_B[f, B(r), 0] = r \text{ sign } (K_1K_2) \quad \text{or} \quad d_B[f, B(r), 0] = -r \text{ sign } (K_1K_2),$$

according to the first or the second condition holds in (17.14).

Proof. As $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and is homogeneous,

$$d_B[f, B(r), 0] = \text{ sign } (K_1K_2) d_B[h, B(r)] = \text{ sign } (K_1K_2) i_K[h, \partial B(r)]$$

and we use formula 17.2 to compute $i_K[h, \partial B(1)]$. Using the parametrization

$$x_1 = \cos s, \quad x_2 = \sin s \quad (s \in [0, 2\pi]),$$

for $\partial B(1)$, and letting

$$H_1(s) := h_1(\cos s, \sin s) = \prod_{i=1}^{r} (\sin s - k_i \cos s),$$

$$H_2(s) := h_2(\cos s, \sin s) = \prod_{j=1}^{r} (\sin s - l_j \cos s),$$

we see that the zeros of $H_1(s)$ are given by

$$s_n = \arctan k_n, \quad s_{r+n} = \arctan k_n + \pi = s_n + \pi, \quad (1 \leq n \leq r),$$

so that

$$H_2(s_n) = (\cos s_n)^r \prod_{j=1}^{r} (k_n - l_j) \quad (1 \leq n \leq r),$$

$$H_2(s_{r+n}) = (\cos(s_n + \pi))^r \prod_{j=1}^{r} (k_n - l_j) \quad (1 \leq n \leq r).$$

Now,

$$H_1'(s) = \sum_{j=1}^{r} \prod_{i \neq j} (\sin s - k_i \cos s)(\cos s + k_j \sin s),$$

so that

$$H_1'(s_n) = (\cos s_n)^r (1 + k_n^2) \prod_{i \neq n} (k_n - k_i) \quad (1 \leq n \leq r),$$

$$H_1'(s_{r+n}) = (\cos(s_n + \pi))^r (1 + k_n^2) \prod_{i \neq n} (k_n - k_i) \quad (1 \leq n \leq r).$$
17.3. USING STURM SEQUENCES

Therefore,

\[
\text{sign } [H_2(s_n)H'_1(s_n)] = \text{sign } \left[ \prod_{j=1}^{r} (k_n - l_j) \prod_{i \neq n} (k_n - k_i) \right]
\]

\[
= \text{sign } [H_2(s_{r+n})H'_1(s_{r+n})] \quad (1 \leq n \leq r).
\]

So formula (17.2) gives

\[
i_K[h, \partial B(1)] = - \sum_{n=1}^{r} \text{sign } \left[ \prod_{j=1}^{r} (k_n - l_j) \prod_{i \neq n} (k_n - k_i) \right]
\]

\[
= - \sum_{n=1}^{r} \left[ \prod_{j=1}^{r} \text{sign } (k_n - l_j) \prod_{i \neq n} \text{sign } (k_n - k_i) \right].
\]

Consequently, if the first condition in (17.14) holds, one has

\[
i_K[h, \partial B(1)] = - \sum_{n=1}^{r} (-1)^{r-n+1}(-1)^{r-n} = r,
\]

and if the second condition holds, one has

\[
i_K[h, \partial B(1)] = - \sum_{n=1}^{r} (-1)^{r-n}(-1)^{r-n} = -r.
\]

**Example 17.2.1** Let us consider again the mapping (17.5). We have

\[
K_1 = -1, K_2 = -2, r = 2, k_1 = -1 < l_1 = 0 < k_2 = +1 < l_2 = +\infty,
\]

and \(d_B[f, B(r), 0] = 2\).

**17.3 Using Sturm sequences**

Let us now consider the nondegenerate multilinear mapping \(f : \mathbb{R}^2 \to \mathbb{R}^2\) defined by formula (17.10), and assume that, say,

\[
m = n \pmod{2} \quad \text{and} \quad m \geq n, \quad a_m \neq 0. \tag{17.17}
\]

If the first condition is not satisfied, the Brouwer degree is zero. The last condition can be realized by permuting the components and/or the variables of \(f\).
Proposition 17.3.1 Let the nondegenerate multilinear mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) given by formula (17.10) satisfy the assumptions (17.17). Let 
\[ k_1 < k_2 < \ldots < k_r \quad (r \leq m) \]
be the real roots of the equation 
\[ f_1(1, y) := a_0 + a_1 y + \ldots + a_m y^m = 0. \]
Then 
\[ d_B[f, B(1), 0] = \frac{1}{2} \sum_{l=1}^{r} \text{sign} \left[ f_2(1, k_l) \right] \left[ \text{sign} \left( f_1(1, k_l - 0) - f_1(1, k_l + 0) \right) \right] \quad (17.18) \]

**Proof.** The points of \( \partial B(1) \) where \( f_1(x_1, x_2) = 0 \) verify the system of equations 
\[ x_1^2 + x_2^2 = 1, \quad x_2 = k_i x_1 \quad (1 \leq i \leq r), \quad (17.19) \]
With the notations of the previous section, those points can be written 
\[ \xi_l = (\cos s_l, \sin s_l), \quad (1 \leq l \leq 2r), \]
where 
\[ s_l = \arctan k_l, \quad s_{r+l} = \arctan k_l + \pi = s_l + \pi, \quad (1 \leq l \leq r), \]
so that 
\[ \xi_{r+l} = -\xi_l \quad (1 \leq l \leq r), \quad \tan s_l = \tan s_{r+l} = k_l \quad (1 \leq l \leq r). \]
Furthermore, 
\[ F_1(s) = f_1(\cos s, \sin s) = (\cos^m s) \ f_1(1, \tan s), \]
\[ F_2(s) = f_2(\cos s, \sin s) = (\cos^n s) \ f_2(1, \tan s). \]
Therefore, using formula (17.8), we get 
\[ i_K[f, \partial B(1)] \]
\[ = \frac{1}{2} \sum_{l=1}^{r} \text{sign} \left[ \cos^n s_l f_2(1, k_l) \right] \text{sign} \left[ \cos^m s_l \left[ f_1(1, k_l - 0) - f_1(1, k_l + 0) \right] \right] \]
\[ + \frac{1}{2} \sum_{l=1}^{r} \text{sign} \left[ \cos^n (s_l + \pi) f_2(1, k_l) \right] \text{sign} \left[ \cos^m (s_l + \pi) \left[ f_1(1, k_l - 0) - f_1(1, k_l + 0) \right] \right] \]
\[ = \frac{1}{2} \sum_{l=1}^{r} \text{sign} \left[ \cos^{n+m} s_l f_2(1, k_l) \right] \text{sign} \left[ f_1(1, k_l - 0) - f_1(1, k_l + 0) \right] \]
\[ + \frac{1}{2} \sum_{l=1}^{r} \text{sign} \left[ \cos^{n+m} (s_l + \pi) f_2(1, k_l) \right] \text{sign} \left[ f_1(1, k_l - 0) - f_1(1, k_l + 0) \right] \]
\[ = \sum_{l=1}^{r} \text{sign} \left[ f_2(1, k_l) \right] \text{sign} \left[ f_1(1, k_l - 0) - f_1(1, k_l + 0) \right] \]
\[ = \frac{1}{2} \sum_{l=1}^{r} \text{sign} \left[ f_2(1, k_l) \right] \left[ \text{sign} \left( f_1(1, k_l - 0) - f_1(1, k_l + 0) \right) \right] \]
17.3. USING STURM SEQUENCES

as \( m + n \) is even. Hence, formula (17.18) follows.

We show now that the right-hand member of formula (17.18) can be computed without an explicit knowledge of the roots \( k_l \). Let us write, for simplicity,

\[
T_0(y) := f_1(1, y), \quad T_1(y) := f_2(1, y).
\]

(17.20)

By the non-degeneracy of \( f \), \( T_0 \) and \( T_1 \) have no common real roots. Using the algorithm introduced in 1836 by C. Sturm [377] to determine the number of real roots of a polynomial in a given interval, define the polynomials \( T_2(y), T_3(y), \ldots, T_p(y) \) by

\[
T_0(y) = \varepsilon_1(y)T_1(y) - T_2(y) \\
T_1(y) = \varepsilon_2(y)T_2(y) - T_3(y) \\
\ldots \ldots \ldots \ldots \\
T_{i-1}(y) = \varepsilon_i(y)T_i(y) - T_{i+1}(y) \\
\ldots \ldots \ldots \ldots \\
T_{p-2}(y) = \varepsilon_{p-1}(y)T_{p-1}(y) - T_p(y) \\
T_{p-1}(y) = \varepsilon_p(y)T_p(y).
\]

(17.21)

If \( T_p \) has a real root \( y_0 \), then, by induction, \( T_0(y_0) = T_1(y_0) = 0 \), contradicting the assumptions, and hence \( T_p(y) \) has a constant sign, as \( T_0 \) and \( T_1 \) has no common real roots.

For each \( y \in \mathbb{R} \), let \( S(y) \) denote the number of sign variations of the sequence \( T_0(y), T_1(y), \ldots, T_p(y) \).

**Lemma 17.3.1** Let \( a < b \) be such that \( T_0(a) \neq 0 \) and \( T_0(b) \neq 0 \). Then

\[
S(b) - S(a) = \frac{1}{2} \sum \text{sign } T_i(k_i)[\text{sign } T_0(k_i - 0) - \text{sign } T_0(k_i + 0),
\]

(17.22)

where the sum is extended to all zeros \( k_i \) of \( T_0 \) located in \([a, b]\).

**Proof.** By the continuity of the \( T_i \), \( S(y) \) can only vary when \( y \) goes through a zero \( k_0 \) of one of the polynomials \( T_i \) \((0 \leq i \leq p - 1)\). Let \( k_0 \) be a zero of \( T_i \) for some \( 1 \leq i \leq p - 1 \). It follows from formulas (17.21) that \( T_{i-1}(k_0) \) and \( T_{i+1}(k_0) \) are different from zero (if not, this would imply, inductively, that \( T_0(k_0) = T_1(k_0) = 0 \), and that

\[
T_{i-1}(k_0)T_{i+1}(k_0) < 0.
\]

By continuity, this inequality holds in a neighbourhood of \( k_0 \). Hence, because \( T_i(k_0 - 0) \) has the sign of one of the \( T_{i-1}(k_0 - 0) \) and \( T_{i+1}(k_0 + 0) \) and \( T_i(k_0 + 0) \) has the sign of one of the \( T_{i-1}(k_0 + 0) \) and \( T_{i+1}(k_0 + 0) \), there is no change of the number of variations of sign of the sequence \( T_0(y), T_1(y), \ldots, T_i(y) \) when \( y \) crosses a zero of \( T_i \), \((k = 1, 2, \ldots, p - 1)\). Let us consider now a zero \( k_0 \) of \( T_0 \). The variation of \( S(y) \) through \( k_0 \) is given by

\[
\Delta S(k_0) = \frac{1}{2} \text{sign } T_i(k_0)[\text{sign } T_0(k_0 - 0) - \text{sign } T_0(k_0 + 0)].
\]
Indeed, when $T_0$ changes sign at $k_0$, $S(y)$ increases by 1 at $k_0$ if sign $T_1(k_0) (=\text{sign } T_1(k_0 - 0))$ and sign $T_0(k_0 - 0)$ are the same, and decreases by 1 if they are opposite. When $T_0$ does not change sign at $k_0$, $S(y)$ does not change at $k_0$.

**Theorem 17.3.1** Let $f$ be given by formula (17.10) be nondegenerate and satisfy conditions (17.17). Let the $T_i(k)$ $(0 \leq i \leq p)$ be defined by formulas (17.20) and (17.21). Then, for each $r > 0$,

$$d_B[f, B(r), 0] = S(+\infty) - S(-\infty),$$

(17.23)

where $S(+\infty)$ (resp. $S(-\infty)$) denotes the number of sign variations of the sequence of symbols

$$\lim_{k \to +\infty} T_0(k), \lim_{k \to +\infty} T_1(k), \ldots, \lim_{k \to +\infty} T_p(k)$$

(resp. $\lim_{k \to +\infty} T_0(k), \lim_{k \to +\infty} T_0(k), \ldots, \lim_{k \to +\infty} T_p(k)$),

or equivalently the number $S(k)$ of sign variations of the sequence $T_0(k), T_1(k), \ldots, T_i(k)$ for positive (resp. negative) $k$ with sufficiently large absolute value.

**Proof.** It is an immediate consequence of formula (17.18), Lemma 17.3.1 and the boundedness of the set of zeros of $T_0(k)$.

**Remark 17.3.1** In (17.23), one can replace $S(+\infty) - S(-\infty)$ by $S(K) - S(-K)$, where $K > \max\{|k| : T_0(k) = 0\}$.

**Example 17.3.1** We still consider the mapping (17.5). Then,

$$T_0(y) = 1 - y^2, \quad T_1(y) = 2y, \quad T_2(y) = -1. $$

<table>
<thead>
<tr>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-\infty$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$+\infty$</td>
<td>$-\infty$</td>
<td>$+\infty$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Hence, for any $r > 0$,

$$d_B[f, B(r), 0] = iK[f, \partial B(1)] = 2 - 0 = 2.$$  

**17.4 Holomorphic functions**

Let $D \subset \mathbb{C}$ be an open bounded set and $f : \overline{D} \subset \mathbb{C} \to \mathbb{C}$ be holomorphic on an open neighbourhood $U$ of $\overline{D}$. By identifying $\mathbb{C}$ to $\mathbb{R}^2$, we can consider $f$ as a mapping defined on the closure of $D$ seen as a bounded open subset of $\mathbb{R}^2$, but use freely the complex notations. So, for each $y \notin f(\partial D)$, $d_B[f, D, y]$ is well defined.

**Proposition 17.4.1** If $D \subset \mathbb{C}$ is bounded by a closed simple Jordan curve $\partial D$ of class $C^2$, and if $y \notin f(\partial D)$, then

$$d_B[f, D, y] = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z) - y} \, dz.$$  

(17.24)
17.4. HOLOMORPHIC FUNCTIONS  

Proof. If \( g : \mathbb{D} \rightarrow \mathbb{C} \) is defined by \( g(z) := f(z) - y \), then \( g \) is holomorphic on an open neighbourhood of \( \mathbb{D} \) and \( d_B[f, D, y] = d_B[g, D, 0] \), and it follows from (3.3) and (2.8) 

\[
d_B[g, D, 0] = i_K[g, \partial D] = \frac{1}{2\pi i} \int_{\partial D} \frac{dg}{g}. \]

Now, using the Cauchy-Riemann equations 
\[
\partial_1 g(z) = \frac{1}{i} \partial_2 g(z) = g'(z),
\]
we get
\[
dg = \partial_1 g \, dx + \partial_2 g \, dy = g' \, dx + ig' \, dy = g' \, dz,
\]
and hence
\[
d_B[g, D, 0] = \frac{1}{2\pi i} \int_{\partial D} \frac{g'(z)}{g(z)} \, dz.
\]

**Example 17.4.1** If \( f(z) = z^n, n \in \mathbb{N} \), and \( r > 0 \), we have \( \frac{f'(z)}{f(z)} = \frac{n}{z} \), and 

\[
d_B[z^n, B(r), 0] = \frac{1}{2\pi i} \int_{\partial B(r)} \frac{n}{z} \, dz = n. \tag{17.25}
\]

An easy consequence of (17.25) is a proof of the **fundamental theorem of algebra**.

**Proposition 17.4.2** Any non constant polynomial \( p : \mathbb{C} \rightarrow \mathbb{C} \) has at least one zero.

Proof. We can write \( p(z) = \sum_{j=0}^{n} a_j z^j \) with \( n \geq 1, a_j \in \mathbb{C} (j = 0, 1, \ldots, n) \) and \( a_n \neq 0 \), so that, without loss of generality, we can assume that \( a_n = 1 \). Now, for any \( \lambda \in [0, 1] \) and \( |z| = R > 0 \), we have 

\[
|P(z, \lambda)| := \left| z^n + \lambda \left( \sum_{j=0}^{n-1} a_j z^j \right) \right| \geq |z|^n - \sum_{j=0}^{n-1} |a_j| |z|^j 
\]

\[
= R^n \left( 1 - \sum_{j=0}^{n-1} |a_j| R^{j-n} \right) > 0,
\]

if we take \( R \) sufficiently large so that \( \sum_{j=0}^{n-1} |a_j| R^{j-n} < 1 \). For such a \( R \), we have, from the homotopy invariance property 3.4.2 and (17.25), 

\[
d_B[p, B(R), 0] = d_B[P(\cdot, 1), B(R), 0] = d_B[P(\cdot, 0), B(R), 0] = n \geq 1,
\]

and the result follows from the existence property 3.4.1. ■
Recall that if \( f : U \subset C \rightarrow C \) is holomorphic not identically zero and if \( a \in U \) is a zero of \( f \), the **multiplicity** of \( a \) is the smallest integer \( m \) such that \( f^{(m)}(a) \neq 0 \). Thus \( m \geq 1 \) and Taylor’s theorem implies that \( m \) is finite. Taylor’s theorem also implies that the zeros of \( f \) are isolated.

**Proposition 17.4.3** Let \( f \) be holomorphic on an open neighborhood of the closure \( \overline{D} \) of an open bounded set \( D \subset C \), and let \( y \notin f(\partial D) \). Then \( d_B[f, D, y] \geq 0 \), and, if \( d_B[f, D, y] > 0 \), \( f^{-1}(y) = \{a^1, \ldots, a^n\} \) is finite and non-empty, and, if \( m_j \) denotes the multiplicity of the zero \( a^j \) of \( f(\cdot) - y \) (\( 1 \leq j \leq n \)), we have

\[
d_B[f, D, y] = \sum_{j=1}^{n} m_j. \tag{17.26}
\]

**Proof.** Because \( f \) is holomorphic and \( f(\cdot) - y \neq 0 \) on \( \partial D \), \( f^{-1}(y) \) is discrete and compact, hence finite (possibly empty). If \( f^{-1}(y) = \emptyset \), \( d_B[f, D, y] = 0 \). If \( f^{-1}(y) = \{a^1, \ldots, a^n\} \), then, for each \( 1 \leq j \leq n \), it follows from Taylor’s formula that

\[
f(z) - y = (z - a^j)^{m_j} g_j(z),
\]

for some \( g_j \) holomorphic on \( D \) and such that \( g_j(a^j) \neq 0 \). Hence, for \( r > 0 \) sufficiently small, using formula (17.24), we get

\[
d_B[f, D, y] = \sum_{j=1}^{n} i_B[f, a^j] = \sum_{j=1}^{n} d_B[f, B_{a^j}(r), 0]
\]

\[
= \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\partial B_{a^j}(r)} \frac{f(z)}{f(z) - y} dz = \sum_{j=1}^{n} \int_{\partial B_{a^j}(r)} \left[ \frac{m_j}{z - a^j} + \frac{g_j'(z)}{g_j(z)} \right] dz
\]

\[
= \sum_{j=1}^{n} m_j.
\]

\[ \blacksquare \]

**Corollary 17.4.1** Let \( f \) be holomorphic on an open neighborhood of the closure \( \overline{D} \) of an open bounded set \( D \subset C \), and let \( y \notin f(\partial D) \). Then \( d_B[f, D, y] = 1 \) if and only if \( f^{-1}(y) \cap D = \{a\} \) and \( f'(a) \neq 0 \), i.e. if and only if \( f(\cdot) - y \) has a unique zero in \( D \) which is of multiplicity one.

**Proof.** **Sufficiency.** If \( f^{-1}(y) \cap D = \{a\} \) and \( a \) has multiplicity one, formula (17.26) implies that \( d_B[f, D, y] = 1 \).

**Necessity.** From formula (17.26), we have, if \( f^{-1}(y) \cap D = \{a^1, \ldots, a^n\} \), \( 1 = \sum_{j=1}^{n} m_j \), which implies \( n = 1 \) and \( m_1 = 1 \), i.e. \( f'(a_1) \neq 0 \). \[ \blacksquare \]

**Remark 17.4.1** Proposition 17.4.3 shows that, for a holomorphic function, the degree is an exact count of the number of zeros of \( f \) contained in \( D \), if one take in account their multiplicity.
17.5 An application to stability and control

If $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, it is a standard result in the theory of ordinary differential equations [74, 337] that the zero solution of the linear differential system

$$x' = Ax$$

is asymptotically stable if and only if the eigenvalues of the matrix $A$ are all located in the left-hand side half plane $\{z \in \mathbb{C} : \Re z < 0\}$. Similarly, it is a standard result in the theory of difference equations [104] that the zero solution of the linear difference system

$$x_{k+1} = Ax_k \quad (k \in \mathbb{N})$$

is asymptotically stable if and only if the eigenvalues of the matrix $A$ are all located in the open unit ball $B(1) \subset \mathbb{C}$. Now the eigenvalues are solutions of the characteristic equation $\det(A - zI) = 0$, i.e. the zeros of a polynomial in $z$ of degree $n$. Hence the corresponding asymptotic stability problems are related to having information of the location of the zeros of a polynomial in the complex plane. In the first case, the problem was first considered and solved, independently, by E.H. Routh [338] and A. Hurwitz [178] (to answer a question of A. Stodola). In the second case, the problem was first considered by I. Schur [351] and his results completed by A. Cohn [72]. Notice that if the characteristic equation is explicited as

$$z^n + \sum_{j=1}^{n-1} a_j z^j = 0,$$

any possible solution is such that

$$|z|^n \leq \sum_{j=1}^{n-1} |a_j||z|^j,$$

and hence such that $|z| \leq R$, where $R > 0$ is the positive root of equation

$$R^n - \sum_{j=1}^{n-1} |a_j| R^j = 0. \quad (17.27)$$

Thus the location problem for the roots is always a location problem in a sufficiently large open bounded set.

More generally, let $\Omega \subset \mathbb{C}$ be open and $p : \mathbb{C} \to \mathbb{C}$ be a polynomial.

**Definition 17.5.1** We say that $p$ is $\Omega$-stable if $p(z) \neq 0$ for $z \notin \Omega$.

When $\Omega = \{z \in \mathbb{C} : \Re z < 0\}$, the $\Omega$-stability is called the **Routh-Hurwitz stability**; when $\Omega = B(1) \subset \mathbb{C}$, the $\Omega$-stability is called the **Schur-Cohn stability**.
It follows immediately from Proposition 17.4.3 that $p$ is $\Omega$-stable if and only if $d_B[p, B(\rho) \setminus \overline{\Omega}, 0] = 0$ for any $\rho > R$, the positive root of equation (17.27).

For some applications, it is interesting to consider the corresponding properties for a family of polynomials. Let $\Lambda \subset \mathbb{R}^m$ and $P \in C(\mathbb{C} \times \Lambda, \mathbb{C})$ be such that, for each $\lambda \in \Lambda$, $P(\cdot, \lambda)$ is a polynomial on $\mathbb{C}$.

**Definition 17.5.2** We say that $\{P(\cdot, \lambda)\}_{\lambda \in \Lambda}$ is $\Omega$-stable if, for each $\lambda \in \Lambda$, $P(\cdot, \lambda)$ is $\Omega$-stable, i.e. if $P(z, \lambda) \neq 0$ for $z \notin \Omega$ and $\lambda \in \Lambda$.

We now state and prove a more general version, given in [264], of a zero exclusion principle [13, 329].

**Theorem 17.5.1** Let $\Lambda \subset \mathbb{R}^m$ and $\{P(\cdot, \lambda)\}_{\lambda \in \Lambda}$ be a continuous family of polynomials on $\mathbb{C}$. Assume that the following conditions hold:

1. $\Lambda$ is connected.
2. $P(\cdot, \lambda)$ has algebraic degree $d \geq 1$ for each $\lambda \in \Lambda$.
3. $P(\cdot, \lambda_0)$ is $\Omega$-stable for some $\lambda_0 \in \Lambda$.
4. $P(z, \lambda) \neq 0$ for each $(z, \lambda) \in \partial \Omega \times \Lambda$.

Then $\{P(\cdot, \lambda)\}_{\lambda \in \Lambda}$ is $\Omega$-stable.

**Proof.** By Assumption 2, we can assume, without loss of generality, that

$$P(z, \lambda) = z^d + \sum_{k=1}^{d} a_k(\lambda)z^{d-k},$$

where the $a_k : \Lambda \to \mathbb{C}$ are continuous ($1 \leq k \leq d$). Hence, if $P(z, \lambda) = 0$ for some $(z, \lambda) \in \mathbb{C} \times \Lambda$, we have

$$|z|^d \leq \sum_{k=1}^{d} |a_k(\lambda)|,$$

where $a_k = \max_{\lambda \in \Lambda} |a_k(\lambda)|$, ($1 \leq k \leq d$). Consequently, $|z| \leq R$, where $R$ is the positive root of equation

$$R^d - \sum_{k=1}^{d} a_k R^{d-k} = 0.$$  

If $\Omega_R := \Omega \cap B(R + 1)$, the estimate for $|z|$ above and Assumption 4 imply that $P(z, \lambda) \neq 0$ for each $(z, \lambda) \in \partial \Omega_R \times \Lambda$. Hence, by the connectedness of $\Lambda$ and homotopy invariance property 3.4.2, $d_B[P(\cdot, \lambda), \Omega_R, 0]$ is independent of $\lambda$. On the other hand, it follows from Proposition 17.4.3 that $d_B[P(\cdot, \lambda), \Omega_R, 0]$ is equal to the number of zeros of $P(\cdot, \lambda)$ in $\Omega_R$, counted with their multiplicities. By Assumption 3 and the definition of $\Omega$-stability, any possible zero of $P(\cdot, \lambda_0)$ lies in $\Omega$, and hence in $\Omega_R$. Consequently, $d_B[P(\cdot, \lambda_0), \Omega_R, 0] = d$, and hence $d_B[P(\cdot, \lambda), \Omega_R, 0] = d$ for each $\lambda \in \Lambda$. Thus all the zeros of $P(\cdot, \lambda)$ are in $\Omega_R$, hence in $\Omega$, and each $P(\cdot, \lambda)$ is $\Omega$-stable. \qed
17.5. AN APPLICATION TO STABILITY AND CONTROL

The following notion was introduced by Schur [352].

**Definition 17.5.3** If \( p : \mathbb{C} \rightarrow \mathbb{C} \) is a polynomial, the first Schur transform or paraconjugate of \( p \) is the polynomial \( p^* : \mathbb{C} \rightarrow \mathbb{C} \) defined by

\[
p^*(z) := \overline{p(-\overline{z})}.
\]

Notice that

\[
p^{**}(z) := (p^*)^*(z) = \overline{p^*(-\overline{z})} = \overline{p(z)} = p(z),
\]

and that, for \( c \in \mathbb{C} \) and another polynomial \( q \) over \( \mathbb{C} \), one has

\[
(cp)^*(z) = \overline{cp^*(z)}, \quad (p + q)^*(z) = p^*(z) + q^*(z).
\]

Define, with Zahreddine [404],

\[
N(z) = \frac{1}{2} \left[ p(z) + (-1)^d p^*(z) \right], \quad D(z) = \frac{1}{2} \left[ p(z) - (-1)^d p^*(z) \right],
\]

so that

\[
p(z) = N(z) + D(z), \quad (17.30)
\]

\[
p^*(z) = (-1)^d [N(z) - D(z)]. \quad (17.31)
\]

\[
N^*(z) = \frac{1}{2} \left[ p^*(z) + (-1)^d p(z) \right] = (-1)^d N(z),
\]

\[
D^*(z) = \frac{1}{2} \left[ p^*(z) - (-1)^d p(z) \right] = (-1)^{d+1} D(z).
\]

**Lemma 17.5.1** \( z_0 \) is a common zero to \( N \) and \( D \) if and only if \( z_0 \) and \( -\overline{z_0} \) are zeros of \( p \).

**Proof.** If \( z_0 \) is a common zero to \( N \) and \( D \), then, by (17.30) and (17.31), we have \( p(z_0) = 0 = p(-\overline{z_0}) \). If \( z_0 \) and \( -\overline{z_0} \) are zeros of \( p \), then \( p(z_0) = 0 = p^*(z_0) \), and, by (17.29), we have \( N(z_0) = D(z_0) = 0 \). \( \blacksquare \)

**Corollary 17.5.1** Any zero of \( p \) which lies on the imaginary axis is a common zero of \( N \) and \( D \).

**Proof.** For such a zero \( z_0 \), one has \( z_0 = -\overline{z_0} \). \( \blacksquare \)

The resultant [258] of two polynomials \( p \) and \( q \) over \( \mathbb{C} \) will be denoted by \( \mathcal{R}[p, q] \).

Let us first consider the special case of the Routh-Hurwitz stability of a family of polynomials. Let \( \Lambda \subset \mathbb{R}^n \) be connected and let \( \{P(\cdot, \lambda)\}_{\lambda \in \Lambda} \) be a continuous family of polynomials on \( \mathbb{C} \) such that \( P(\cdot, \lambda) \) has degree \( d \) for each \( \lambda \in \Lambda \). For each \( \lambda \in \Lambda \), let

\[
P(z, \lambda) = N(z, \lambda) + D(z, \lambda)
\]

be the decomposition defined by (17.30). The following result, given in [264], generalizes, with a simpler proof, a result of [404].
Theorem 17.5.2 The family \( \{P(\cdot, \lambda)\}_{\lambda \in \Lambda} \) is Routh-Hurwitz-stable if and only if \( P(\cdot, \lambda_0) \) is Routh-Hurwitz-stable for some \( \lambda_0 \in \Lambda \) and \( \mathcal{R}[N(\cdot, \lambda), D(\cdot, \lambda)] \neq 0 \) for each \( \lambda \in \Lambda \).

Proof. Necessity. If \( \mathcal{R}[N(\cdot, \lambda_0), D(\cdot, \lambda_0)] = 0 \) for some \( \lambda_0 \in \Lambda \), then, by a classical result [258], \( N(\cdot, \lambda_0) \) and \( D(\cdot, \lambda_0) \) have a common zero \( z_0 \), so that, by Lemma 17.5.1,

\[
P(z_0, \lambda_0) = 0 = P(-\overline{z_0}, \lambda_0).
\]

Now \( \Re z_0 \geq 0 \) if and only if \( \Re(-\overline{z_0}) \leq 0 \), and hence \( P(\cdot, \lambda_0) \) is not Routh-Hurwitz-stable.

Sufficiency. Assume now that \( \mathcal{R}[N(\cdot, \lambda), D(\cdot, \lambda)] \neq 0 \) for each \( \lambda \in \Lambda \). Then, for each \( \lambda \in \Lambda \), \( N(\cdot, \lambda) \) and \( D(\cdot, \lambda) \) have no common zeros, and Corollary 17.5.1 implies that \( P(\cdot, \lambda) \) has no zero on the imaginary axis, which is the boundary of \( \{z \in \mathbb{C} : \Re z < 0\} \). The conclusion follows then from Theorem 17.5.1. \( \blacksquare \)

We obtain immediately as a special case Zahreddine’s result. Let

\[
p_0(z) = z^d + \sum_{k=1}^d a_k z^{d-k}, \quad p_1(z) = z^d + \sum_{k=1}^d a_k z^{d-k}
\]

be two monic Routh-Hurwitz-stable polynomials. Let

\[
p_j(z) = N_j(z) + D_j(z), \quad (j = 0, 1)
\]

be their respective decompositions (17.30) and let

\[
P(z, \lambda) = (1 - \lambda)p_0 + \lambda p_1, \quad (0 \leq \lambda \leq 1),
\]

be their convex combinations. It is immediate to check that if, for each \( \lambda \in [0, 1] \), \( P(z, \lambda) = N(z, \lambda) + D(z, \lambda) \) is the decomposition (17.30) of \( p(\cdot, \lambda) \), then

\[
N(z, \lambda) = (1 - \lambda)N_0(z) + \lambda N_1(z), \quad D(z, \lambda) = (1 - \lambda)D_0(z) + \lambda D_1(z)
\]

\( (0 \leq \lambda \leq 1) \).

Corollary 17.5.2 If \( p \) and \( q \) are Routh-Hurwitz-stable, their convex combinations \( (1 - \lambda)p + \lambda q \) (\( \lambda \in [0, 1] \)), are Routh-Hurwitz-stable if and only if \( \mathcal{R}[(1 - \lambda)N_0 + \lambda N_1, (1 - \lambda)D_0 + \lambda D_1] \neq 0 \) for each \( \lambda \in [0, 1] \).

Let us now introduce the second Schur transform of a polynomial and its properties [351].

Definition 17.5.4 Given a monic polynomial \( p(z) = z^d + \sum_{k=1}^d a_k z^{d-k} \) on \( \mathbb{C} \), the second Schur transform of \( p \) is the polynomial \( p^\# \) on \( \mathbb{C} \) defined by

\[
p^\#(z) := z^d p \left( \frac{1}{z} \right), \quad (17.32)
\]
Notice that
\[
p^\#(z) := (p^\#)(z) = z^d p^\# \left( \frac{1}{z} \right) = z^d \frac{1}{z^d} p(z) = p(z),
\]
that \( p^\#(0) = 1 \) and that, for \( c \in \mathbb{C} \) and another monic polynomial \( q \) over \( \mathbb{C} \), one has
\[
(c p^\#)(z) = \overline{c} p^\#(z), \quad (p + q)^\#(z) = p^\#(z) + q^\#(z).
\]
Define, with Zahreddine [404],
\[
H(z) = \frac{1}{2} [p(z) + p^\#(z)], \quad K(z) = \frac{1}{2} [p(z) - p^\#(z)],
\]
so that
\[
p(z) = H(z) + K(z), \tag{17.34}
\]
\[
p^\#(z) = H(z) - K(z). \tag{17.35}
\]

**Lemma 17.5.2** If \( z_0 \) is a common zero to \( H \) and \( K \) then \( z_0 \neq 0 \), and \( z_0 \) and \( \frac{1}{z_0} \) are zeros of \( p \). If \( z_0 \neq 0 \) and \( \frac{1}{z_0} \) are zeros of \( p \), then \( z_0 \) is a common zero to \( H \) and \( K \).

**Proof.** If \( z_0 \) is a common zero to \( H \) and \( K \), then, by (17.34) and (17.35), we have
\[
0 = p(z_0) = p^\#(z_0), \quad \text{so that} \quad z_0 \neq 0 \quad \text{and} \quad p \left( \frac{1}{z_0} \right) = 0.
\]
If \( z_0 \neq 0 \) and \( \frac{1}{z_0} \) are zeros of \( p \), then \( p(z_0) = 0 = p^\#(z_0) \), and, by (17.33), we have \( H(z_0) = K(z_0) = 0 \).

**Corollary 17.5.3** Any zero \( z_0 \) of \( p \) which lies on the unit circle is a common zero of \( H \) and \( K \).

**Proof.** For such a zero \( z_0 \), one has \( 0 \neq z_0 = \frac{1}{z_0} \).

Let us now consider the Schur-Cohn-stability of a family of polynomials. Let \( \Lambda \subset \mathbb{R}^m \) be connected and let \( \{ P(\cdot, \lambda) \}_{\lambda \in \Lambda} \) be a continuous family of polynomials on \( \mathbb{C} \) such that \( P(\cdot, \lambda) \) has degree \( d \) for each \( \lambda \in \Lambda \). For each \( \lambda \in \Lambda \), let
\[
P(z, \lambda) = H(z, \lambda) + K(z, \lambda)
\]
be the decomposition defined by (17.34). The following result, given in [271], generalizes, with a simpler proof, a result of [404].
Theorem 17.5.3 The family \( \{ P(\cdot, \lambda) \}_{\lambda \in \Lambda} \) is Schur-Cohn-stable if and only if polynomial \( P(\cdot, \lambda_0) \) is Schur-Cohn-stable for some \( \lambda_0 \in \Lambda \) and \( R[H(\cdot, \lambda), K(\cdot, \lambda)] \neq 0 \) for each \( \lambda \in \Lambda \).

Proof. Necessity. If \( R[H(\cdot, \lambda_0), K(\cdot, \lambda_0)] = 0 \) for some \( \lambda_0 \in \Lambda \), then, by a classical result [258], \( H(\cdot, \lambda_0) \) and \( K(\cdot, \lambda_0) \) have a common zero \( z_0 \), so that by Lemma 17.5.2, \( z_0 \neq 0 \) and

\[
p(z_0, \lambda_0) = 0 = p\left(\frac{1}{z_0}, \lambda_0\right).
\]

Now \( |z_0| \geq 1 \) if and only if \( \left| \frac{1}{z_0} \right| \leq 1 \), and hence \( P(\cdot, \lambda_0) \) is not Schur-Cohn-stable.

Sufficiency. Assume now that \( R[H(\cdot, \lambda), K(\cdot, \lambda)] \neq 0 \) for each \( \lambda \in \Lambda \). Then, for each \( \lambda \in \Lambda \), \( K(\cdot, \lambda) \) and \( H(\cdot, \lambda) \) have no common zeros, which, by Corollary 17.5.3, implies that \( P(\cdot, \lambda) \) has no zero on \( \partial B(1) \). The conclusion follows then from Theorem 17.5.1.

We obtain immediately as a special case a result of Zahreddine [404]. Let

\[
p_0(z) = z^d + \sum_{k=1}^{d} a_k z^{d-k}, \quad p_1(z) = z^d + \sum_{k=1}^{d} a_k z^{d-k}
\]

be two monic Schur-Cohn-stable polynomials. Let \( p_j(z) = H_j(z) + K_j(z) \) (\( j = 0, 1 \)) be their respective decompositions (17.34) and let

\[
p(z, \lambda) = (1 - \lambda)p_0 + \lambda p_1, \quad (0 \leq \lambda \leq 1),
\]

be their convex combinations. It is immediate to check that if, for each \( \lambda \in [0, 1] \), \( p(z, \lambda) = H(z, \lambda) + K(z, \lambda) \) is the decomposition (17.34) of \( p(\cdot, \lambda) \), then

\[
H(z, \lambda) = (1 - \lambda)H_0(z) + \lambda H_1(z), \quad K(z, \lambda) = (1 - \lambda)K_0(z) + \lambda K_1(z)
\]

(\( 0 \leq \lambda \leq 1 \)).

Corollary 17.5.4 Assume that \( p \) and \( q \) are Schur-Cohn-stable. Then their convex combinations \( (1 - \lambda)p + \lambda q \) (\( \lambda \in [0, 1] \)), are Schur-Cohn-stable if and only if \( R[(1 - \lambda)H_0 + \lambda H_1, (1 - \lambda)K_0 + \lambda K_1] \neq 0 \) for each \( \lambda \in [0, 1] \).
Chapter 18

Periodic solutions of some planar systems

18.1 Autonomous systems

Let $J$ denotes the $2 \times 2$ symplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. $$

Recall that $J^2 = -I$, $JJ^T = J^TJ = I$, $(Jv, v) = 0 \ (v \in \mathbb{R}^2).$ (18.1)

Let $H \in C^1(\mathbb{R}^2, \mathbb{R})$ be positive on $\mathbb{R}^2 \setminus \{0\}$, positively homogeneous of degree 2, namely

$$H(\sigma u) = \sigma^2 H(u) \quad (\sigma \geq 0, \ u \in \mathbb{R}^2) \quad (18.2)$$

and such that $\nabla H : \mathbb{R}^2 \to \mathbb{R}^2$ locally Lipschitzian. It follows from (18.2) that $\nabla H$ is positively homogeneous of degree one, namely

$$\nabla H(\sigma u) = \sigma H(u) \quad (\sigma \geq 0, \ u \in \mathbb{R}^2), \quad (18.3)$$

It follows from (18.2) and (18.3) that

$$H(u) = 0 \iff u = 0, \quad \nabla H(u) = 0 \iff u = 0. \quad (18.4)$$

Furthermore, for all $u \in \mathbb{R}^2 \setminus \{0\}$, one has

$$\alpha \|u\|^2 \leq H(u) = \|u\|^2 H(u/\|u\|) \leq \beta \|u\|^2, \quad (18.5)$$

where

$$0 < \alpha := \min_{\|v\|=1} H(v) \leq \max_{\|v\|=1} H(v) := \beta. \quad (18.6)$$

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Let us consider the corresponding planar Hamiltonian system
\[ Ju' = \nabla H(u). \] (18.7)

The identity
\[ 0 = \langle Ju'(t), u'(t) \rangle = \langle \nabla H(u(t)), u'(t) \rangle = [H(u(t))]' \]
satisfied by all solutions of (18.7) implies the existence of the first integral
\[ H(u(t)) = E \quad (E \geq 0). \] (18.8)

Thus, (18.5) implies that any solution \( u \) of energy \( E \) is contained in the annulus centered at 0 and of radii \( 2E/\beta \) and \( 2E/\alpha \).

Furthermore, for each \( E > 0 \), the curve of equation \( H(u) = E \) surrounds the origin and intersects each half-line issued from the origin in a unique point, because, for each \( v \in \mathbb{R}^2 \) with \( \|v\| = 1 \) the equation
\[ H(\sigma v) = E \iff \sigma^2 H(v) = E \]
has a unique positive solution \( \sigma = [E/H(v)]^{1/2} \).

So, for \( E > 0 \), the curves \( H(u) = E \) are closed curves around the origin, which therefore is a center.

If we fix a reference solution \( \varphi : \mathbb{R} \to \mathbb{R}^2 \) of (18.7) such that
\[ H(\varphi(t)) = 1/2 \quad (t \in \mathbb{R}), \] (18.9)
we have, using Euler’s identity
\[ \langle J \varphi'(t), \varphi(t) \rangle = \langle \nabla H(\varphi(t)), \varphi(t) \rangle = 2H(\varphi(t)) = 1 \quad (t \in \mathbb{R}), \] (18.10)
and
\[ \langle \varphi'(t), J \varphi(t) \rangle = \langle J^T J \varphi'(t), J \varphi \rangle = -\langle J \varphi'(t), \varphi(t) \rangle = -1. \] (18.11)

Hence, as the action of \( J \) on a vector means a rotation of \( \pi/2 \) in the counterclockwise direction, the orbit of \( \varphi(t) \) is described in the clockwise direction when \( t \) increases.

It follows from the positive homogeneity of degree one of \( \nabla H \) and of its autonomous character that all solutions of (18.7) are given by
\[ u(t) = A \varphi(t + \theta) \quad (A \geq 0, \theta \in \mathbb{R}). \] (18.12)

Therefore, if \( \tau > 0 \) denotes the time required for a solution of energy 1/2 to describe once the orbit \( H(u) = 1/2 \), i.e. the smallest period of \( \varphi \), all solutions of (18.7) will have the same period \( \tau \), which means that 0 is an isochronous center.

### 18.2 Forced system : nonresonance

Keeping the notations and assumptions of Section 8.1 for \( H \), let \( f : \mathbb{R} \to \mathbb{R}^2 \) be a continuous function of period \( T > 0 \) (\textbf{T-periodic function}) and let us consider the forced planar Hamiltonian system
\[ Ju' = \nabla H(u) + f(t). \] (18.13)
18.2. FORCED SYSTEM : NONRESONANCE

Because of the fact that $\nabla H(u)$ is locally Lipschitzian and has a growth at most linear due to its positive homogeneity of degree one, all solutions $u(t; u_0)$ of (18.13) with initial condition $u_0$ at $t = 0$ are uniquely defined for all $t \in \mathbb{R}$.

Let us assume that

$$T \neq n\tau$$

for all positive integers $n$, (18.14)

so that $N\tau < T < (N + 1)\tau$ for some nonnegative integer $N$. This implies that the only $T$-periodic solution of system (18.7) is $u \equiv 0$.

**Lemma 18.2.1** There exists $R > 0$ such that, for each $\lambda \in [0, 1]$, each possible $T$-periodic solution of system

$$Ju' = \nabla H(u) + \lambda p(t)$$

(18.15)

is such that $\|u\|_\infty < R$.

**Proof.** If it is not the case, there exist a sequence $(\lambda_k)$ in $[0, 1]$, and a sequence $(u_k)$ of $T$-periodic solutions of

$$Ju'_k = \nabla H(u_k) + \lambda_k p(t) \quad (k \in \mathbb{N}).$$

such that

$$\lim_{k \to \infty} \|u_k\|_\infty = +\infty.$$

Hence, letting $v_k = \frac{u_k}{\|u_k\|_\infty}$ and using the positive homogeneity of $\nabla H$, we have

$$Jv'_k = \nabla H(v_k) + \lambda_k \frac{p(t)}{\|u_k\|_\infty} \quad (k \in \mathbb{N}).$$

Hence, for all $k$ such that $\|u_k\|_\infty \geq 1$, we have

$$\|v_k\|_\infty = 1, \quad \|v'_k\|_\infty \leq \max_{\|u\| = 1} \|\nabla H(u)\| + \|f\|_\infty := M,$$

so that Ascoli-Arzelá’s theorem implies, going if necessary to a subsequence, that $(v_k)$ converges uniformly on $\mathbb{R}$ to a $T$-periodic function $v$ such that $\|v\|_\infty = 1$.

Furthermore, from the identity

$$Jv_k(t) = Jv_k(0) + \int_0^t \left[ \nabla H(v_k(s)) + \lambda_k \frac{p(s)}{\|u_k\|_\infty} \right] ds \quad (t \in \mathbb{R}, \ k \in \mathbb{N})$$

we deduce that

$$Jv(t) = Jv(0) + \int_0^t \nabla H(v(s)) \, ds \quad (t \in \mathbb{R}),$$

and hence $v \in C^1(\mathbb{R})$ and

$$Jv' = \nabla H(v).$$

Thus $v$ is a $T$-periodic solution of (18.7) with $\|v\|_\infty = 1$, a contradiction to (18.14).
Now, let us consider the orbit of (18.7) defined by
\[ H(u) = \beta R^2 \]  
(18.16)
where \( \beta \) is defined in (18.6) and \( R \) given by Lemma 18.2.1. If the open bounded set \( D_R \) is defined by
\[ D_R := \{ u \in \mathbb{R}^2 : H(u) < \beta R^2 \}, \]
we have
\[ \partial D_R = \{ u \in \mathbb{R}^2 : H(u) = \beta R^2 \}, \quad B(R) \subset D_R. \]

The following result is proved in a different way in [127].

**Theorem 18.2.1** For each T-periodic forcing \( f \), problem (18.13) has at least one T-periodic solution.

**Proof.** For each \( \lambda \in [0, 1] \) and each \( u_0 \in \mathbb{R}^2 \), let us denote by \( u(t; u_0, \lambda) \) the unique solution of (18.13) equal to \( u_0 \) at \( t = 0 \), and let \( P_T : \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2 \) be Poincaré’s operator associated to the T-periodic solutions of (18.15) defined by
\[ P_T(u_0, \lambda) = u(T; u_0, \lambda) \quad (u_0 \in \mathbb{R}^2, \ \lambda \in [0, 1]). \]

\( P_T(\cdot, 1) \) is Poincaré’s operator associated to the T-periodic solutions of (18.13) and \( P_T(\cdot, 0) \) Poincaré’s operator associated to the T-periodic solutions of (18.7). From Lemma 18.2.1, it follows that, for any \( \lambda \in [0, 1] \), each possible fixed point \( u_0 \) of \( P_T(\cdot, \lambda) \) belongs to \( B(R) \subset D_R \). Hence, for each \( \lambda \in [0, 1] \) and each \( u_0 \in \partial D_R \), one has \( u_0 \neq P_T(u_0, \lambda) \), and the homotopy invariance of Brouwer degree implies that
\[ d_B[P_T(\cdot, 1) - I, D_R, 0] = d_B[P_T(\cdot, 0) - I, D_R, 0]. \]
(18.17)
Now, as \( \partial D_R \) is an orbit for (18.7), \( P_T(u_0, 0) \in \partial D_R \) when \( u_0 \in \partial D_R \), and, by the nonresonance condition (18.14), \( P_T(u_0, 0) \neq u_0 \) for each \( u_0 \in \partial D_R \). Furthermore, as \( D_R \) is strictly starshaped with respect to 0, \( \mu P_T(u_0, 0) \in \mu D_R \subset D_R \) for each \( \mu \in [0, 1] \) and each \( u_0 \in \partial D_R \). Hence, using again homotopy invariance of Brouwer degree, we get
\[ d_B[P_T(\cdot, 0) - I, D_R, 0] = d_B[-I, D_R, 0] = 1, \]
so that, using (18.17),
\[ d_B[P_T(\cdot, 1) - I, D_R, 0] = 1, \]
and \( P_T(\cdot, 1) \) has a fixed point. \( \blacksquare \)
18.3 Forced system: equivalent formulation

In order to obtain existence conditions for the T-periodic solutions of (18.13) when the nonresonance condition (18.14) does not hold, it is convenient to write (18.13) in an equivalent form.

Using the method of variations of constants, let us introduce the new unknown functions \( \rho(t) \) and \( \theta(t) \) by the relation

\[ u(t) = \rho(t) \varphi(t + \theta(t)) \quad (t \in \mathbb{R}^2) \quad (18.18) \]

where \( \rho(t) > 0 \) and \( \varphi \) is the particular solution of (18.7) introduced in Section 8.1. If \( u \) is a solution of (18.13), then, using the positive homogeneity of \( \nabla H \),

\[ \rho'(t)J\varphi(t + \theta(t)) + \rho(t)J\varphi'(t + \theta(t))(1 + \theta'(t)) = \nabla H(\rho(t)\varphi(t + \theta(t))) + f(t). \]

Hence, using the definition of \( \varphi \), we get

\[ \rho'(t)J\varphi(t + \theta(t)) + \rho(t)J\varphi'(t + \theta(t))\theta'(t) = f(t). \quad (18.19) \]

Taking the inner product of both members of (18.19) respectively with \( \varphi'(t + \theta(t)) \) and \( \varphi(t + \theta(t)) \), and using relations (18.1), we obtain, as long as \( \rho(t) > 0 \),

\[ \rho'(t) = -\langle f(t), \varphi'(t + \theta(t)) \rangle, \quad \theta'(t) = \frac{1}{\rho(t)} \langle f(t), \varphi(t + \theta(t)) \rangle. \quad (18.20) \]

Let \( \rho_0 > 0 \) and \( \theta_0 \in [0, \tau] \) be given, and let \( \rho(t; \rho_0, \theta_0) \) and \( \theta(t; \rho_0, \theta_0) \) denote the solutions of (18.20) such that

\[ \rho(0; \rho_0, \theta_0) = \rho_0, \quad \theta(0; \rho_0, \theta_0) = \theta_0. \]

**Lemma 18.3.1** There exists \( C_1 = C_1(\varphi, f) \) such that, for each \( t \in [0, T] \) one has

\[ |\rho(t; \rho_0, \theta_0) - \rho_0| \leq C_1T. \quad (18.21) \]

In particular, if \( \rho_0 \geq C_1T + 1 \), then \( \rho(t; \rho_0, \theta_0) \geq 1 \) for all \( t \in [0, T] \).

**Proof.** It follows from the first equation in (18.20) that, for \( t \in [0, T] \),

\[
|\rho(t; \rho_0, \theta_0) - \rho_0| = \left| \int_0^t \langle f(s), \varphi'(s + \theta(s; \rho_0, \theta_0)) \rangle \, ds \right|
\leq \int_0^T \|f(s)\| \|\varphi'(s + \theta(s; \rho_0, \theta_0))\| \, ds
\leq \max_{[0, T]} \|f\| \max_{[0, \tau]} \|\varphi'\| T := C_1T.
\]

\[ \blacksquare \]
Lemma 18.3.2 If $\rho_0 \geq C_1 T + 1$, then there exists $C_2 = C_2(\varphi, f)$ such that, for each $t \in [0, T]$, one has

$$|\theta(t; \rho_0, \theta_0) - \theta_0| \leq \frac{C_2 T}{\rho_0 - C_1 T}.$$  \hspace{1cm} (18.22)

Proof. It follows from the second equation in (18.20) and from (18.21) that, for $t \in [0, T]$,

$$|\theta(t; \rho_0, \theta_0) - \theta_0| = \left| \int_0^t \frac{1}{\rho(t; \rho_0, \theta_0)} (f(s), \varphi(s + \theta(s; \rho_0, \theta_0))) \, ds \right|$$

$$\leq \frac{1}{\rho_0 - C_1 T} \int_0^T \|f(s)||\varphi(s + \theta(s; \rho_0, \theta_0))\| \, ds$$

$$\leq \frac{1}{\rho_0 - C_1 T} \max_{[0,T]} \|f\| \max_{[0,T]} \|\varphi\| T := \frac{C_2 T}{\rho_0 - C_1 T}.$$  \hspace{1cm} (18.23)

In order to apply Poincaré’s operator in the search of $T$-periodic solutions of (18.13), given $\rho_0 \geq C_1 T + 1$ and $\theta_0 \in [0, \tau]$, we have to estimate

$$\rho(T; \rho_0, \theta_0) = \rho_0 - \int_0^T (f(s), \varphi'(s + \theta(s; \rho_0, \theta_0))) \, ds$$

$$\theta(T; \rho_0, \theta_0) = \theta_0 + \int_0^T \frac{(f(s), \varphi(s + \theta(s; \rho_0, \theta_0)))}{\rho(s; \rho_0, \theta_0)} \, ds.$$  \hspace{1cm} (18.24)

Define the $\tau$-periodic mapping $\Gamma : [0, \tau] \simeq S^1 \to \mathbb{R}^2$ by

$$\Gamma_1(\theta_0) := -\int_0^T (f(s), \varphi(s + \theta_0)) \, ds$$

$$\Gamma_2(\theta_0) := \int_0^T (f(s), \varphi'(s + \theta_0)) \, ds = -\Gamma_1'(\theta_0).$$  \hspace{1cm} (18.25)

Lemma 18.3.3 For $\rho_0 \geq C_1 T + 1$ and $\theta_0 \in [0, \tau]$, we have

$$\theta(T; \rho_0, \theta_0) = \theta_0 + \frac{1}{\rho_0} [-\Gamma_1(\theta_0) + R_1(\rho_0, \theta_0)]$$

$$\rho(T; \rho_0, \theta_0) = \rho_0 - \Gamma_2(\theta_0) + R_2(\rho_0, \theta_0).$$  \hspace{1cm} (18.26)

where

$$\lim_{\rho_0 \to +\infty} R_j(\rho_0, \theta_0) = 0 \quad (j = 1, 2),$$

uniformly in $\theta_0 \in [0, \tau]$.
Now, from Lemmas 18.3.1 and 18.3.2, we have
\[ R_2(\rho_0, \theta_0) = \int_0^T (f(s), \varphi'(s + \theta(s; \rho_0, \theta_0)) - \varphi'(s + \theta_0)) \, ds, \]
and, by (18.22),
\[ \lim_{\rho_0 \to +\infty} \theta(t; \rho_0, \theta_0) = \theta_0 \]
uniformly in \( t \in [0, T] \) and \( \theta_0 \in [0, \tau] \). Hence,
\[ \lim_{\rho_0 \to +\infty} R_2(\rho_0, \theta_0) = \int_0^T (f(s), \lim_{\rho_0 \to +\infty} [\varphi'(s + \theta(s; \rho_0, \theta_0)) - \varphi'(s + \theta_0))] \, ds = 0, \]
uniformly in \( \theta_0 \in [0, \tau] \).
On the other hand,
\[ R_1(\rho_0, \theta_0) = \rho_0 \int_0^T \frac{(f(s), \varphi(s + \theta(s; \rho_0, \theta_0)))}{\rho(s; \rho_0, \theta_0)} \, ds - \int_0^T (f(s), \varphi(s + \theta_0)) \, ds 
\[ = \int_0^T \rho_0 \int_0^T (f(s), \varphi(s + \theta(s; \rho_0, \theta_0)) - \varphi(s + \theta_0)) \, ds 
\[ + \int_0^T \rho_0 \int_0^T \frac{\rho_0}{\rho(s; \rho_0, \theta_0)} - 1 \int (f(s), \varphi(s + \theta_0)) \, ds. \]
Now, from Lemmas 18.3.1 and 18.3.2, we have
\[ \lim_{\rho_0 \to +\infty} \rho(s; \rho_0, \theta_0) = 1, \quad \lim_{\rho_0 \to +\infty} \theta(s; \rho_0, \theta_0) = \theta_0, \]
uniformly in \( t \in [0, T] \) and \( \theta_0 \in [0, \tau] \), and the result follows in an analogous way. \( \blacksquare \)

18.4 **Forced system : resonance**

From now on, let us assume that there exists a positive integer \( N \) such that
\[ T = N\tau. \quad (18.27) \]
Each nonzero initial condition \( u_0 \in \mathbb{R}^2 \) can be univoquely written as
\[ u_0 = \rho_0 \varphi(\theta_0) \]
for some \( \rho_0 > 0 \) and \( \theta_0 \in [0, \tau] \). On the other hand, if \( u(t; u_0) \) denotes the unique solution of (18.13) such that \( u(0; u_0) = u_0 \), and if \( P_T \) defined by \( P_T(u_0) = u(T; u_0) \)
is Poincaré’s operator, then, with the notations of the previous section,
\[ P_T(\rho_0 \varphi(\theta_0)) = u(T; \rho_0 \varphi(\theta_0)) = \rho(T; \rho_0, \theta_0) \varphi(T + \theta(T; \rho_0, \theta_0)) 
\[ = \rho(T; \rho_0, \theta_0) \varphi(N\tau + \theta(T; \rho_0, \theta_0)) = \rho(T; \rho_0, \theta_0) \varphi(\theta(T; \rho_0, \theta_0)), \]
Hence it follows from the definition of Brouwer degree that

\[ \Pi(\rho_0, \theta_0) := \rho(T; \rho_0, \theta_0)\varphi(\theta(T; \rho_0, \theta_0)) - \rho_0\varphi(\theta_0), \]  

(18.28)

where we have identified \([0, \tau],\) taking in account the \(\tau\)-periodicity, to \(S^1\).

In order to study the Brouwer degree of this map, let us introduce, for each \(\rho_0 > 0,\) the open bounded sets \(\Omega_{\rho_0} \subset \mathbb{R}^2\) defined by

\[ D_{\rho_0} := \{ \sigma \varphi(\theta_0) : \theta_0 \in [0, \tau], \sigma \in [0, \rho_0] \}, \]  

(18.29)

so that \(\partial D_{\rho_0} = \{ \rho_0 \varphi(\theta_0) : \theta_0 \in [0, \tau] \}\) is an orbit of (18.7), and is oriented clockwise. Consequently, if

\[ U : [0, +\infty] \times S^1 \to \mathbb{R}^2, \quad (\rho_0, \theta_0) \mapsto \rho_0 \varphi(\theta_0), \]

then \(\Pi = (P_T - I) \circ U, \) \(D_{\rho_0} = U([0, \rho_0[ \times S^1)\)

\[ J_U(\rho_0, \theta_0) = \rho_0[\varphi_1(\theta_0)\varphi'_2(\theta_0) - \varphi_2(\theta_0)\varphi'_1(\theta_0)] = -\rho_0 (\varphi'_{1}(\theta_0), \varphi(\theta_0)) = -\rho_0 < 0. \]

Hence it follows from the definition of Brouwer degree that

\[ d_B[P_T - I, D_{\rho_0}, 0] = -d_B[(P_T - I) \circ U, U^{-1}(D_{\rho_0})] \]  

(18.30)

\[ = -d_B[\Pi, [0, \rho_0] \times S^1, 0]. \]  

(18.31)

We need a few lemmas to compute this last degree. The first one is due to M.A. Krasnosel’kii [209, 220].

Lemma 18.4.1 Let \(D \subset \mathbb{R}^2\) be open, bounded and such that \(\partial D\) is a smooth Jordan curve. Let \(g^1, g^2 \in C^1(\overline{D}, \mathbb{R}^2)\) be such that \([g^1(x), g^2(x)]\) is a base for each \(x \in \partial D\). Let \(f \in C^1(\overline{D}, \mathbb{R}^2)\) be such that \(f(x) \neq 0\) on \(\partial D\), and \(h : \overline{D} \to \mathbb{R}^2\) be defined by

\[ h(x) = f_1(x)g^1(x) + f_2(x)g^2(x). \]

Then

\[ i_K[f, \partial D] = (\text{sign det } G) \cdot \{i_K[h, \partial D] - i_K[g^1, \partial D]\}, \]  

(18.32)

where \(G\) is the \(2 \times 2\)-matrix given by

\[ G = \begin{pmatrix} g^1_1(x^0) & g^2_1(x^0) \\ g^1_2(x^0) & g^2_2(x^0) \end{pmatrix}. \]  

(18.33)

Proof. By assumption, we have

\[ (1 - \lambda)g_1^1(x) + \lambda g^2_2(x) \neq 0 \]
for all $\lambda \in [0,1]$ and all $x \in \partial D$, so that
\[ i_K[g^1, \partial D] = d_B[g^1, D] = d_B[g^2, D] = i_K[g^2, \partial D]. \] (18.34)
Let $x^0 \in \partial D$ be fixed, and let $h^0: \overline{D} \to \mathbb{R}^2$ be defined by
\[ h^0(x) = f_1(x)g^1(x^0) + f_2(x)g^2(x^0). \]

Let us denote by $\gamma(x)$ the positively oriented angle between $h(x)$ and $g^1(x)$, by $\theta(x)$ the positively oriented angle between $h(x)$ and $g^1(x^0)$, and by $\alpha$ the positively oriented angle between $g^1(x)$ and $g^1(x^0)$. We have therefore
\[ d\theta = d\gamma + d\alpha, \]
and hence, by definition of Kronecker index,
\[ i_K[h, \partial D] = i_K[h^0, \partial D] + i_K[g^1, \partial D]. \] (18.35)

On the other hand, in terms of the canonical basis $(e^1, e^2)$ of $\mathbb{R}^2$, we have
\[ h^0(x) = f_1(x)g^1(x^0) + f_2(x)g^2(x^0) \]
\[ = [f_1(x)g_1^1(x^0) + f_2(x)g_1^2(x^0)]e^1 + [f_1(x)g_2^1(x^0) + f_2(x)g_2^2(x^0)]e^2, \]
so that
\[ h^0(x) = Gf(x), \]
where $G$ is given in (18.33). Consequently,
\[ i_K[h, \partial D] = d_B[h, D, 0] = (\text{sign det } G) d_B[f, D, 0] = (\text{sign det } G) i_K[f, D], \] (18.36)
and the result follows from (18.35) and (18.36).

The following lemma computes $d_B[P_T - I, D_{\rho_0}, 0]$ in terms of $i_K[\Gamma, S^1]$ when $\rho_0$ is large.

**Lemma 18.4.2** Assume that
\[ \Gamma(\theta_0) \neq 0 \quad (\theta_0 \in [0,\tau]). \] (18.37)
Then, for all $\rho_0 > 0$ sufficiently large, one has
\[ d_B[P_T - I, D_{\rho_0}, 0] = 1 - i_K[\Gamma, S^1]. \] (18.38)

**Proof.** By Lemma 18.3.3, we have
\[ \rho(T; \rho_0, \theta_0) \varphi(\theta(T; \rho_0, \theta_0)) \]
\[ = \rho_0 \varphi(\theta_0) + \rho_0 [\varphi(\theta_0) + \rho_0^{-1} \varphi(\theta_0) - \Gamma(\theta_0) + R_1(\rho_0, \theta_0)] \]
\[ = \rho_0 \varphi(\theta_0) + \rho_0 [\varphi(\theta_0) + \rho_0^{-1} \varphi(\theta_0) - \Gamma(\theta_0) + R_1(\rho_0, \theta_0)] - \varphi(\theta_0) \]
\[ = [\varphi(\theta_0) - \rho_0 \varphi(\theta_0)] - \Gamma(\theta_0) + R_1(\rho_0, \theta_0) - \varphi(\theta_0). \]
Hence, using (18.26) and definition (18.28)
\[
\lim_{\rho_0 \to +\infty} \Pi(\rho_0, \theta_0) = -\Gamma_1(\theta_0)\varphi'(\theta_0) - \Gamma_2(\theta_0)\varphi(\theta_0) := -\Psi(\theta_0). \tag{18.39}
\]
As $\varphi(\theta_0)$ and $\varphi'(\theta_0)$ are linearly independent, assumption (18.37) implies that $\Psi(\theta_0) \neq 0$ for all $\theta_0 \in [0, \tau]$, and Rouché’s theorem together with (18.39) implies that, for all $\rho_0 > 0$ sufficiently large,
\[
d_B[\Pi, [0, \rho_0] \times S^1, 0] = d_B[-\Psi, [0, \rho_0] \times S^1, 0] = d_B[\Psi, [0, \rho_0] \times S^1, 0] \tag{18.40}
\]
\[
= d_B[\Psi, [0, 1] \times S^1, 0] = i_K[\Psi, S^1], \tag{18.41}
\]
Taking in account the fact that $\Psi$ does not depend upon $\rho_0$. For each $\theta_0 \in [0, \tau],$
\[
\langle J\varphi'(\theta_0), \varphi(\theta_0) \rangle = 1.
\]
In particular, when $\varphi_1(\theta_0) = 0$, $\varphi_1'(\theta_0)\varphi_2(\theta_0) = 1 > 0$, and, as $\varphi_1$ vanishes twice on $[0, \tau[$, we have, from formula (17.2),
\[
i_K[\varphi, S^1] = -1,
\]
and hence
\[
i_K[\varphi', S^1] = i_K[J\varphi', S^1] = i_K[\varphi, S^1] = -1. \tag{18.42}
\]
Furthermore,
\[
\det \begin{pmatrix} \varphi_1'(\theta_0) & \varphi_1(\theta_0) \\ \varphi_2'(\theta_0) & \varphi_2(\theta_0) \end{pmatrix} = \langle J\varphi'(\theta_0), \varphi(\theta_0) \rangle = +1. \tag{18.43}
\]
Therefore, using Lemma 18.4.1 with $g_1 = \varphi'$, $g_2 = \varphi$, $f = \Gamma = (\Gamma_1, \Gamma_2)$, $h = \Psi$, and (18.42), we obtain
\[
i_K[\Gamma, S^1] = i_K[\Psi, S^1] - i_K[\varphi', S^1] = i_K[\Psi, S^1] + 1. \tag{18.44}
\]
Hence, for all sufficiently large $\rho_0 > 0$, using (18.30) and (18.44),
\[
d_B[P_T - I, D_{\rho_0}, 0] = -i_K[\Pi, S^1] = -i_K[\Psi, S^1] = 1 - i_K[\Gamma, S^1].
\]

We can now state and prove an existence theorem for $T$-periodic solutions of (18.13).

**Theorem 18.4.1** Assume that the $\tau$-periodic function
\[
\Gamma_1 : [0, \tau] \to \mathbb{R}, \theta_0 \mapsto \int_0^T (f(s), \varphi(s + \theta_0)) \, ds
\]
does not vanish or has more than four zeros, which are all simple. Then system (18.13) has at least one $T$-periodic solution.
**Proof.** If $\Gamma_1(\theta_0) \neq 0$ for all $\theta_0 \in [0, \tau]$, or if all its zeros are simple, then $\Gamma(\theta_0) = (\Gamma_1(\theta_0), -\Gamma'_1(\theta_0)) \neq 0$ for all $\theta_0 \in [0, \tau]$. Hence, by Lemma 18.4.2, we have, for all $\rho_0 > 0$ sufficiently large,

$$d_B[P_T - I, D_{\rho_0}, 0] = 1 - i_K[\Gamma, S^1].$$  \hspace{1cm} (18.45)

If $\Gamma_1(\theta_0) \neq 0$ for all $\theta_0 \in [0, \tau]$, then $i_K[\Gamma, S^1] = 0$ and, by (18.45), $d_B[P_T - I, D_{\rho_0}, 0] = 1$, so that $P_T$ has a fixed point in $\Omega_{\rho_0}$. If $\Gamma_1$ has zeros, all simple, their number is finite, and, if we denote them by $\theta_{0,k}$ ($1 \leq k \leq m$), we have, using formula (17.2),

$$i_K[\Gamma, S^1] = \frac{1}{2} \sum_{k=1}^{m} \text{sign} \Gamma'_1(\theta_{0,k}) \text{sign} \Gamma'_1(\theta_{0,k}) = \frac{m}{2}.$$  

This shows that $m$ is even, and, furthermore, using (18.45),

$$d_B[P_T - I, D_{\rho_0}, 0] = 1 - \frac{m}{2} \neq 0$$

if $m \geq 4$.  \hspace{1cm} ■

### 18.5 Asymmetric piecewise-linear oscillators

The free linear oscillator

$$x'' + \alpha x = 0 \quad (\alpha > 0)$$

can be generalized to the **free asymmetric oscillator**

$$x'' + \mu x^+ - \nu x^- = 0 \quad (\mu > 0, \nu > 0)$$  \hspace{1cm} (18.46)

where $x^+ := \max(x, 0)$, $x^- := \max(-x, 0)$. The linear restoring force $-\alpha x$ is replaced by the piecewise linear one $-\mu x^+ + \nu x^-$. Letting

$$x = u_1, \quad x' = u_2,$$

we obtain the equivalent Hamiltonian system

$$Ju' = \nabla H(u)$$  \hspace{1cm} (18.47)

where

$$H(u) = \frac{u_2^2}{2} + \mu \frac{(u_1^+)^2}{2} + \nu \frac{(u_1^-)^2}{2}$$  \hspace{1cm} (18.48)

is of class $C^1$, nonnegative, positively homogeneous of degree two, and $\nabla H(u)$ is positively homogeneous of degree one and globally Lipschitzian.
CHAPTER 18. PERIODIC SOLUTIONS OF SOME PLANAR SYSTEMS

It easily follows from the energy integral

\[ H(u) = E \quad (E \geq 0), \]

and from the fact the the system corresponds to the linear oscillator

\[ u''_1 + \mu u_1 = 0 \quad (\text{resp.} \quad u''_1 + \nu u_1 = 0) \]

in the half phase-plane \( \{u_1 \geq 0\} \) (resp. \( \{u_1 < 0\} \)) that the orbits of (18.47) in the phase plane \( \mathbb{R}^2 \) are egg-shaped closed curves around the origin made by half-ellipses of equation

\[ \frac{u_2^2}{2} + \mu \frac{u_1^2}{2} = E \]

in the right half plane and

\[ \frac{u_2^2}{2} + \nu \frac{u_1^2}{2} = E \]

in the left half plane. Hence, the origin is an isochronous center with minimal period

\[ \tau = \pi \left( \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} \right). \]

Given a fixed period \( T > 0 \) that, without loss of generality and for simplicity of notations we take equal to \( 2\pi \), it follows that equation (18.46) or equivalently system (18.47) has a \( 2\pi \)-periodic solution if and only if there exists a positive integer \( N \) such that \( 2\pi = N\tau \), i.e. such that

\[ \left( \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} \right) = \frac{2}{N}. \] (18.49)

This is a family indexed by \( N \) of hyperbolic-like curves independently introduced by Fučík [136] and Dancer [82], and usually called the Fučík curves. Their points are also called the positive Fučík eigenvalues of problem (18.46) or (18.47) for period \( 2\pi \). Notice that the Fučík curves intersect the diagonal in the \( (\mu, \nu) \)-plane at the point \( (n^2, n^2) \), and the \( n^2 \) are the usual eigenvalues of the linear oscillator \( x'' + \lambda x = 0 \) for the period \( 2\pi \).

If one fixes the positive integer \( N \), and \( (\mu, \nu) \) satisfies condition (18.49), one can define the corresponding Fučík eigenfunction for (18.46) by

\[ \sigma(t) := \begin{cases} \frac{1}{\sqrt{\mu}} \sin(\sqrt{\mu}t) & \text{for } t \in [0, \frac{2\pi}{\sqrt{\mu}}], \\ -\frac{1}{\sqrt{\nu}} \sin\left(\sqrt{\nu}(t - \frac{2\pi}{\sqrt{\nu}})\right) & \text{for } t \in \left[\frac{2\pi}{\sqrt{\nu}}, \frac{2\pi}{\sqrt{\mu}}\right]. \end{cases} \] (18.50)

The corresponding solution of (18.47) is given by

\[ \varphi(t) = (\sigma(t), \sigma'(t)), \] (18.51)
and it is easy to check that its satisfies relations (18.9) and (18.10). They are 2(\pi/N)-periodic. Then the non-trivial solutions of (18.46) are given by

\[ x(t) = A\sigma(t + \theta) \quad (A > 0, \, \theta \in [0,2\pi/N]), \]

and the corresponding non-trivial solutions of (18.47) are given by

\[ u(t) = A\varphi(t + \theta) \quad (A > 0, \, \theta \in [0,2\pi/N]). \]

Consider now the forced asymmetric oscillator

\[ x'' + \mu x^+ - \nu x^- = e(t) \quad (\mu > 0, \, \nu > 0), \]

where \( e : \mathbb{R} \to \mathbb{R} \) is continuous and 2\pi-periodic. With the change of variables \((u_1, u_2) = (x, x')\), it is equivalent to the forced Hamiltonian system

\[ J\dot{u}' = \nabla H(u) + f(t) \]

where \( H \) is given in (18.48) and

\[ f(t) = (-e(t), 0). \]

Then, with the notations of the previous section, the corresponding function \( \Gamma_1(\theta_0) \) takes here the form \( \Gamma_1 = \Delta \), where \( \Delta \) is given by

\[ \Delta(\theta_0) := \int_0^{2\pi} e(t)\sigma(t + \theta_0) \, dt \quad (\theta \in [0,2\pi/N]), \]

and is (2\pi/N)-periodic. It was first introduced for (18.52) by Dancer [82] in 1976.

The following existence result for 2\pi-periodic solutions of (18.52) in the non-resonance case, first proved in a different way by Fučík [136] in 1976, is a direct consequence of Theorem 18.2.1.

**Theorem 18.5.1** If \((\mu, \nu)\), with \((\mu > 0, \, \nu > 0)\), is not a Fučík eigenvalue, then then equation (18.52) has at least one 2\pi-periodic solution for each 2\pi-periodic continuous forcing term \( e(t) \).

Because of this result, the set of Fučík eigenvalues is also called the **Fučík spectrum** for the 2\pi-periodic solutions of (18.52).

The existence result for 2\pi-periodic solutions of (18.52) in the resonant case was first proved by Dancer [82] (when \( \Delta \) does not change sign), and by Fabry and Fonda [106] (when \( \Delta \) changes sign). It is a direct consequence of Theorem 18.4.1.

**Theorem 18.5.2** If \((\mu, \nu)\) satisfies (18.49) for some positive integer \( N \), and if the 2\pi-periodic forcing \( e(t) \) is such that the function

\[ \Delta(\theta_0) := \int_0^{2\pi} e(t)\sigma(t + \theta_0) \, dt, \]

with \( \sigma \) given in (18.50), does not vanish or has at least four zeros, all simple, then equation (18.52) has at least one 2\pi-periodic solution.
Chapter 19

Computing degree in higher dimensions

19.1 Cartesian products of mappings

Let $n_1 \geq 1, \ldots, n_m \geq 1$ be integers with $\sum_{j=1}^{m} n_j = n$, $D^j \subset \mathbb{R}^{n_j}$ be open bounded sets, $f^j \in C(D^j, \mathbb{R}^{n_j}) \cap C^2(D^j, \mathbb{R}^{n_j})$ and $z^j$ be such that $z^j \notin f^j(\partial D^j) \ (1 \leq j \leq m)$. As an easy consequence, $D = D^1 \times \ldots \times D^m$ is an open bounded subset of $\mathbb{R}^n$ and the mapping $f$ defined by

$$
f : D = D^1 \times \ldots \times D^m \to \mathbb{R}^n, x \mapsto f(x) = (f^1(x^1), \ldots, f^m(x^m)) \quad (19.1)
$$

belongs to $C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n)$ and is such that $z^j \notin f(\partial D)$.

**Lemma 19.1.1** Under the above assumptions, if $z$ is a regular value for $f$ in $D$, then

$$
d_B[f, D, z] = \prod_{j=1}^{m} d_B[f^j, D^j, z^j]. \quad (19.2)
$$

**Proof.** As $z$ is a regular value, $f^{-1}(z)$ is a finite set. Now, with $x = (x^1, \ldots, x^m)$,

$$
f(x) = z \iff f^1(x^1) = z^1, \ldots, f^m(x^m) = z^m,
$$

and hence each $(f^j)^{-1}(z^j)$ is finite, namely

$$(f^j)^{-1}(z^j) = \{x^{j,1}, \ldots, x^{j,q_j}\} \quad (1 \leq j \leq m),$$

so that

$$f^{-1}(z) = \{(x^{1,l_1}, \ldots, x^{m,l_m}) : 1 \leq l_1 \leq q_1, \ldots, 1 \leq l_m \leq q_m\}.$$
Furthermore, an easy computation shows that, for \( x \in D \),

\[
J_f(x) = \prod_{j=1}^{m} J_{f_j}(x^j),
\]

so that, in particular, each \( z^j \) must be a regular value for \( f^j \) (1 \( \leq \) \( j \) \( \leq \) \( m \)). Consequently,

\[
d_B[f, D, z] = \sum_{l_1=1}^{q_1} \cdots \sum_{l_m=1}^{q_m} \text{sign} \ J_f(x^{1,l_1}, \ldots, x^{m,l_m})
\]

\[
= \sum_{l_1=1}^{q_1} \text{sign} \ J_{f_1}(x^{1,l_1}) \cdots \left[ \sum_{l_m=1}^{q_m} \text{sign} \ J_{f_m}(x^{m,l_m}) \right]
\]

\[
= d_B[f^1, D^1, z^1] \cdots d_B[f^m, D^m, z^m] = \prod_{j=1}^{m} d_B[f^j, D^j, z^j].
\]

The result is easily extended to the situation of \( f \) continuous and \( z \not\in f(\partial D) \) arbitrary.

**Theorem 19.1.1** If \( f \in C(\overline{D}, \mathbb{R}^n) \) is given by (19.1) and if \( z^j \not\in \partial D^j \) (1 \( \leq \) \( j \) \( \leq \) \( m \)) then formula (19.2) holds.

**Proof.** Take \( z' \) regular for \( f \) sufficiently close to \( z \) so that

\[
d_B[f, D, z] = d_B[f, D, z'] = \prod_{j=1}^{m} d_B[f^j, D^j, z'^j] = \prod_{j=1}^{m} d_B[f^j, D^j, z^j].
\]

As an example, let us consider the mapping \( p^0 : \mathbb{C}^m \rightarrow \mathbb{C}^m \) defined by

\[
p^0(z_1, \ldots, z_m) = (p^0_1(z_1), \ldots, p^0_m(z_m))
\]

where the functions \( p^0_j : \mathbb{C} \rightarrow \mathbb{C} \) are defined by

\[
p^0_j(z) = z^{k_j} \quad (1 \leq j \leq m)
\]

(19.3)

with \( k_j \geq 1 \) is an integer (1 \( \leq j \) \( \leq m \)). \( p \) can also be seen as a mapping from \( \mathbb{R}^{2m} \) into \( \mathbb{R}^{2m} \). We know from example 17.4.1 that, for each \( r > 0 \),

\[
d_B[p^0_j, B(r), 0] = k_j \quad (1 \leq j \leq m),
\]

so that Theorem 19.1.1 implies that

\[
d_B[p^0, B(r) \times \ldots \times B(r), 0] = \prod_{j=1}^{m} k_j.
\]

(19.4)
19.2. HOMOGENEOUS POLYNOMIALS MAPPINGS

Now, \( p^0(z) = 0 \) if and only if \( z = 0 \), and hence, using excision property 3.4.1 we deduce from (19.4) and from the definition of Brouwer index 5.1.1 that, for any open bounded neighborhood \( D \) of 0 in \( \mathbb{C}^m \), and given by (19.3), we have

\[
d_B[p^0, D, 0] = i_B[p^0, 0] = \prod_{j=1}^{m} k_j \tag{19.5}
\]

19.2 Homogeneous polynomials mappings

We consider now the more general situation of mappings \( p : \mathbb{C}^m \to \mathbb{C}^m \) where each component \( p_j^{[k_j]} \) is a homogeneous polynomial in \((z_1, \ldots, z_n)\) of degree \( k_j \geq 1 \) \((1 \leq j \leq m)\). We assume that \( p^{-1}(0) = \{0\} \) and compute \( i_B[p, 0] \). The following results were first obtained in 1953 by J. Cronin [78] (see also [81]).

**Proposition 19.2.1** Let the mapping \( p : \mathbb{C}^m \to \mathbb{C}^m \) be such that each component \( p_j^{[k_j]} \) is a homogeneous polynomial in \((z_1, \ldots, z_n)\) of degree \( k_j \geq 1 \) \((1 \leq j \leq m)\). If \( p^{-1}(0) = \{0\} \), then, for any open bounded neighborhood \( D \) of 0, one has

\[
d_B[p, D, 0] = i_B[p, 0] = \prod_{j=1}^{m} k_j. \tag{19.6}
\]

**Proof.** Given \( m \) homogeneous polynomial \( p_1, \ldots, p_m : \mathbb{C}^m \to \mathbb{C} \) of respective degrees \( k_1, \ldots, k_m \), let us order their coefficients in a sequence \((a_1, \ldots, a_p)\) considered as an element of \( \mathbb{C}^p \). The resultant \( \mathcal{R}(a_1, \ldots, a_p) \) of the polynomials \( p_j \) \((j = 1, \ldots, m)\) is a polynomial in \((a_1, a_2, \ldots, a_p)\) with integer coefficients, such that \( \mathcal{R}(a_1, \ldots, a_p) \neq 0 \) if and only if 0 is the unique common zero of the \( p_j \) \((j = 1, \ldots, m)\).

Let \( b_1, b_2, \ldots, b_p \) denote the corresponding sequence of the coefficients of the polynomials \( p_1^{[m_1]}, \ldots, p_m^{[m_p]} \), and \( (c_1, \ldots, c_p) \) the corresponding sequence of the coefficients of the polynomial \( p_1^{[m_1]}, \ldots, p_m^{[m_p]} \) defined in (19.3). By assumption and definition, \( \mathcal{R}(b_1, \ldots, b_p) \neq 0 \) and \( \mathcal{R}(c_1, \ldots, c_p) \neq 0 \). Define the polynomial \( \phi : \mathbb{C} \to \mathbb{C} \) by

\[
\phi(t) = \mathcal{R}[(1-t)b_1 + tc_1, \ldots, (1-t)b_p + tc_p].
\]

Since \( \phi(0) = \mathcal{R}(b_1, \ldots, b_p) \neq 0 \) and \( \phi(1) = \mathcal{R}(c_1, \ldots, c_p) \neq 0 \), the polynomial \( \phi(t) \) is not identically zero, and hence has a finite number of zeros, different from 0 and 1. Consequently, there is a continuous path \( \tau : [0, 1] \to \mathbb{C} \) such that \( \tau(0) = 0, \tau(1) = 1 \) and \( \phi(\tau(\lambda)) \neq 0 \) for all \( \lambda \in [0, 1] \). Consequently, the homotopy \( P : \mathbb{C}^m \times [0, 1] \to \mathbb{C}^m \) where \( P(z, \lambda) = (P_1(z, \lambda), \ldots, P_m(z, \lambda)) \), and \( P_1(z, \lambda), \ldots, P_m(z, \lambda) \) are the homogeneous polynomials of respective degrees \( k_1, \ldots, k_m \) associated to the sequence of coefficients

\[
(1 - \tau(\lambda))b_1 + \tau(\lambda)c_1, \ldots, (1 - \tau(\lambda))b_p + \tau(\lambda)c_p
\]
is such that \( P(z, 0) = p(z), \) \( P(z, 1) = p^0(z), \) and
\[
R[(1 - \tau(\lambda))b_1 + \tau(\lambda)c_1, \ldots, (1 - \tau(\lambda))b_p + \tau(\lambda)c_p] \neq 0
\]
for all \( \lambda \in [0, 1]. \) In other words, \([P(\cdot, \lambda)]^{-1}(0) = \{0\}\) for all \( \lambda \in [0, 1], \) and hence, using the homotopy invariance property 3.4.2 and (19.5), we get, for any open bounded neighborhood \( D \) of \( 0, \)
\[
i_B[p, 0] = d_B[p, D, 0] = d_B[P(\cdot, 0), D, 0] = d_B[p^0, D, 0] = i_B[p^0, 0]\]
\[
= \prod_{j=1}^m k_j.
\]
\[\blacksquare\]

Combining Proposition 19.2.1 with Propositions 5.2.2 and 5.3.1, we obtain immediately the following results.

**Proposition 19.2.2** Let \( U \subset \mathbb{C}^m \) be an open neighborhood of \( 0, \) and \( f \in C(U, \mathbb{C}^m) \) be such that, for some mapping \( p: \mathbb{C}^m \to \mathbb{C}^m \) whose each component \( p_j\big|_{k_j} \) is a homogeneous polynomial in \((z_1, \ldots, z_n)\) of degree \( k_j \geq 1 \) \((1 \leq j \leq m)\), one has
\[
\lim_{z \to 0} \frac{f_j(z) - p_j(z)}{\|z\|^{k_j}} = 0 \quad (1 \leq j \leq m).
\]
If \( p^{-1}(0) = \{0\}, \) then
\[
i_B[f, 0] = \prod_{j=1}^m k_j.
\]

**Proposition 19.2.3** Let \( f \in C(\mathbb{C}^m, \mathbb{C}^m) \) be such that, for some mapping \( p: \mathbb{C}^m \to \mathbb{C}^m \) whose each component \( p_j|_{k_j} \) is a homogeneous polynomial in \((z_1, \ldots, z_n)\) of degree \( k_j \geq 1 \) \((1 \leq j \leq m)\), one has
\[
\lim_{z \to \infty} \frac{f_j(z) - p_j(z)}{\|z\|^{k_j}} = 0 \quad (1 \leq j \leq m).
\]
If \( p^{-1}(0) = \{0\}, \) then, for any \( z \in \mathbb{C}^m, \)
\[
i_B[f, \infty, z] = \prod_{j=1}^m k_j.
\]

We can complete Proposition 19.2.2 by a result, also due to Cronin [78, 81] which only requires the degree of \( f \) to be defined on some ball.
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Proposition 19.2.4 Let $U \subset \mathbb{C}^m$ be an open neighborhood of 0, and $f \in C(U, \mathbb{C}^m)$ be such that, for some mapping $p : \mathbb{C}^m \to \mathbb{C}^m$ whose each component $p_j^{[k_j]}$ is a homogeneous polynomial in $(z_1, \ldots, z_n)$ of degree $k_j \geq 1$ $(1 \leq j \leq m)$, one has
\[ \lim_{z \to 0} \frac{f_j(z) - p_j(z)}{\|z\|^{k_j}} = 0 \quad (1 \leq j \leq m). \]
If $0 \notin f(\partial B(R))$ for some $R > 0$, then
\[ d_B[f, B(R), 0] \geq \prod_{j=1}^m k_j. \]

Proof. Let us first assume that the resultant $R$ of $p_1^{[k_1]}, \ldots, p_m^{[k_m]}$ is nonzero. By Proposition 19.2.2, if $\varepsilon > 0$ is sufficiently small, $d_B[f, B(\varepsilon), 0] = \prod_{j=1}^m k_j$. Take $\varepsilon \leq R$. Using Sard’s theorem, there exists $y$ arbitrarily close to zero such that $y$ is not a critical value of $f$ and
\[ d_B[f, B(\varepsilon), y] = d_B[f, B(\varepsilon), y] = d_B[f, B(R), y]. \]
Consequently, as the Jacobian of $f$ is positive on $f^{-1}(y)$,
\[ \prod_{j=1}^m k_j = d_B[f, B(\varepsilon), y] = \# f^{-1}(y) \cap B(\varepsilon) \leq \# f^{-1}(y) \cap B(R) \]
\[ = d_B[f, B(R), y]. \]
If the resultant $R$ of $p_1^{[k_1]}, \ldots, p_m^{[k_m]}$ is zero, then, there exist arbitrary small changes of the coefficients of those polynomials giving polynomials $q_1^{[k_1]}, \ldots, q_m^{[k_m]}$ the resultant of which is nonzero. Using Rouche’s property 3.4.2, we can take them so that the mapping $g$ obtained in replacing $p$ in $f$ by $q$ is such that
\[ d_B[f, B(R), 0] = d_B[g, B(R), 0] \geq \prod_{j=1}^m k_j, \]
by applying the first part of the proof to $g$.

19.3 Orientation preserving mappings

We shall consider in this section and the following ones some classes of mappings which always have a nonnegative Brouwer degree. They contain in particular, in dimension two, the mappings associated to holomorphic functions of $\mathbb{C}$ into $\mathbb{C}$, and considered in Chapter 17. We will see that they contain various classes of mappings between higher dimensional spaces.

Definition 19.3.1 The linear mapping $L : \mathbb{R}^n \to \mathbb{R}^n$ is said to be orientation-preserving if $\det L \geq 0$. 
The following definitions were introduced in 1970 by F.E. Browder [46] under the name of mapping of strict analytic type, and renamed by the same author strictly orientation-preserving mappings in [47]. Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^n$.

**Definition 19.3.2** We say that $f$ is strictly orientation-preserving if the following conditions hold:

(a) $f \in C^2(U, \mathbb{R}^n)$ and, for each $x \in U$, its differential $f'(x)$ is orientation preserving;

(b) The set $\{x \in U : J_f(x) = 0\}$ is nowhere-dense in $U$.

Again, the following class of mapping was first introduced in [46] under the name of mappings of analytic type and renamed orientation-preserving mappings in [47].

**Definition 19.3.3** We say that $f \in C(U, \mathbb{R}^n)$ is orientation-preserving if, for each open subset $V$ with $\overline{V} \subset U$ and compact, there exists a sequence of mappings $(f_k)$, such that $f_k : V \to \mathbb{R}^n$ of strictly orientation-preserving, such that $f_k \to f$ uniformly on $V$.

We prove some general properties of the degree of orientation-preserving mappings first proved by F.E. Browder in [46, 47]. We begin by proving them for strictly orientation-preserving mappings.

**Lemma 19.3.1** Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ a strictly orientation-preserving mapping, $D \subset \mathbb{R}^n$ an open bounded set such that $\overline{D} \subset U$, and $z \notin f(D)$. Then:

(a) $d_B[f, D, z] \geq 0$.

(b) $d_B[f, D, z] > 0$ if and only if $z \in f(D)$.

(c) If $d_B[f, D, z] = 1$, $f^{-1}(z) \cap D$ is connected.

(d) If $f(D)$ contains a point of some component $C$ of $\mathbb{R}^n \setminus f(\partial D)$, $C \subset f(D)$.

**Proof.** (a) If follows from (3.1) and definitions 19.3.1 and 19.3.2 that $d_B[f, D, z] \geq 0$.

(b) If $d_B[f, D, z] > 0$, $z \in f(D)$ by the existence property 3.4.1. If $z \in f(D)$, let $C$ be the connected component of $\mathbb{R}^n \setminus f(\partial D)$ containing $z$. By Definition 19.3.2, the closed set $\{x \in \overline{D} : J_f(x) = 0\}$ has empty interior, and hence, if $x_0 \in D$ is such that $f(x_0) = z$, there exists a sequence $(x_n)$ in $D$ converging to $x_0$ such that $J_f(x_n) > 0$ for all integers $n \geq 1$. By continuity, $f(x_n) \to f(x_0) = z$, and hence there exists $m \in \mathbb{N}^*$ such that $f(x_n) \in C$ for $n \geq m$. By the inverse mapping theorem $x_m$ is an isolated solution of equation $f(x) = f(x_m)$, and there exists $r > 0$ such that $B_r(x_m)$ only contains that solution. Consequently, using the additivity property 3.4.3 and the property 3.4.6, we get

$$d_B[f, D, z] = d_B[f, D, f(x_m)] = d_B[f, B_r(x_m), f(x_m)] + d_B[f, D \setminus B_r(x_m), f(x_m)] \geq d_B[f, B_r(x_m), f(x_m)] = i_B[f, x_m] = \text{sign } J_f(x_m) = 1.$$
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(c) Suppose that $d_B[f, D, z] = 1$ and that $f^{-1}(z) \cap D$ is not connected. Then there exists two disjoint open sets $N_1$ and $N_2$ of $V$ with disjoint closures such that $f^{-1}(z) \cap D \subset N_1 \cup N_2$, each $N_j$ containing points of $f^{-1}(z)$. By conclusion (b),

$$d_B[f, N_1, z] \geq 1, \quad d_B[f, N_2, z] \geq 1,$$

and by the additivity property 3.4.3 of degree,

$$d_B[f, D, z] = d_B[f, N_1, z] + d_B[f, N_2, z] \geq 2,$$

a contradiction.

(d) Suppose that for a given component $C$ of $\mathbb{R}^n \setminus f(\partial D)$, we have a point $y \in f(D) \cap C$. From conclusion (b), $d_B[f, D, y] > 0$, and by property 3.4.6, $d_B[f, D, z] = d_B[f, D, y]$ for any $z \in C$, so that, by the existence property 3.4.1, $C \subset f(D)$. $\blacksquare$

We now extend the conclusions of Lemma 19.3.1 to orientation-preserving mappings.

**Theorem 19.3.1** Let $U \subset \mathbb{R}^n$ be open, $f \in C(U, \mathbb{R}^n)$ an orientation-preserving mapping, $D \in \mathbb{R}^n$ an open bounded set such that $D \subset U$, and $z \notin f(\partial D)$. Then:

(a) $d_B[f, D, z] \geq 0$.

(b) $d_B[f, D, z] > 0$ if and only if $z \in f(D)$.

(c) If $d_B[f, D, z] = 1$, $f^{-1}(z) \cap D$ is connected.

(d) If $f(D)$ contains a point of some component $C$ of $\mathbb{R}^n \setminus f(\partial D)$, $C \subset f(D)$.

**Proof.** Let the open subset $V$ be such that $\overline{V} \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. By definition of $f$, there exists a sequence $(f_k)$ of mappings of $V$ in $\mathbb{R}^n$ strictly orientation-preserving and converging uniformly to $f$ on $V$ as $k \to \infty$. So the conclusions of Lemma 19.3.1 are valid for the $f_k$.

(a) Let $z \in \mathbb{R}^n \setminus f(\partial D)$, $d_0 = \min_{x \in \partial D} \|f(x) - z\|$ and $k_0$ sufficiently large so that $\max_{x \in \partial D} \|f_k(x) - f(x)\| < d_0$. Then, from Rouché’s property 3.4.2, we have $d_B[f, D, z] = d_B[f_k, D, z] \geq 0$ $(k \geq k_0)$ and the result follows letting $k \to \infty$.

(b) If $d_B[f, D, z] > 0$, then $z \in f(D)$ by the existence property 3.4.1. If $z \in f(D)$, and if $d_0 = \min_{x \in \partial D} \|f(x) - z\|$, let $x_0 \in D$ be such that $f(x_0) = z$. Choose $k_0$ so large that for $k > k_0$, one has $\max_{x \in \overline{D}} \|f_k(x) - f(x)\| < \frac{d_0}{2}$. Then, for each $x \in \partial D$, one has

$$\|f_k(x) - z\| \geq \|f(x) - z\| - \|f_k(x) - f(x)\| > \frac{d_0}{2},$$

which implies that $B_y(d_0/2)$ is contained in the same component of $\mathbb{R}^n \setminus f_k(\partial D)$ as $y$. On the other hand

$$\|f_k(x_0) - z\| = \|f_k(x_0) - f(x_0)\| < \frac{d_0}{2},$$
and hence \( f_k(x_0) \) lies in the same component of \( \mathbb{R}^n \setminus f_k(\partial D) \) as \( y \). By conclusion (b) of Lemma 19.3.1, we see that \( d_B[f_k, D, z] > 0 \), and hence, by Rouché’s property, we have, for all \( k > k_0 \),

\[
d_B[f, D, z] = d_B[f_k, D, z] > 0.
\]

(c) and (d) Follow exactly the same argument as the proof of (c) and (d) in Lemma 19.3.1.

\[
\text{19.4 Monotone mappings}
\]

As a first example, let us denote by \((\cdot, \cdot)\) the usual inner product in \( \mathbb{R}^n \), and assume that, for some open \( U \subset \mathbb{R}^n \), \( f : U \to X \) is \textbf{monotone}, i.e. that \( (f(x) - f(y), x - y) \geq 0 \) for any \( x, y \in U \).

**Proposition 19.4.1** If \( U \subset \mathbb{R}^n \) is open and \( f : U \to \mathbb{R}^n \) is monotone, then \( f \) is orientation-preserving on \( U \).

**Proof.** Let \( V \) be an open set with compact closure \( \overline{V} \subset U \), then \( \text{dist}(\overline{V}, \partial U) > 0 \), and there exists \( \delta > 0 \) such that, for any \( x \in V, B_x(\delta) \subset U \). If \( t > 0 \) is sufficiently small and \( q \) is a \( C^2 \) scalar nonnegative function, with support contained in \( B_x(r) \) for some \( 0 < r < \delta \), and such that \( \int_{\mathbb{R}^n} q(x) \, dx = 1 \), we define \( g_t : V \to \mathbb{R}^n \) by

\[
g_t(u) = \int_{\mathbb{R}^n} |f(u - v) + t(u - v)|q(v) \, dv = \int_{B_s(v)} |f(w + tw)|q(u - w) \, dw.
\]

Hence, \( g_t \in C^2(V, \mathbb{R}^n) \), and, using the monotonicity of \( f \),

\[
(g_t(x) - g_t(y), x - y) = \int_{\mathbb{R}^n} ((f(x - v) - f(y - v) + t(x - y)), x - y)q(v) \, dv \\
\geq t \int_{\mathbb{R}^n} ||x - y||^2 q(v) \, dv = t||x - y||^2.
\]

Taking \( x = y + sv \) with \( s \neq 0 \) sufficiently small, we get

\[
\frac{(g_t(y + sv) - g_t(y), v)}{s} \geq t||v||^2
\]

for all \( s \neq 0 \) sufficiently small and all \( v \in \mathbb{R}^n \), and hence

\[
((g_t)'(y)v, v) \geq t||v||^2
\]

for all \( y \in V \) and \( v \in \mathbb{R}^n \). Hence, all real eigenvalues of \((g_t)'(y)\) are positive and, as complex eigenvalue occur in conjugate pairs, \( J_{g_t}(y) \), equal to the product of the eigenvalues of \((g_t)'(y)\), is positive for all \( t > 0 \) and \( u \in \mathbb{R}^n \). Finally, it is standard that \( g_t \to f \) uniformly on \( V \) when \( t \to 0 + \).

\[\square\]
19.5 Holomorphic mappings

As a second example, let us consider the class of holomorphic mappings from \( \mathbb{C}^m \) into \( \mathbb{C}^m \), viewed as mappings from \( \mathbb{R}^{2m} \) into \( \mathbb{R}^{2m} \). We first recall a few definitions \([56]\). Let \( U \subset \mathbb{C}^m \), \( f : U \to \mathbb{C}^m \) and \( a \in U \).

**Definition 19.5.1** We say that \( f \) is holomorphic at \( a \) if there exists a \( \mathbb{C} \)-linear mapping \( L \) and a mapping \( r : (U - a) \setminus \{0\} \to \mathbb{C}^m \) such that \( \lim_{h \to 0} r(h) = 0 \) and, for \( h \in U - a \),

\[
f(a + h) = f(a) + L(h) + \|h\| r(h). \tag{19.7}
\]

If it is the case, \( L \) is unique and noted \( f'(a) \) or \( df(a) \), the limits

\[
\partial z_j f(a) := \lim_{t \in \mathbb{C}, t \to 0} \frac{f(a + te^j) - f(a)}{t} \quad (1 \leq j \leq m) \tag{19.8}
\]

exist, are equal to \( f'(a)(e^j) \) (\( j = 1, 2, \ldots, m \)) and

\[
f'(a)h = \sum_{j=1}^{m} h_j \partial z_j f(a). \tag{19.9}
\]

In the terminology of differentials, (19.9) gives, for \( f(z) = z \) the identity

\[
dz = \sum_{j=1}^{m} h_j e^j = h, \quad \text{i.e.} \quad dz_j = h_j \quad (j = 1, 2, \ldots, m),
\]

and hence (19.9) can be written

\[
df = \sum_{j=1}^{m} \partial z_j f dz_j. \tag{19.10}
\]

If we write \( z_j = x_j + iy_j \) with \( x_j, y_j \in \mathbb{R} \) (\( j = 1, 2, \ldots, m \)) it follows from (19.7) that \( f \), considered as a mapping from \( \mathbb{R}^{2m} \) to \( \mathbb{R}^{2m} \) by taking the real and imaginary parts of the variables and of the components of the function, is differentiable, and

\[
df = \sum_{j=1}^{m} \left( \partial x_j f dx_j + \partial y_j f dy_j \right). \tag{19.11}
\]

Using the relations

\[
\begin{align*}
dz_j &= dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j, \\
dx_j &= \frac{1}{2}(dz_j + d\bar{z}_j), \quad dy_j = \frac{1}{2i}(dz_j - d\bar{z}_j) \quad (j = 1, 2, \ldots, m),
\end{align*}
\]

we can write (19.11) as

\[
df = \sum_{j=1}^{m} \left[ \frac{1}{2}(\partial x_j f - i\partial y_j f)dz_j + \frac{1}{2}(\partial x_j f + i\partial y_j f) d\bar{z}_j \right]. \tag{19.12}
\]
This implies in particular that
\[
\overline{df} = \sum_{j=1}^{m} \left[ \frac{1}{2}(\partial_{x_j} \overline{f} - i\partial_{y_j} \overline{f}) \, dz_j + \frac{1}{2}(\partial_{x_j} \overline{f} + i\partial_{y_j} \overline{f}) \, d\overline{z_j} \right]
\]
\[
= \sum_{j=1}^{m} \left[ \frac{1}{2}(\partial_{x_j} f + i\partial_{y_j} f) \, dz_j + \frac{1}{2}(\partial_{x_j} f - \partial_{y_j} f) \, d\overline{z_j} \right]
\]
\[
= \overline{df},
\] (19.13)
and that \( f \) is holomorphic on \( U \) if and only if
\[
\partial_{x_j} f := \frac{1}{2}(\partial_{x_j} f + i\partial_{y_j} f) = 0 \quad (j = 1, 2, \ldots, m).
\]

The following result is fundamental. For \( f : U \rightarrow \mathbb{C}^m \) holomorphic, we denote by \( J_f(z) = \det f'(z) \) the (complex) Jacobian associated to the \( \mathbb{C} \)-differential of \( f \), by \( \varphi : U \subset \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m} \) the associated real mapping defined by
\[
\varphi(x_1, y_1, \ldots, x_m, y_m) = (u_1(x_1, \ldots, y_m), v_1(x_1, \ldots, y_m), \ldots, u_m(x_1, \ldots, y_m), v_m(x_1, \ldots, y_m)),
\] (19.14)
and by \( J_\varphi(z) \) the (real) Jacobian associated to the \( \mathbb{R} \)-differential of \( \varphi \).

**Lemma 19.5.1** If \( U \subset \mathbb{C}^m \) is open, and \( f : U \rightarrow \mathbb{C}^m \) is holomorphic in \( U \),
\[
J_\varphi(x_1, y_1, \ldots, x_m, y_m) = |J_f(z)|^2
\] (19.15)
for all \( z = (z_1, \ldots, z_m) = (x_1, y_1, \ldots, x_m, y_m) \in U \).

**Proof.** Using the rules of exterior calculus, we have
\[
df_1 \wedge \ldots \wedge df_m = \left( \sum_{j_1=1}^{m} \partial_{z_{j_1}} f_1 \, dz_{j_1} \right) \wedge \ldots \wedge \left( \sum_{j_m=1}^{m} \partial_{z_{j_m}} f_m \, dz_{j_m} \right)
\]
\[
= \left[ \sum_{\Pi=(j_1,\ldots,j_m)} (\text{sign } \Pi) \partial_{z_{j_1}} f_1 \ldots \partial_{z_{j_m}} f_m \right] \, dz_1 \ldots \wedge \, dz_m
\]
\[
= J_f \, dz_1 \wedge \ldots \wedge \, dz_m,
\] (19.16)
where the sum is taken over all the permutations \( \Pi = \{j_1, \ldots, j_m\} \) of \( \{1, \ldots, m\} \). Therefore,
\[
\overline{df}_1 \wedge \ldots \wedge \overline{df}_m = \overline{df}_1 \wedge \ldots \wedge \overline{df}_m = J_f \, d\overline{z}_1 \wedge \ldots \wedge \, d\overline{z}_m.
\] (19.17)
Similarly,
\[
du_1 \wedge dv_1 \wedge \ldots \wedge du_m \wedge dv_m = J_\varphi \, dx_1 \wedge dy_1 \wedge \ldots \wedge dx_m \wedge dy_m.
\] (19.18)
On the other hand
\[
dz_j \wedge \overline{dz}_j = (dx_j + idy_j) \wedge (dx_j - idy_j) = -2i \, dx_j \wedge dy_j
\]
and, using (19.16), (19.17), (19.18) and (19.19), this gives
\[
df_j \wedge d\overline{f}_j = (du_j + idv_j) \wedge (du_j - idv_j) = -2i \, du_j \wedge dv_j \quad (j = 1, 2, \ldots, m).
\]
Hence,
\[
df_1 \wedge d\overline{f}_1 \wedge \ldots \wedge df_m \wedge d\overline{f}_m = (-2i)^m \, du_1 \wedge dv_1 \wedge \ldots \wedge du_m \wedge dv_m,
\]
or, using the same permutations in the left and right-hand members,
\[
df_1 \wedge \ldots \wedge df_m \wedge d\overline{f}_1 \wedge \ldots \wedge d\overline{f}_m
\]
\[
= (-2i)^m du_1 \wedge \ldots \wedge du_m \wedge dv_1 \wedge \ldots \wedge dv_m,
\]
and, using (19.16), (19.17), (19.18) and (19.19), this gives
\[
J_f \cdot J_{\overline{f}} \, dz_1 \wedge \ldots \wedge dz_m \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_m
\]
\[
= (-2i)^m J_\varphi \, dx_1 \wedge \ldots \wedge dx_m \wedge dy_1 \wedge \ldots \wedge dy_m,
\]
and hence (19.15), if we notice that, from (19.19)
\[
dz_1 \wedge \ldots \wedge dz_m \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_m = (-2i)^m \, dx_1 \wedge \ldots \wedge dx_m \wedge dy_1 \wedge \ldots \wedge dy_m.
\]

We need a second preliminary result to show that the holomorphic mappings are orientation-preserving. We use the notations of Lemma 19.5.1.

**Lemma 19.5.2** Let \( U \subset \mathbb{C}^m \) be connected and open, and \( f : U \to \mathbb{C}^m \) holomorphic. Given \( \delta_0 > 0 \), there exists \( \varepsilon \in [0, \delta_0] \) such that, if \( f_\varepsilon(z) = f(z) + \varepsilon z \), the set \( S_\varepsilon := \{ x \in U : J_\varphi(x) = 0 \} \) is nowhere dense in \( U \).

**Proof.** By Lemma 19.5.1, it suffices to show that the set of \( z \in U \) such that \( J_{f_\varepsilon}(z) = 0 \) is nowhere dense in \( U \). Because the partial \( \mathbb{C} \)-derivatives of an holomorphic functions are holomorphic (see e.g. [56]), \( J_{f_\varepsilon} \) is a holomorphic function of \( z \), and, for fixed \( z = z_0 \) a polynomial in \( t \) whose top coefficient is one. Hence, given any \( \delta_0 > 0 \), there is some \( \varepsilon \in [0, \delta_0] \) such that \( J_{f_{\varepsilon}}(z_0) \neq 0 \), and hence \( J_{f_{\varepsilon}}(z_0) > 0 \). Suppose that, for this value of \( \varepsilon \), \( S_\varepsilon \) is dense in a non-trivial open subset of \( U \). Since \( S_\varepsilon \) is closed in \( U \), then \( J_{f_\varepsilon} = 0 \) on a non-trivial open subset of \( U \). Since \( J_{f_\varepsilon} \) is analytic and \( U \) is connected, we must have \( J_{f_\varepsilon}(z) = 0 \) on \( U \), a contradiction with \( J_{f_\varepsilon}(z_0) \neq 0 \).

**Proposition 19.5.1** Let \( U \subset \mathbb{C}^m \) be open, and \( f : U \to \mathbb{C}^m \) holomorphic. Then the associated mapping \( \varphi : U \subset \mathbb{R}^{2m} \to \mathbb{R}^{2m} \) is orientation-preserving.

**Proof.** By Lemmas 19.5.1 and 19.5.2, on each connected component of \( U \), there exists a sequence \((\varphi_k)\) of strictly orientation-preserving mappings (corresponding to the \( f_{\varepsilon_k} \) for a suitable sequence \( \varepsilon_k \to 0 \)) which converges uniformly to \( \varphi \) on any bounded subset of the connected component.
Let $U \subset \mathbb{C}^m$ be open, $f : U \to \mathbb{C}^m$ be holomorphic, $\varphi : U \subset \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ the associated real mapping defined in (19.14), $D$ open bounded such that $\overline{D} \subset U$, and $y \not\in f(\partial D)$. Notice that we use the same notations for corresponding points and corresponding sets in $\mathbb{C}^m$ and $\mathbb{R}^{2m}$. We write $d_B[f, D, y]$ for $d_B[\varphi, D, y]$. From Proposition 19.5.1 and Theorem 19.3.1, we know that $d_B[f, D, y] \geq 0$ and $d_B[f, D, y] > 0$ if and only if $y \in f(\partial D)$. A more precise relation between the number of zeros of $f$ and its topological degree has been obtained in 1973 by P. Rabinowitz [326] (see also [328]). It requires proving some preliminary algebraic result.

**Lemma 19.5.3** Given $\zeta_1, \ldots, \zeta_k$ in $\mathbb{C}^m \setminus \{0\}$, one can find a basis of $\mathbb{C}^m$ such that, in this basis, all coordinates $\zeta_{jl}$ ($1 \leq j \leq k; 1 \leq l \leq m$) are different from zero.

**Proof.** Let $\mu \in \mathbb{C}^m$ and $M$ the matrix with equal lines all made of the coordinates of $\overline{\mu}$. Choose $\mu$ sufficiently small so that the matrix $I + M$ is invertible. Consider the basis transformation defined by the matrix $(I + M)$. If $w_j = \zeta_j + M \zeta_j$, i.e., for any $1 \leq l \leq m$, $w_{jl} = \zeta_{jl} + (\zeta_j)\mu$, where $(\cdot | \cdot)$ denotes the standard inner product in $\mathbb{C}^m$. If we define the hyperplanes $H_{jl} \subset \mathbb{C}^m$ by $H_{jl} := \{z \in \mathbb{C}^m : (\zeta_j, z) = -\zeta_{jl}(1 \leq j \leq k; 1 \leq l \leq m)\}$, then $\mathbb{C}^m \setminus \bigcup_{1 \leq j \leq k; 1 \leq l \leq m} H_{jl}$ is everywhere dense in $\mathbb{C}^m$, and hence we can choose $\mu \in \mathbb{C}^m \setminus \bigcup_{1 \leq j \leq k; 1 \leq l \leq m} H_{jl}$ sufficiently small so that $I + M$ is invertible. With this choice, $w_{jl} \neq 0$ for all $1 \leq j \leq k$ and $1 \leq l \leq m$.

**Theorem 19.5.1** If $U \subset \mathbb{C}^m$ is open, $f : U \to \mathbb{C}^m$ holomorphic, $D$ open bounded such that $\overline{D} \subset U$ $y \not\in f(\partial D)$, and $d_B[f, D, y] = k$, then equation $f(z) = y$ has at most $k$ distinct solutions in $D$.

**Proof.** The proof is by recurrence over $k$. For $k = 0$, it follows from assertion (b) of Theorem 19.3.1 that equation $f(z) = y$ has no solution in $D$. Assume now that the result is true for $k - 1$ and all $f, D, y$ satisfying the assumptions, and suppose that the result is not true for $k$, i.e., that there exists some $f, D, y$ such that $d_B[f, D, y] = k$ and $f^{-1}(y) \cap D$ contains at least $k + 1$ distinct elements $\zeta_0, \zeta_1, \ldots, \zeta_k$. By a translation of the origin of coordinates and a translation of $f$, which do not change the Brouwer degree, we can assume without loss of generality that $\zeta_0 = 0$ and $y = 0$. Also, because Brouwer degree is independent of the chosen basis in $\mathbb{R}^{2m}$ and hence in $\mathbb{C}^m$, we can use Lemma 19.5.3 to choose a basis in which none of the coordinates of $\zeta_1, \ldots, \zeta_k$ is zero, i.e., such that $\zeta_{jl} \neq 0$ ($1 \leq j \leq k; 1 \leq l \leq m$). Let us introduce then the application $g : U \to \mathbb{C}^m$ defined by

$$g_j(z) = f_j(z) + \varepsilon \prod_{j=0}^{k} (z_l - \zeta_{jl}) \quad (1 \leq l \leq m),$$

where $\varepsilon > 0$, $g$ is holomorphic on $U$ and $g(\zeta_j) = 0$ for all $j = 0, 1, \ldots, k$. Furthermore, it follows from Rouché’s property 3.4.2 that, for $\varepsilon > 0$ sufficiently small, $d_B[g, D, y] = d_B[f, D, y] = k$. Now $J_g(0)$ is a polynomial in $\varepsilon$ with coefficient of the highest power equal to $(-1)^{km}\prod_{1 \leq j \leq k; 1 \leq l \leq m} \zeta_{jl} \neq 0$. Consequently we can find $\varepsilon > 0$ arbitrarily small such that $J_g(0) \neq 0$, which implies that 0 is an isolated zero...
of $g$, and hence, using the additivity property 3.4.3, we get, for some sufficiently small $\delta > 0$,

$$d_B[g, D, 0] = d_B[g, B(\delta), 0] + d_B[g, D \setminus \overline{B}(\delta), 0]$$

$$= iv[g, 0] + d_B[g, D \setminus \overline{B}(\delta), 0] = 1 + d_B[g, D \setminus \overline{B}(\delta), 0],$$

and hence $d_B[g, D \setminus \overline{B}(\delta), 0] = k - 1$. Using the recurrence assumption, $g$ has at most $k - 1$ distinct zeros in $D \setminus \overline{B}(\delta)$, which contradicts the assumption that $g$ at has at least $k + 1$ distinct zeros in $D$.

\[\text{Corollary 19.5.1} \text{ If } U \subset \mathbb{C}^m \text{ is open, } f : U \to \mathbb{C}^m \text{ holomorphic, } D \text{ open bounded such that } \overline{D} \subset U \text{ and } f(\partial D), \text{ and } d_B[f, D, y] = 1, \text{ then equation } f(z) = y \text{ has a unique solution in } D.\]

\[\text{Remark 19.5.1} \text{ The converse of Corollary 19.5.1 is not true under the sole hypothesis that the solution is unique. Indeed, for } m = 1, \text{ and } p \geq 2 \text{ an integer, the equation } z^p = 0 \text{ has the unique solution } z = 0 \text{ and, according to example 17.4.1, } d_B[z^p, B(R), 0] = p > 1 \text{ for any } R > 0.\]

Corollary 19.5.1 allows a sharpening of the conclusion in Brouwer’s fixed point theorem for holomorphic mappings.

\[\text{Corollary 19.5.2} \text{ Let } U \subset \mathbb{C}^m \text{ be open, } f : U \to \mathbb{C}^m \text{ holomorphic and assume that there exists an open, bounded, convex set } D \text{ such that } \overline{D} \subset U \text{ and } f(\partial D) \subset \overline{D}. \text{ If } f \text{ has no fixed point on } \partial A \text{ (in particular if } f(\overline{A}) \subset A), \text{ then } f \text{ has a unique fixed point in } A.\]

\[\text{Proof.} \text{ Because the degree is invariant by translation, we can assume, without loss of generality, that } 0 \in D. \text{ Let us introduce the homotopy } H : \overline{D} \times [0,1] \to \mathbb{C}^m \text{ defined by } H(z, \lambda) = z - \lambda f(z). \text{ By assumption, } 0 \not\in H(\partial D, 1), \text{ and, for } z \in \partial D \text{ and } \lambda \in [0,1], \text{ we have, using convexity, } \lambda g(z) \in A, \text{ and hence } H(z, \lambda) \neq 0. \text{ By the homotopy invariance property 3.4.2, we get}

$$d_B[I - f, D, 0] = d_B[H(\cdot, 1), D, 0] = d_B[H(\cdot, 0), D, 0] = d_B[I, D, 0] = 1,$$

and the result follows from Corollary 19.5.1.\]

\[\text{Corollary 19.5.3} \text{ Let } G \subset \mathbb{C}^m \text{ be open and bounded, } f : G \to \mathbb{C}^m \text{ be holomorphic, } y \in \mathbb{C}^m \text{ and } \zeta \in G \text{ such that } f(\zeta) = y. \text{ Let us denote by } C \text{ the connected component of } f^{-1}(y) \text{ containing } \zeta. \text{ Then either } \zeta \text{ is an isolated solution of equation } f(z) = y \text{ (and hence } C = \{y\}), \text{ or } C \text{ meets any neighborhood of } \partial G.\]

\[\text{Proof.} \text{ Assume there exists a neighborhood of } \partial G \text{ which does not meet } C. \text{ Then one can find an open set } V \text{ such that } C \subset V \subset \overline{V} \subset G. \text{ Now, } f^{-1}(y) \cap \partial V \text{ and } C \text{ are two disjoint compact disjoint subsets of } f^{-1}(y) \cap \overline{V}. \text{ If we apply Whyburn’s lemma 7.2.1, we obtain two compact subsets } K \text{ and } K' \text{ of } f^{-1}(y) \cap \overline{V} \text{ such that } K \supset C,\]
Consequently, that 

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functions

Proof.

After possible translations, we can assume, without loss of
generality, that \( \zeta = 0 \) and \( y = 0 \). Define \( f_j : G \to \mathbb{C} \) by \( f_j = 0 \) (\( j = p + 1, \ldots, m \)), and \( \tilde{f} = (f_1, \ldots, f_m) \), so that \( f \) and \( \tilde{f} \) have the same zeros. Arguing by contradiction, assume that \( C \) does not meet some neighborhood of \( \partial D \). It follows from Corollary 19.5.3 that 0 is an isolated zero of \( \tilde{f} \), and hence there is some \( \delta > 0 \) such that 0 \( \tilde{f}(\partial B(\delta)) \).

Consequently, \( d_B[\tilde{f}, B(\delta), 0] \) is defined and equal to \( k > 0 \). Define \( g : G \to \mathbb{C}^m \) by

\[
g_j(z) = \begin{cases} 
    f_j(z) + \varepsilon z_j & \text{if } 1 \leq j \leq p, \\
    0 & \text{if } p + 1 \leq j \leq m.
\end{cases}
\]

For sufficiently small \( \varepsilon > 0 \), it follows from Rouché’s property 3.4.2 that

\[
d_B[g, B(\delta), 0] = d_B[\tilde{f}, B(\delta), 0] = k > 0.
\]

Hence Theorem 19.5.3 implies that 0 is an isolated zero of \( g \). But, on the other hand, the Jacobian of \( (g_1, \ldots, g_p) \) with respect to \( z_1, \ldots, z_m \) computed at 0 is a polynomial in \( \varepsilon \) whose coefficient of the highest power is 1. Hence, for arbitrary small values of \( \varepsilon \), this Jacobian is different from zero. Using the implicit function theorem for complex mappings, this implies, for each of those \( \varepsilon \), the existence of \( p \) functions \( \varphi_j(\tilde{z}) \) (\( 1 \leq j \leq p \)), where \( \tilde{z} = (z_{p+1}, \ldots, z_m) \), which are holomorphic near \( \tilde{z} = 0 \), such that \( \varphi_j(0) = 0 \) (\( j = 1, \ldots, p \)), and

\[
g_j[\varphi_1(\tilde{z}), \ldots, \varphi_p(\tilde{z}), \tilde{z}] = 0 \quad (1 \leq j \leq m)
\]
on a suitable neighborhood of \( \tilde{z} = 0 \). But this contradicts the isolated character of 0 as a zero of \( g \).

\[
\text{Remark 19.5.2} \quad \text{Recall that an analytic manifold can be defined as the set of common zeros of a family of holomorphic functions } \{f_i\}_{i \in I} \text{ on some open set } U \text{ of } \mathbb{C}^n. \text{ So Corollaries 19.5.3 and 19.5.4 provide results on the structure of analytic manifolds when } \#I \leq m. \text{ In particular, for } \#I = m, \text{ a compact analytic manifold only has a finite number of points, a result which is also true if } \#I \geq m, \text{ as shown for example in [157].}
\]
19.5. **HOLOMORPHIC MAPPINGS**

The following improvement of Corollary 19.5.1 is also due to Rabinowitz [326] (see also [328]).

**Theorem 19.5.2** Let \( U \subset \mathbb{C}^m \) be open, \( f: U \to \mathbb{C}^m \) holomorphic, \( D \subset \mathbb{C} \) open, bounded and such that \( \overline{D} \subset U \), and \( y \not\in f(\partial D) \). Then \( d_B[f, D, y] = 1 \) if and only if the following conditions hold:

(i) There exists a unique \( \zeta \in D \) such that \( f(\zeta) = y \).

(ii) \( J_f(\zeta) \neq 0 \).

**Proof.** Sufficiency. If conditions (i) and (ii) holds, it follows from Corollary 5.2.1 that \( d_B[f, D, y] = \text{sign } J_f(\zeta) = 1 \).

Necessity. If \( d_B[f, D, y] = 1 \), Corollary 19.5.1 implies that condition (i) holds. We now prove condition (ii). Without loss of generality, after translations, we may assume that \( \zeta = 0 = y \). Let us assume by contradiction that \( J_f(0) = 0 \). Then 0 is an eigenvalue of \( f'(0) \), and hence its Jordan canonical form contains 0 in its main diagonal, say \( \partial z_m f_m(0) = 0 \). Without loss of generality, we may assume that 0 is a simple eigenvalue of \( f'(0) \) and that \( 2a := \partial^2 z_m f_m(0) \neq 0 \). Indeed, if it were not the case, we could perturb \( f \) in the following way by letting

\[
(f_\varepsilon)_i = \begin{cases} f_i(z) + \varepsilon z_i & \text{if } 1 \leq i \leq m - 1, \\ f_m(z) + \varepsilon z_m^2 & \text{if } i = m, \end{cases}
\]

and choosing \( \varepsilon > 0 \) sufficiently small so that \( d_B[f_\varepsilon, D, 0] = d_B[f, D, 0] = +1 \) (so that 0 remains the unique solution of \( f_\varepsilon(z) = 0 \) contained in \( D \)). We notice that, by construction,

\[
\partial_{z_i}(f_\varepsilon)_i(0) \neq 0 \quad (i = 1, \ldots, m - 1), \quad \partial_{z_m}(f_\varepsilon)_m(0) = 0, \quad \partial^2_{z_m}(f_\varepsilon)_m(0) \neq 0.
\]

Thus \( f'(0) \) has the form

\[
\begin{pmatrix}
L & 0 \\
0 & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & \cdots & 0
\end{pmatrix}
\]

with \( L \) a non-singular Jordan matrix of dimension \((m - 1) \times (m - 1)\). Set \( \hat{f}(z) = (f_1(z), \ldots, f_{m-1}(z)) \), so that \( f = (\hat{f}, f_m) \), and \( \hat{z} = (z_1, \ldots, z_{m-1}) \), so that \( z = (\hat{z}, z_m) \). Because of the holomorphy of \( f \) near 0, we can write for all \( z \in \overline{B(\rho_1)} \), \( (\sigma \text{ a multi-index}), \)

\[
f_j(z) = \sum_{|\sigma| \geq 1} f_j \sigma z^\sigma \quad (1 \leq j \leq m).
\]

Hence, we have

\[
\hat{f}(z) = L\hat{z} + g(z),
\]
for some mapping $g$ such that $\|g(z)\| \leq M_1 \|z\|^2$ for all $z \in \overline{B}(\rho_1)$, and

$$|f_m(z) - \sum_{|\sigma| = 2} f_m \sigma z^\sigma| \leq M_2 \|z\|^3,$$

where $M_1$ and $M_2$ are positive constants. If we define the homotopy $F(\cdot, \lambda)$ by

$$\hat{F}(z, \lambda) = L\hat{z} + \lambda g(z), \quad F_m(z, \lambda) = (1 - \lambda)a z_m^2 + \lambda \sum_{|\sigma| \geq 2} f_m \sigma,$$

we have

$$|F_m(z, \lambda)| \geq |a| |z_m|^2 - \left| \sum_{|\sigma| = 2} f_m \sigma z^\sigma - az_m^2 \right| - M_2 \|z\|^3,$$

Now,

$$\left| \sum_{|\sigma| = 2} f_m \sigma z^\sigma - az_m^2 \right| \leq M_3 \|\hat{z}\| |z_m| + M_4 \|\hat{z}\|^2,$$

where $M_3$ and $M_4$ are positive constants. Let $\alpha > 0$ be sufficiently small so that $M_3\alpha + M_4\alpha^2 \leq \frac{|a|}{4}$, and define the subsets of $C^m$

$$P_\alpha := \{ z \in C^m : \|\hat{z}\| \leq \alpha |z_m| \}, \quad Q_\alpha := \{ z \in C^m : \|\hat{z}\| \geq \alpha |z_m| \},$$

so that $P_\alpha \cup Q_\alpha = C^m$. If $z \in P_\alpha \cap \overline{B}(\rho_1)$, we have

$$\left| \sum_{|\sigma| = 2} f_m \sigma z^\sigma - az_m^2 \right| \leq \frac{|a|}{4} |z_m|^2.$$

If, furthermore, $z \in \overline{B}(\rho_2)$, with $\rho_2 = \frac{|a|}{4M_2(1 + \alpha^2)}$, we have

$$M_2 \|z\|^3 \leq \frac{|a|}{4} |z_m|^2,$$

and hence, for $z \in P_\alpha \cap \overline{B}(\min\{\rho_1, \rho_2\})$, we have

$$|F_m(z, \lambda)| \geq \frac{|a|}{2} |z_m|^2 \geq \frac{|a|}{2(1 + \alpha^2)} \|z\|^2. \quad (19.20)$$

We now estimate $\hat{F}(z, \lambda)$. For $z \in \overline{B}(\rho_1)$, we have

$$\|\hat{F}(z, \lambda)\| \geq \|L\hat{z}\| - \|g(z)\| \geq M_5 \|\hat{z}\| - M_1 \|z\|^2,$$
where $M_3$ is a positive constant. If $z \in Q_\alpha \cap \overline{B}(\rho_3)$, where $\rho_3 = \frac{M_3}{2M_1 \sqrt{1 + \frac{1}{2\alpha^2}}}$, we obtain
\[
\|\hat{F}(z, \lambda)\| \geq \frac{M_3}{2}\|\hat{z}\| \geq \frac{M_3}{2\sqrt{1 + \frac{1}{2\alpha^2}}}\|z\|.
\] (19.21)

Define $\rho := \min\{\rho_1, \rho_2, \rho_3\}$. Because of $P_0 \cup Q_\alpha = \mathbb{C}^m$, the estimates (19.20) and (19.21) imply that $0 \notin F(\partial B(\rho) \times [0, 1])$, and using successively the excision property 3.4.1, the homotopy invariance property 3.4.2, and property 3.4.6, we obtain
\[
1 = d_B[f, D, 0] = d_B[f, B(\rho), 0] = d_B[F(\cdot, 1), B(\rho), 0] = d_B[F(\cdot, 0), B(\rho), 0]
\] (19.22)
when $|\theta|$ is sufficiently small. Taking $\theta = (0, \ldots, 0, \beta)$, with $\beta \neq 0$ sufficiently small, we see that the equation $F(z, 0) = \theta$, i.e. $Lz = 0$, $az_\beta^2 = \beta$, has two distinct solutions $\zeta_\pm = (0, \ldots, 0, \pm \sqrt{\beta})$, with $J_{F(\cdot, 0)}(\zeta_\pm) = \pm (\det L) \cdot (2a\sqrt{\beta}) \neq 0$. Hence, $d_B[F(\cdot, 0), B(\rho), 0] = +2$, a contradiction with (19.22).

A consequence of Theorem 19.5.2 is an injectivity condition first proved in 1913 by G.R. Clements [70] in a different way.

**Corollary 19.5.5** Let $U \subset \mathbb{C}^m$ be open, $f : U \rightarrow \mathbb{C}^m$ holomorphic. A necessary and sufficient condition for $f$ to be one-to-one in a neighborhood of $\zeta \in U$ is that $J_f(\zeta) \neq 0$.

**Proof. Sufficiency.** Follows directly from the local inversion theorem.

**Necessity.** Let $f$ be one-to-one on a neighborhood of $\zeta$, and let $\overline{B}_\zeta(\rho)$ be contained in this neighborhood. Hence $f(\zeta) \notin f(\partial B_\zeta(\delta))$ and $d_B[f, B_\zeta(\delta), f(\zeta)]$ is defined and strictly positive by Theorem 19.3.1. From Sard's theorem, there exists $y$ arbitrary closed to $f(\zeta)$ such that $d_B[f, B_\zeta(\delta), y] = d_B[f, B_\zeta(\delta), f(\zeta)]$ and $J_f(\zeta') \neq 0$, where $\{\zeta'\} = f^{-1}(y) \cap B_\zeta(\delta)$. This last set is non-empty as $d_B[f, B_\zeta(\delta), y] > 0$ and is a singleton because of the injectivity assumption. Hence $d_B[f, B_\zeta(\delta), y] = 1$, so that $d_B[f, B_\zeta(\delta), f(\zeta)] = 1$, and hence, using Theorem 19.5.2, $J_f(\zeta) \neq 0$.

**Remark 19.5.3** The conclusion of Theorem 19.5.2 is obviously false in the real case. For example, with the real function $f(x) = x^3$, one has $d_B[f, \cdot] = 1$, $d_B[f', 0] = 0$.

**Remark 19.5.4** Theorem 19.5.2 implies that the Brouwer index of an isolated solution $\zeta$ of equation $f(z) = y$ is greater or equal to two if $J_f(\zeta) = 0$. This is a special case of Cronin’s proposition 19.2.4 when $0$ is an isolated zero of $f$. Notice however that the proof of Theorem 19.5.2 avoids the use of the resultant of polynomials.
19.6 Quaternionic monomials

Another class of orientation-preserving mappings is that of **quaternionic monomials**. We refer to [98] for the required definitions and properties of quaternions. For \( q = q_0 + q_1i + q_2j + q_3k \in \mathbb{H} \), we write

\[
\tilde{q} = \Im q = q_1i + q_2j + q_3k, \quad \overline{q} = q_0 - \Im q, \quad \|q\| = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}.
\]

Let \( J_1 \cup J_2 \) be a partition of \( \{1, 2, \ldots, n\} \) (one of the \( J_i \) can be empty) and, for each \( 1 \leq j \leq n \), let \( c_j : \mathbb{H} \to \mathbb{H} \) be defined by

\[
c_j(q) = q \text{ if } j \in J_1, \quad c_j(q) = \overline{q} \text{ if } j \in J_2; \tag{19.23}
\]

and, given \( a_j \in \mathbb{H} \setminus \{0\}, (0 \leq j \leq n) \), let \( f : \mathbb{H} \to \mathbb{H} \) be the quaternionic monomial defined by

\[
f(q) = a_0c_1(q)a_1c_2(q)a_2 \ldots c_n(q)a_n. \tag{19.24}
\]

If we identify \( \mathbb{H} \) with \( \mathbb{R}^4 \), \( f : \mathbb{R}^4 \to \mathbb{R}^4 \) is an homogeneous mapping vanishing only at 0, and hence \( d_B[f, B(R), 0] \) is well defined and independent of \( R > 0 \). The following lemma [274] is useful for its computation.

**Lemma 19.6.1** Let \( n \geq 1 \) be an integer and \( p_n : \mathbb{H} \to \mathbb{H}, q \mapsto q^n \) considered as a mapping from \( \mathbb{R}^4 \) into \( \mathbb{R}^4 \). Then

\[
J_{p_n}(q) = \begin{cases} 
4^n\|q\|^4(n-1) & \text{if } \Im q = 0 \\
2^n\|q\|^{2(n-1)} \left( \frac{\Im(q^n)}{\Im q} \right)^2 & \text{if } \Im q \neq 0.
\end{cases} \tag{19.25}
\]

**Proof.** An elementary induction shows that the value at \( h \in \mathbb{H} \) of the total derivative \( Dp_n(q) \) of \( p_n \) at \( q \in \mathbb{H} \) is given by

\[
Dp_n(q)h = \sum_{m=0}^{n-1} q^m h q^{n-1-m}. \tag{19.26}
\]

We prove the result by choosing an appropriate basis for the quaternions. If \( q \) is real (i.e. \( \tilde{q} = 0 \)), then by commutativity of \( q \) and \( h \), we get \( Dp_n(q)h = nq^{n-1}h \) and hence

\[
J_{p_n}(q) = n^4\|q\|^4(n-1).
\]

If \( q \) is not real, \( q \) generates a 2-dimensional subfield of \( \mathbb{H} \) which can be identified to \( \mathbb{C} \), namely \( \mathbb{C} = \{a + bI : a, b \in \mathbb{R}\} \), where \( I = \tilde{q}/\|\tilde{q}\| \). From (19.26), it is immediate to check that \( \mathbb{C} \) is invariant under the action of \( Dp_n(q) \). The same is true for a fixed complementary subspace \( D \) of \( \mathbb{H} \) spanned by quaternions \( I, J, K \) such that \( I, J, K \) verify the usual relations. By analogy with the discussion of Section 6, one has \( Dp_n(q)c = nq^{n-1}c \) for \( c \in \mathbb{C} \), and the Jacobian of the restriction of \( Dp_n(q) \) on \( \mathbb{C} \)
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is given by $n^2\|q\|^{2(n-1)}$. On the other hand, for $c \in \mathbb{C}$ and $d \in \mathbb{D}$, we have $dc = \overline{c}d$, where $\overline{c}$ is the complex conjugate of $c$ in $\mathbb{C}$, and hence

$$Dp_n(q)d = \left(\sum_{m=0}^{n-1} q^{n-m-1}q^{-m}\right) d = \frac{q^n - \overline{q}^n}{q - \overline{q}} d = \frac{\Im(q^n)}{3q} d.$$ 

So the Jacobian of the restriction of $Dp_n(q)$ on $\mathbb{D}$ is $\left[\frac{\Im(q^n)}{3q}\right]^2$. Consequently, for $q$ not real one has

$$J_{p_n}(q) = n^2\|q\|^{2(n-1)} \left[\frac{\Im(q^n)}{3q}\right]^2.$$ 

\[\]

**Corollary 19.6.1** With the notations of Lemma 19.6.1, $J_{p_n}(q) \geq 0$ for all $q \in \mathbb{H}$, and $J_{p_n}(q) > 0$ outside of the set $\mathcal{N}$ of $n$-measure zero defined by

$$\{q \in \mathbb{H} : \left(\sin^2 \frac{k\pi}{n}\right) q_0^2 = \left(\cos^2 \frac{k\pi}{n}\right) (q_1^2 + q_2^2 + q_3^2) \quad (1 \leq k \leq n-1)\}. \quad (19.27)$$

In particular, $p_n$ is a strictly orientation-preserving mapping.

**Proof.** If $\Im q = 0$, it follows from (19.25) that $J_{p_n}(q) > 0$ for $q \neq 0$. If $\Im q \neq 0$, one can write $q = \|q\| (\cos \theta + i \sin \theta)$ with

$$I = \frac{q}{\|q\|}, \quad \cos \theta = \frac{q_0}{\|q\|}, \quad \sin \theta = \frac{\|q\|}{\|q\|},$$

so that $q^n = \|q\|^n (\cos n\theta + i \sin n\theta)$. By (19.25), $J_{p_n}(q) = 0$ if and only if $\Im q^n = 0$, i.e. if and only if $\theta = \frac{k\pi}{n} \quad (1 \leq k \leq 2n-1)$. The set of those zeros is contained in the $n - 1$ cones

$$\{q \in \mathbb{H} : \left(\sin^2 \frac{k\pi}{n}\right) q_0^2 = \left(\cos^2 \frac{k\pi}{n}\right) (q_1^2 + q_2^2 + q_3^2) \quad (1 \leq k \leq n-1)\}.$$ 

So $\mathcal{N}$ has $n$-measure zero.

We now compute $d_B[p_m, B(R), 0]$ and $d_B[\overline{p_m}, B(R), 0]$, where $m \geq 1$ is an integer.

**Lemma 19.6.2** For each $R > 0$, one has

$$d_B[p_m, B(R), 0] = m. \quad (19.28)$$

**Proof.** By homotopy invariance, we have, for all $\varepsilon > 0$ sufficiently small

$$d_B[p_m, B(R), 0] = d_B[p_m, B(R), \varepsilon i].$$
Now, the equation \( q^m = \varepsilon^i \) has \( m \) roots, which are the \( m \) complex roots of \( i \), namely
\[
q_j = \varepsilon^{1/m} \left[ \cos \frac{(4k+1)\pi}{2m} + i \sin \frac{(4k+1)\pi}{2m} \right], \quad (0 \leq k \leq m-1).
\] (19.29)

None of those \( q_j \) is real, and hence, using formula 19.25, we obtain
\[
J_{p_m}(q_k) = m^2 \varepsilon^2 \sin^2 \frac{(4k+1)\pi}{2m} > 0,
\]
so that, using formula 3.4, we get formula (19.28).

\[ \text{Lemma 19.6.3} \]
For each \( r > 0 \), one has
\[
d_B[p_m, B(R), 0] = -m.
\] (19.30)

\[ \text{Proof.} \]
By homotopy invariance, we have, for all \( \varepsilon > 0 \) sufficiently small
\[
d_B[p_m, B(R), 0] = d_B[p_m, B(R), -\varepsilon i].
\]

Now,
\[
\overline{q^m} = -\varepsilon^i \leftrightarrow q^m = \varepsilon^i \leftrightarrow q = q_j \quad (0 \leq k \leq m-1),
\]
where \( q_j \) is given in formula 19.29. Furthermore, if we define the linear mapping
\( C : H \to \mathbb{H}, q \mapsto \overline{q} \),
then \( \det C = -1 \), and \( p_m = p_m \circ C \), so that
\[
J_{p_m}(q) = J_{p_m}(\overline{q}) \cdot \det C = -J_{p_m}(\overline{q}),
\]
and formula (19.30) follows from Corollary 3.1.1 and Lemma 19.6.2.

\[ \text{Theorem 19.6.1} \]
If the \( c_j \) are defined by formula (19.23) \( (1 \leq j \leq n) \), \( a_j \in \mathbb{H} \setminus \{0\} \), and \( f \) is defined by formula (19.24), then, for each \( R > 0 \), one has
\[
d_B[f, B(R), 0] = \# J_1 - \# J_2.
\] (19.31)

\[ \text{Proof.} \]
Let \( A_j : [0, 1] \to \mathbb{H} \setminus \{0\} \) be continuous and such that
\[
A_j(0) = 1, \quad A_j(1) = a_j, \quad (0 \leq j \leq n),
\]
and let \( g : H \to \mathbb{H} \) be defined by
\[
g(q) = c_1(q)c_2(q) \ldots c_n(q).
\]
By the homotopy invariance of degree, we have
\[
d_B[f, B(R), 0] = d_B[g, B(R), 0].
\]
Notice now that, as \( q\overline{q} = \overline{q}q \), we have \( c_j(q)c_k(q) = c_k(q)c_j(q) \) for all \( 1 \leq j, k \leq n \). Hence, of we write \( m_1 = \# J_1, \ m_2 = \# J_2 \),
\[
g(q) = \begin{cases} 
q^{m_1-m_2} |q|^{2m_2} & \text{if } m_1 > m_2 \\
|q|^{m_2} & \text{if } m_1 = m_2 \\
q^{m_2-m_1} |q|^{2m_1} & \text{if } m_1 < m_2.
\end{cases}
\]
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Noticing that, on $\partial B(R)$, one has

$$g(q) = \begin{cases} 
q^{m_1-m_2}R^{2m_2} & \text{if } m_1 > m_2 \\
R^{m_2} & \text{if } m_1 = m_2 \\
q^{m_2-m_1}R^{2m_1} & \text{if } m_1 < m_2,
\end{cases}$$

and using the boundary dependence property and Corollary 3.1.1, we obtain

$$d_B[g,B(R),0] = \begin{cases} 
d_B[p_{m_1-m_2},B(R),0] & \text{if } m_1 > m_2 \\
0 & \text{if } m_1 = m_2 \\
d_B[p_{m_2-m_1},B(R),0] & \text{if } m_1 < m_2,
\end{cases}$$

and the proof is complete.}

As an application, we prove a slight extension of Eilenberg-Niven’s fundamental theorem of algebra for quaternions [101] (see also [274] for a more elementary proof based upon differential forms). Let $n \geq 1$ be an integer and $g : \mathbb{H} \to \mathbb{H}$ a continuous function such that

$$\lim_{\|q\| \to \infty} \frac{\|g(q)\|}{\|q\|^n} = 0. \quad (19.32)$$

**Theorem 19.6.2** With the notations of (19.27), if $a_0, a_1, \ldots, a_n$ are non-zero quaternions, if $\#J_1 \neq J_2$, and if condition (19.32) holds, then equation

$$p(q) := a_0c_1(q)a_1c_2(q)a_2 \ldots c_n(q)a_n + g(q) = 0 \quad (19.33)$$

has at least one solution in $\mathbb{H}$.

**Proof.** By assumption,

$$\beta := \|a_0a_1a_2 \ldots a_n\| = \|a_0\|\|a_1\| \ldots \|a_n\| > 0,$$

and, by condition (19.32), there exists $R > 0$ such that

$$\|g(q)\| \leq \beta \frac{\|q\|^n}{2}. \quad (19.34)$$

whenever $\|q\| \geq R$. We prove that equation (19.33) has at least one solution in $B(R) \subset \mathbb{H}$ by showing that $d_B[p,B(R),0] \neq 0$. Let us define the homotopy $F : B(R) \times [0,1] \to \mathbb{H}$ by

$$F(q,\lambda) = a_0c_1(q)a_1c_2(q)a_2 \ldots c_n(q)a_n + \lambda g(q).$$

For $(q, \lambda) \in \partial B(R) \times [0,1]$, we have using (19.34),

$$\|a_0c_1(q)a_1c_2(q)a_2 \ldots c_n(q)a_n + \lambda g(q)\| \geq \beta \|q\|^n - \|g(q)\| \geq \beta \frac{R^n}{2} > 0.$$

Hence, using the homotopy invariance property 3.4.2 and Theorem 19.6.1, we get

$$d_B[p,B(R),0] = d_B[F(\cdot,1),B(R),0] = d_B[F(\cdot,0),B(R),0] = d_B[f,B(R),0] = \#J_1 - \#J_2 \neq 0,$$

and the result follows from the existence property 3.4.1.
Corollary 19.6.2 If $a_0, a_1, \ldots, a_n$ are non-zero quaternions and if (19.32) holds, then equation

$$f(q) := a_0 a_1 q a_2 \ldots q a_n + g(q) = 0$$

has at least one solution in $\mathbb{H}$. This is in particular the case if $g(q)$ is a polynomial in $q$ of degree lower or equal to $n$. 
Chapter 20

Degree of some symmetric mappings

20.1 Odd mappings

The simplest non-trivial symmetry in $\mathbb{R}^n$ is associated to the group $\mathbb{Z}_2$.

**Definition 20.1.1** $E \subset \mathbb{R}^n$ is said to be **symmetric** with respect to the origin if $E = -E$, and $f : E \to \mathbb{R}^n$ is said to be **odd** if $f(-x) = -f(x)$ for all $x \in E$.

The following result is due to Karel Borsuk [36], and the given proof is due to Gromes [155].

**Theorem 20.1.1** Let $D \subset \mathbb{R}^n$ be an open bounded symmetric neighborhood of 0. If $f : D \to \mathbb{R}^n$ is continuous, odd, and such that $0 \not\in f(\partial D)$, then $d_B[f, D, 0]$ is odd.

**Proof.** We first show that we may assume that $f \in C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n)$ and $J_f(0) \neq 0$. For this, observe that if $g \in C(D) \cap C^1(D)$ is a $C^1$ approximation of $f$ (to be fixed later), and $h(x) = (1/2)[g(x) - g(-x)]$ is its odd part, let us choose $\delta$ which is not an eigenvalue of $h'_0$. Then $\tilde{f} := h - \delta I \in C^1(D, \mathbb{R}^n) \cap C(D, \mathbb{R}^n)$, is odd, and such that $\tilde{f}'_0 = h'_0 - \delta I$ is invertible, so that $J_{\tilde{f}}(0) \neq 0$. Now, if $\mu := \min_{\partial D} \|f\|$, and $g \in C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n)$ chosen in such a way that $\max_{\partial D} \|\tilde{f} - g\| < \mu/2$, we have, for all $x \in D$,

$$\|f(x) - h(x)\| = (1/2)\|f(x) - g(x) - [f(-x) - g(-x)]\| < (1/2)(\mu + \mu) = \mu,$$

and, by Rouché’s property 3.4.2, $d_B[f, D, 0] = d_B[h, D, 0]$.

Let now $f \in C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n)$ be odd and such that $J_f(0) \neq 0$. If we show the existence of an odd mapping $g \in C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n)$ sufficiently close to $f$ and such that 0 is a regular value for $g$, then Rouché’s property 3.4.2 and Corollary 5.2.1 imply that

$$d_B[f, D, 0] = d_B[g, D, 0] = \text{sign } J_f(0) + \sum_{0 \neq x \in g^{-1}(0)} \text{sign } J_g(x). \quad (20.1)$$
Now \( \{ x \in D \setminus \{0\} : g(x) = 0 \} \) is symmetric with respect to the origin, and \( J_g(-x) = J_g(x) \), which shows that the right-hand member in (20.1) is an odd integer.

We construct such a mapping \( g \) by induction. For each integer \( 1 \leq k \leq n \), we define
\[
D_k := \{ x \in D : x_i \neq 0 \text{ for some } 1 \leq i \leq k \},
\]
and we introduce an odd mapping \( \varphi \in C^1(\mathbb{R}) \) such that \( \varphi'(0) = 0 \) and \( \varphi(t) = 0 \) if and only if \( t = 0 \). We first consider the mapping \( \overline{f}(x) = f(x)/\varphi(x_1) \) on the open bounded set \( D_1 := \{ x \in D : x_1 \neq 0 \} \). Notice that
\[
\overline{f}_x = [\varphi(x_1)]^{-2}[\varphi(x_1)f'_x - \varphi'(x_1)f(x)p_1]
\]
with \( p_1(h) = h_1 \). By Sard’s theorem, we can find a regular value \( y^1 \) for \( \overline{f} \) on \( D_1 \) with \( \| y^1 \| \) as small as we want. In this way, if \( g_1(x) := f(x) - \varphi(x_1)y^1 \) on \( D_1 \), and \( g_1(x) = 0 \) for some \( x \in D_1 \), we obtain, using formula (20.2), that
\[
\overline{f}_x = [\varphi(x_1)]^{-1}[f'_x - y^1p_1],
\]
i.e. that \( (g_1)_x' = \varphi(x_1)f'_x \), so that \( 0 \) is a regular value for \( g_1 \) on \( D_1 \). Suppose now that we have found an odd \( g_k \in C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n) \), close to \( f \) on \( \overline{D} \), and such that \( 0 \) is a regular value of \( g_k \) on \( D_k \) for some integer \( 1 \leq k < n \). Then we define \( g_{k+1} = g_k(x) - \varphi(x_{k+1})y^{k+1}_x \) with \( \| y^{k+1} \| \) small and such that \( 0 \) is a regular value for \( g_{k+1} \) on \( \{ x \in D : x_{k+1} \neq 0 \} \). Of course, \( g_{k+1} \in C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n) \) is odd and close to \( f \) on \( \overline{D} \). If \( x \in D_{k+1} \) and \( x_{k+1} = 0 \), then \( x \in D_k \), \( g_{k+1}(x) = g_k(x) \) and \( (g_{k+1})'_x = (g_k)'_x \), so that \( J_{g_{k+1}}(x) \neq 0 \) and \( 0 \) is a regular value for \( g_{k+1} \) on \( D_{k+1} \). So, finally, \( g := g_n \) is odd, close to \( f \) in \( \overline{D} \), and such that \( 0 \) is a regular value of \( g \) on \( D \setminus \{0\} \), since \( D_n = D \setminus \{0\} \). By the induction step, we also have \( g_0'(1) = (g_1)'_0 = \ldots = (g_n)'_0 \), and hence \( 0 \) is a regular value for \( g \) in \( D \).

We immediately deduce an easy generalization.

**Corollary 20.1.1** Let \( D \subset \mathbb{R}^n \) be an open bounded symmetric neighborhood of \( 0 \), and \( f \in C(\overline{D}, \mathbb{R}) \) be such that \( 0 \notin f(\partial D) \) and
\[
f(-x) = \mu f(x) \quad \text{for all } \mu \geq 1 \text{ and } x \in \partial D.
\]
Then \( d_B[f, D, 0] \) is odd, and \( f(D) \) contains a neighborhood of \( 0 \).

**Proof.** Let us define the homotopy \( F \) by
\[
F(x, \lambda) = f(x) - \lambda f(-x), \quad \lambda \in [0, 1],
\]
so that \( F(\cdot, 1) \) is odd, \( F(\cdot, 0) = f \), and, by (20.3) with \( \mu = 1/\lambda \) and condition \( 0 \notin f(\partial D) \), we have \( 0 \notin F(\partial D \times [0, 1]) \), and hence
\[
d_B[f, D, 0] = d_B[F(\cdot, 1), D, 0] = 1 \mod 2.
\]
The last part of the statement follows from Corollary 3.4.7. 

\[\square\]
20.1. ODD MAPPINGS

A special case of Corollary 20.1.1 for mapping having different directions at antipodal points goes as follows.

**Corollary 20.1.2** Let \( D \subset \mathbb{R}^n \) be an open bounded symmetric neighborhood of 0, and \( f \in C(\overline{D}, \mathbb{R}^n) \) be such that \( 0 \notin f(\partial D) \) and

\[
\frac{f(-x)}{||f(-x)||} \neq \frac{f(x)}{||f(x)||} \quad \text{for all } x \in \partial D.
\] (20.4)

Then \( d_B[f, D, 0] \) is odd, and \( f(D) \) contains a neighborhood of 0.

**Proof.** If one had \( f(-x) = \mu f(x) \), for some \( \mu \geq 1 \) and \( x \in \partial D \), then \( ||f(-x)|| = \mu ||f(x)|| \) and hence \( \frac{f(-x)}{f(x)} = \frac{f(x)}{f(x)} \), which is a contradiction to (20.4). Thus the result follows from Corollary 20.1.1.

A consequence of Corollary 20.1.2 is the simultaneous existence of a zero and of a fixed point for mappings odd on \( \partial D \).

**Corollary 20.1.3** Let \( D \subset \mathbb{R}^n \) be an open bounded symmetric neighborhood of 0 and \( f \in C(\overline{D}, \mathbb{R}^n) \) be odd on \( \partial D \). Then there exists \( x^* \in \overline{D} \) such that \( f(x^*) = 0 \), and \( \tilde{x} \in D \) such that \( \tilde{x} = f(\tilde{x}) \).

**Proof.** If \( 0 \in f(\partial D) \), the first part of the result is proved. If not, it follows from Corollary 20.1.2 that \( d_B[f, D, 0] = 1 \ (\text{mod } 2) \) and, by the existence property 3.4.1, \( f \) has a zero in \( D \). Now, the mapping \( g \in C(\overline{D}, \mathbb{R}^n) \) defined by \( g(x) = x - f(x) \) has the same properties than \( f \) and the first part of the proof applies.

**Remark 20.1.1** Corollary 20.1.3 implies the no-retraction of a closed ball onto its boundary, because if \( R : \overline{B}(R) \to \partial B(R) \) is such a retraction, \( -R \) would have a fixed point \( x^* \) by Corollary 20.1.3, which should be on \( \partial B(R) \) and such that \( x^* = -x^* \), which is contradictory.

We end this section by some other non-existence properties.

**Definition 20.1.2** If \( X, Y \) are Hausdorff topological spaces, \( h \in C(X, Y) \) is called nullhomotopic if there exists a homotopy \( H \in C(X \times [0, 1], Y) \) such that \( H(\cdot, 0) = h \) and \( H(\cdot, 1) \) is constant.

The following characterization of nullhomotopic mapping on a sphere is of interest.

**Proposition 20.1.1** Let \( Y \) be a Hausdorff topological space, Then \( h \in C(S^n, Y) \) is nullhomotopic if and only if \( h \) has a continuous extension \( \tilde{h} \in C(\overline{B}(1), Y), \) where \( \overline{B}(1) \subset \mathbb{R}^{n+1} \).

**Proof.** If \( h \) has a continuous extension \( \tilde{h} \) to \( \overline{B}(1) \), the homotopy \( H \in C(S^n \times [0, 1], Y) \) defined by \( H(x, t) = \tilde{h}(tx) \) shows that \( h \) is nullhomotopic. If \( h \) is nullhomotopic and
CHAPTER 20. DEGREE OF SOME SYMMETRIC MAPPINGS

$H \in C(S^n \times [0, 1], Y)$ is a corresponding homotopy, one can obtain an extension $\tilde{h} \in C(\mathcal{B}(1), Y)$ by

$$
\tilde{h}(x) = \begin{cases} 
H(S^n, 0) & \text{if } 0 \leq \|x\| \leq 1/2, \\
H(x/\|x\|, 2\|x\| - 1) & \text{if } 1/2 \leq \|x\| \leq 1.
\end{cases}
$$

**Corollary 20.1.4** Let $D \subset \mathbb{R}^n$ be an open bounded symmetric neighborhood of 0 and $g \in C(\partial D, \mathbb{R}^n \setminus \{0\})$ be odd on $\partial D$. Then $g$ is not null homotopic.

**Proof.** Let $\tilde{g} \in C(\mathcal{D}, \mathbb{R}^n)$ be a continuous extension of $g$. Then, by Corollary 20.1.2, $d_B[\tilde{g}, D, 0] = 1 \pmod{2}$. If $g \in C(\mathcal{D}, \mathbb{R}^n)$ is nullhomotopic, $G$ is the corresponding homotopy, $\tilde{G} \in C(\mathcal{D} \times [0, 1], \mathbb{R}^n)$ is a continuous extension of $G$, and $g_0$ the constant value of $\tilde{G}(\cdot, 1)$ on $\partial D$, then, using homotopy invariance property 3.4.2,

$$
1 \pmod{2} = d_B[\tilde{g}, D, 0] = d_B[\tilde{G}(\cdot, 0), D, 0] = d_B[\tilde{G}(\cdot, 1), D, 0] = d_B[g_0, D, 0] = 0,
$$

a contradiction. ■

In particular, a sphere cannot be continuously deformed into a point.

**Corollary 20.1.5** There exists no homotopy $H \in C(S^n \times [0, 1], S^n)$ such that $H(\cdot, 0) = I$ and $H(\cdot, 1)$ is constant on $S^n$.

**Proof.** Apply Corollary 20.1.4 to $D = B(1) \subset \mathbb{R}^{n+1}$ and $g = I$. ■

### 20.2 Borsuk-Ulam’s theorem

The following lemma is a consequence of Corollary 20.1.2 with $n$ replaced by $n + 1$.

**Lemma 20.2.1** If $D \subset \mathbb{R}^{n+1}$ is an open, bounded symmetric neighborhood of 0, $g \in C(\partial D, \mathbb{R}^{n+1})$ is odd and $g(\partial D) \subset \mathbb{R}^n$, then there exists some $x^* \in \partial D$ such that $g(x^*) = 0$.

**Proof.** If $0 \not\in g(\partial D)$ and $\tilde{g} \in C(\mathcal{D}, \mathbb{R}^n)$ is an extension of $g$, then $\tilde{g}(-x) = -\tilde{g}(x)$ for all $x \in \partial D$, and Corollary 20.1.2 implies that $\tilde{g}(D)$ contains a neighborhood of 0 in $\mathbb{R}^{n+1}$, a contradiction with $\tilde{g}(D) \subset \mathbb{R}^n$. ■

The following non-existence result is of interest.

**Corollary 20.2.1** If $n > m$, and there exists no odd mapping $g \in C(S^n, S^m)$.

**Proof.** As $g(S^n) \subset S^m \subset \mathbb{R}^{m+1}$ and $m + 1 < n + 1$, Lemma 20.2.1 implies that $g(x^*) = 0$ for some $x^* \in S^n$, a contradiction with $g(S^n) \subset S^m$. ■
20.3. ELLIPTIC DIFFERENTIAL OPERATORS

The following consequence of Lemma 20.2.1, called Borsuk-Ulam’s theorem, was conjectured by S. Ulam and proved by Borsuk [36].

**Theorem 20.2.1** If $D \subset \mathbb{R}^{n+1}$ is an open, bounded symmetric neighborhood of 0, and $f \in C(\partial D, \mathbb{R}^{n+1})$ is such that $f(\partial D) \subset \mathbb{R}^n$, there exists $x^* \in \partial D$ such that $f(x^*) = f(-x^*)$.

**Proof.** Apply Lemma 20.2.1 to the continuous odd mapping $g$ defined on $\partial D$ by $f(x) = g(x) - g(-x)$.

In particular, for the Earth identified with $S^2$, assuming that temperature and pressure continuously depend upon the position, there always exists two antipodal points having the same temperature and pressure.

20.3 Elliptic differential operators

Let us deduce here an interesting consequence of Borsuk-Ulam’s theorem 20.2.1 for elliptic differential operators with constant coefficients. Let $n, m \geq 1$ be integers and

$$P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$$

be a differential operator with constant (real or complex) coefficients $c_\alpha$. Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_k \in \mathbb{N}$ ($k = 1, 2, \ldots, n$), $|\alpha| = \sum_{k=1}^n \alpha_k$, and $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$. The symbol of $P$ is the polynomial (also denoted by $P$)

$$P(\xi) := \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha,$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$. We say that $P$ is of order $m$ if there exists $\alpha$ such that $|\alpha| = m$ and $c_\alpha \neq 0$. If $P$ is of order $m$, the polynomial $P_m(\xi) := \sum_{|\alpha|=m} c_\alpha \xi^\alpha$ is called the principal symbol of $P$. $P_m$ is a homogeneous polynomial of degree $m$, namely

$$P_m(t\xi) = t^m P_m(\xi) \text{ for all } \xi \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

The differential operator $P(\partial)$ is said to be elliptic if $P_m(\xi) \neq 0$ whenever $\xi \in \mathbb{R}^n \setminus \{0\}$.

**Proposition 20.3.1** Let $n, m \geq 1$ be integers. Then

1. If $P(\partial)$ is elliptic with real coefficients and $n \geq 2$, then $m$ is even.
2. If $P(\partial)$ is elliptic and $n \geq 3$, then $m$ is even.

**Proof.** 1) Let $f : \partial B(1) \subset \mathbb{R}^n \to \mathbb{R}^{n-1}$ be defined by

$$f(\xi) = (P_m(\xi), 0, \ldots, 0)$$
where \( P_m(\xi) \) is followed by \( n - 2 \) zeros. Clearly \( f \) is continuous and Borsuk-Ulam’s theorem 20.2.1 implies the existence of \( \xi^* \in \partial B(1) \) such that \( f(\xi^*) = f(-\xi^*) \), and hence such that

\[
P_m(\xi^*) = P_m(-\xi^*) = (-1)^m P_m(\xi^*).
\]

Because \( P_m(\xi^*) \neq 0, m \) must be even.

2) Let \( Q_m = \mathbb{RP}_{m}, R_m = \exists P_{m} \), so that, for all \( \xi \in \mathbb{R}^n \) and \( t \in \mathbb{R} \) we have

\[
Q_{m}(t\xi) = \frac{1}{2}[P_{m}(t\xi) + \overline{P}(t\xi)] = \frac{1}{2}[t^m P_{m}(\xi) + t^m \overline{P}_{m}(\xi)] = t^m Q_{m}(\xi),
\]

and similarly for \( R_{m} \). Let us define the continuous mapping \( g : \partial B(1) \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} \) by

\[
g(\xi) = (Q_{m}(\xi), R_{m}(\xi), 0, \ldots, 0)
\]

with \( R_{m}(\xi) \) followed by \((n - 3)\) zeros. By Borsuk-Ulam’s theorem 20.2.1, there exists \( \xi^* \in \partial B(1) \) such that \( g(\xi^*) = g(-\xi^*) \), and hence such that

\[
Q_{m}(\xi^*) = Q_{m}(-\xi^*) = (-1)^m Q_{m}(\xi^*),
\]

\[
R_{m}(\xi^*) = R_{m}(-\xi^*) = (-1)^m R_{m}(\xi^*).
\]

Consequently, \( P_{m}(\xi^*) = (-1)^m P_{m}(\xi^*) \), and, as \( P_{m}(\xi^*) \neq 0, m \) must be even. ■

Remark 20.3.1 For \( n = 1 \), the thesis fails. Indeed, taking \( P(\partial) = \frac{d}{dx} \), we have \( P(\xi) = \xi = P_{m}(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \setminus \{0\} \) and \( P(\partial) \) is elliptic.

Remark 20.3.2 For \( n = 2 \), and \( P \) not real, the thesis fails. Indeed, taking \( P(\partial) = \partial_1 + i \partial_2 \), we have \( P(\xi) = \xi_1 + i \xi_2 = P_{1}(\xi) \neq 0 \) for all \( \xi \in \mathbb{R}^2 \setminus \{0\} \) and \( P(\partial) \) is elliptic.

20.4 Lusternik-Schnirel’mann’s covering theorem

Another interesting consequence of Borsuk-Ulam’s theorem with \( D = B(1) \) is Lusternik-Schnirel’mann’s covering theorem proved in 1930 by L. Lusternik and L. Schnirelmann [256] and in 1933 by Borsuk [36]. We denote by \( \alpha : S^n \rightarrow S^n \) the antipodal mapping defined by \( \alpha(x) = -x \).

Corollary 20.4.1 In any closed covering \( \{M_1, \ldots, M_{n+1}\} \) of \( S^n \) by \( n + 1 \) subsets of \( S^n \), there is at least one \( M_i \) containing a pair of antipodal points.

Proof. If there is some closed covering \( \{M_1, \ldots, M_{n+1}\} \) of \( S^n \) with no \( M_i \) containing a pair of points \( x \) and \( -x \), then we have \( M_i \cap \alpha(M_i) = \emptyset \) for all \( i = 1, \ldots, n + 1 \). Hence, Urysohn’s function \( g_i \in C(S^n, [0, 1]) \) defined by

\[
g_i(x) = \frac{dist(x, M_i)}{dist(x, M_i) + dist(x, \alpha(M_i))}
\]
Hence, if we define \( g_i(x) = 0 \) if \( x \in M_i \) and \( g_i(x) = 1 \) if \( x \in \alpha(M_i) \) for \( i = 1, 2, \ldots, n + 1 \). Hence, if we define \( g : S^n \to \mathbb{R}^n \) by \( g(x) = (g_1(x), \ldots, g_n(x)) \), it follows from Borsuk-Ulam's theorem 20.2.1 that there exists \( x^* \in S^n \) such that \( g(x^*) = g(\alpha(x^*)) \), i.e.

\[
g_1(x^*) = g_1(\alpha(x^*)), \ldots, g_n(x^*) = g_n(\alpha(x^*)).
\]

Consequently, we must have \( x^* \not\in M_i \) (\( i = 1, 2, \ldots, n \)) and \( \alpha(x^*) \not\in M_i \) (\( i = 1, 2, \ldots, n \)). Hence \( x^* \in M_{n+1} \cup \alpha(M_{n+1}) \), a contradiction. \( \blacksquare \)

Define an antipode preserving mapping \( f : S^n \to S^k \) by the condition \( f(x) = f(-x) \) for all \( x \in S^n \).

**Corollary 20.4.2** An antipode preserving mapping \( f \in C(S^n, S^n) \) is not nullhomotopic.

**Proof.** If it were the case, using Proposition 20.1.1, \( f \) would have a continuous extension \( \tilde{f} : \overline{B}(1) \subset \mathbb{R}^{n+1} \to S^n \). Let

\[
S_{n+1}^n = \{ x \in S^{n+1} : x_{n+1} \geq 0 \},
\]

\[
\pi : S^{n+1} \to \overline{B}(1), \quad (x_1, \ldots, x_{n+2}) \mapsto (x_1, \ldots, x_{n+1})
\]

and define \( g : S^{n+1} \to S^n \) by

\[
g(x) = \begin{cases} 
\tilde{f} \circ \pi(x) & \text{if } x \in S_{n+1}^n, \\
-\tilde{f} \circ \pi \circ \alpha(x) & \text{if } x \in S_{n+1}^n,
\end{cases}
\]

then \( g \in C(S^{n+1}, S^n) \) is an odd mapping, a contradiction to Corollary 20.2.1. \( \blacksquare \)

Another way to formulate Corollary 20.4.2, using Proposition 20.1.1, is the following one.

**Corollary 20.4.3** If \( f \in C(S^n, S^n) \) is antipode preserving, any continuous extension \( \tilde{f} : \overline{B}(1) \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) of \( f \) to \( \overline{B} \) vanishes at some \( x^* \in B(1) \).

**Proof.** If there exists a continuation \( \tilde{f} \in C(\overline{B}(1), \mathbb{R}^{n+1} \setminus \{0\}) \) of \( f \) to \( \overline{B}(1) \subset \mathbb{R}^{n+1} \), then \( g = \tilde{f} / ||\tilde{f}|| \in C(\overline{B}(1), S^n) \) is a continuous extension of \( f \), and, by Proposition 20.1.1, \( f \) is nullhomotopic, a contradiction to Corollary 20.4.2.

### 20.5 Measure of non-compactness of the unit sphere in a Banach space

A nice application of Lusternik-Schnirel'mann’s covering result 20.4.1, which is due to M. Furi and A. Vignoli [141] and to R. Nussbaum [299] is a computation of Kuratowski’s measure of non-compactness of the unit sphere in an infinite-dimensional Banach space.

Let \((X, d)\) be a metric space and \(B\) be the collection of all bounded subsets of \(X\). In 1930, K. Kuratowski [229] has introduced as follows the concept of **measure of non-compactness** \(\alpha : B \to [0, +\infty[\). For any \(B \in B\),
\[ \alpha(B) := \inf \{ \varepsilon > 0 : B \text{ can be covered with a finite number of subsets of diameter smaller of equal to } \varepsilon \}. \]

Let us denote by \( \delta(B) \) the diameter of \( B \).

We list here without proofs some fundamental properties of the mapping \( \alpha \) (notice that the first two ones, used in the proof of theorem 20.5.1 below, are direct consequences of the definition of \( \alpha \)):

(i) \( \alpha(B) \leq \delta(B) \).

(ii) \( \alpha(B) = 0 \) if and only if \( B \) is relatively compact.

(iii) If \( B_1 \subset B_2 \) belong to \( \mathcal{B} \), then \( \alpha(B_1) \leq \alpha(B_2) \).

(iv) for any \( B \in \mathcal{B} \), \( \alpha(\overline{B}) = \alpha(B) \).

(v) for any \( B_1, B_2 \) belonging to \( \mathcal{B} \), \( \alpha(B_1 \cup B_2) = \max \{ \alpha(B_1), \alpha(B_2) \} \).

If we now assume that \( X \) is a real Banach space, we have the following supplementary properties:

(vi) for any \( B \in \mathcal{B} \), \( \alpha(\text{co } B) = \alpha(B) \).

(vii) for any \( B, C \) belonging to \( \mathcal{B} \), \( \alpha(B + C) \leq \alpha(B) + \alpha(C) \).

(viii) for any \( B \in \mathcal{B} \) and any \( \lambda \in \mathbb{R} \), \( \alpha(\lambda B) = |\lambda| \alpha(B) \).

For the proofs and for further properties, see e.g. [89] or [254].

Now, since a Banach space is finite-dimensional if and only if \( \partial B(1) \) is compact, if follows from Property (ii) that \( \alpha(\partial B(1)) = 0 \) if \( \text{dim } X < +\infty \), while \( \alpha(\partial B(1)) > 0 \) and, according to Property (i), \( \alpha(\partial B(1)) \leq 2 \) if \( \text{dim } X = \infty \). This result can be precised as follows.

**Theorem 20.5.1** If \( X \) be an infinite-dimensional real Banach space, \( \alpha(\partial B(1)) = 2 \).

**Proof.** Assume by contradiction that \( \alpha(\partial B(1)) < 2 \). Then we can write \( \partial B(1) = S_1 \cup S_2 \cup \ldots \cup S_n \) with \( \delta(S_i) < 2 \) for all \( i = 1, 2, \ldots, n \). Taking \( \overline{S_i} \) instead of \( S_i \) if necessary, we may assume that the \( S_i \) are closed. Let \( X_n \) be a \( n \)-dimensional vector subspace of \( X \). Then \( \partial B(1) \cap X_n \) is covered with \( \{ S_1 \cap X_n, S_2 \cap X_n, \ldots, S_n \cap X_n \} \).

Lusternik-Schnirelmann’s covering property 20.4.1 implies the existence of at least one of the \( S_i \cap X_n \), say \( S_k \cap X_n \), for some \( k \in \{1, 2, \ldots, n\} \), containing a pair of antipodal points, i.e. \( x^* \in S_k \cap S_n \) with \( -x^* \in S_k \cap S_n \). This implies that

\[ 2 = \|x^* - (-x^*)\| \leq \delta(S_k \cap X_n) \leq \delta(S_k), \]


a contradiction. \( \Box \)

**Remark 20.5.1** By the properties (iv) and (v) of the measure of non-compactness, we deduce from Theorem 20.5.1 that \( X \) is a real infinite-dimensional Banach space if and only if

\[ \alpha(\partial B(1)) = \alpha(B(1)) = \alpha(\overline{B}(1)) = 2. \]
20.6 Genus

Let \( E \) be a real Banach space, and denote by \( \Sigma(E) \) the set of all closed subsets of \( E \) which do not contain 0 and are symmetric with respect to 0. For example \( \partial B(R) \) and \( \{-x, x\} \) for \( x \in E \setminus \{0\} \) belong to \( \Sigma(E) \). The concept of genus was first introduced in 1952 by M.A. Krasnosel’skii [211] (see also [213]) as an alternative of the Lusternik-Schnirelmann category in the minimax approach of critical points. This genus measure the ‘size’ of some subsets of \( E \) which are symmetric with respect to the origin.

**Definition 20.6.1** The genus \( \gamma(A) \) of a non-empty set \( A \in \Sigma(E) \) is the smallest integer \( n \in \mathbb{N}^* \) for which there exists an odd mapping \( g \in C(A, \mathbb{R}^n \setminus \{0\}) \). We write \( \gamma(A) = +\infty \) if no such mapping exists, and \( \gamma(\emptyset) = 0 \).

**Example 20.6.1** If \( A = \overline{B}_x(r) \cup \overline{B}_{-x}(r) \) with \( 0 < r < \|x\| \), then \( \gamma(A) = 1 \), because the mapping \( g : A \to \mathbb{R}^n \setminus \{0\} \) defined by

\[
g(x) = \begin{cases} 
-1 & \text{if } x \in \overline{B}_{-x}(r) \\
+1 & \text{if } x \in \overline{B}_x(r)
\end{cases}
\]

is continuous. More generally, any non connected set in \( \Sigma(E) \) has genus equal to 1.

**Proposition 20.6.1** If \( D \subset \mathbb{R}^n \) is a bounded open symmetric neighborhood of 0, then \( \gamma(\partial D) = n \).

**Proof.** First, \( \gamma(\partial D) \leq n \), because \( I \in C(\partial D, \mathbb{R}^n \setminus \{0\}) \) is odd. If \( \gamma(\partial D) = m < n \), then there exists an odd mapping \( g \in C(\partial D, \mathbb{R}^m \setminus \{0\}) \), a contradiction to Lemma 20.2.1.

We give now the main properties of the genus. First, taking the image of a set in \( \Sigma \) by a continuous odd mapping does not decrease its genus.

**Proposition 20.6.2** If \( A, B \in \Sigma(E) \) and \( f \in C(A, B) \) is odd, then \( \gamma(A) \leq \gamma(B) \).

**Proof.** The result is trivial if \( \gamma(B) = +\infty \). If \( \gamma(B) = n \), there exists \( g \in C(B, \mathbb{R}^n \setminus \{0\}) \) and odd, and hence \( g \circ f \in C(A, \mathbb{R}^n \setminus \{0\}) \) is odd, so that \( \gamma(A) \leq n \).

The genus is not decreasing with respect to inclusion.

**Proposition 20.6.3** If \( A, B \in \Sigma(E) \) and \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \).

**Proof.** Take \( f = I \) in Proposition 20.6.2.

The genus is invariant under an odd homeomorphism.

**Proposition 20.6.4** If \( h : A \to B \) is an odd homeomorphism of \( A \) onto \( B \), then \( \gamma(A) = \gamma(B) \).

**Proof.** A consequence of Proposition 20.6.2 with \( f = h \) and \( f = h^{-1} \).
The genus is subadditive with respect to union.

**Proposition 20.6.5** If $A, B \in \Sigma(E)$, then $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

*Proof.* If $\gamma(A)$ or $\gamma(B)$ is equal to $+\infty$, the result is trivial. If $\gamma(A) = n$ and $\gamma(B) = m$, there exists odd mappings $g \in C(A, \mathbb{R}^n \setminus \{0\})$ and $h \in C(A, \mathbb{R}^m \setminus \{0\})$, and, by Tietze theorem, those mappings have continuous extensions $\tilde{g} \in C(E, \mathbb{R}^n)$ and $\tilde{h} \in C(E, \mathbb{R}^m)$, which can be assumed to be odd by taking if necessary their odd part. Then, the mapping $f = (\tilde{g}|_{A \cup B}, \tilde{h}|_{A \cup B})$ is odd and $f \in C(A \cup B, \mathbb{R}^{n+m} \setminus \{0\})$, because one of the components is always non zero on $A \cup B$. Hence $\gamma(A \cup B) \leq n + m$.

The genus of the difference of two sets is larger than the difference of their genus.

**Corollary 20.6.1** If $A, B \in \Sigma(E)$ and $\gamma(B) < +\infty$, then $\gamma(A \setminus B) \geq \gamma(A) - \gamma(B)$.

*Proof.* The relation $A \subset (A \setminus B \cup B)$ and Propositions 20.6.3 and 20.6.5 imply that $\gamma(A) \leq \gamma(A \setminus B) + \gamma(B)$.

The genus of a compact set is always finite.

**Proposition 20.6.6** If $A \in \Sigma(E)$ is compact, then $\gamma(A) < +\infty$.

*Proof.* The family of open sets 

$$
\{B_x(\|x\| < 2) \cup B_{-x}(\|x\|/2) : x \in A\}
$$

is an open covering of $A$, and hence contains a finite subcovering 

$$
\{B_x(\|x^j\|/2) \cup B_{-x}(\|x^j\|/2) : j = 1, 2, \ldots, m\}.
$$

Using Proposition 20.6.5 and Example 20.6.1, we get 

$$
\gamma(A) \leq \sum_{j=1}^{m} \gamma[B_{x^j}(\|x^j\|/2) \cup B_{-x^j}(\|x^j\|/2)] = m.
$$

Finally, the genus of some neighborhood of a compact $A \subset \Sigma$ is the same as that of $A$.

**Proposition 20.6.7** If $A \in \Sigma(E)$ is compact, there exists $\delta > 0$ such that if $N_\delta(A) = \{x \in E : \text{dist}(x, A) \leq \delta\}$, one has $\gamma(N_\delta(A)) = \gamma(A)$.

*Proof.* The result is trivial by Corollary 20.6.3 if $\gamma(A) = +\infty$. If $\gamma(A) = n$, there exists an odd mapping $g : (A, \mathbb{R}^n \setminus \{0\})$. Let $\tilde{g} \in C(E, \mathbb{R}^n)$ be an odd extension of $g$. $A$ being compact and $\tilde{g}$ non zero on $A$ and uniformly continuous on $A$, there exists $\delta > 0$ such that $\tilde{g}(x) \neq 0$ for all $x \in N_\delta(A)$. Hence $\tilde{g} \in C(N_\delta(A), \mathbb{R}^n \setminus \{0\})$ and is odd, so that $\gamma(N_\delta(A)) \leq n$. On the other hand, it follows from Proposition 20.6.3 that $m = \gamma(A) \leq \gamma(N_\delta(A))$. 


The concept of genus allows a strengthening of Lemma 20.2.1.

**Theorem 20.6.1** Let \( n > m \), \( D \subset \mathbb{R}^{n+1} \) be an open, bounded and symmetric neighborhood of 0, \( g \in C(\partial D, \mathbb{R}^{m+1}) \) be an odd mapping, and \( A = \{ x \in \partial D : g(x) = 0 \} \). Then \( \gamma(A) \geq n - m \).

**Proof.** If follows from Proposition 20.6.7 that there exists \( \delta > 0 \) such that \( \gamma(A) = \gamma(N_\delta(A)) \). Furthermore, there exists \( \varepsilon_0 \) such that, for each \( \varepsilon \in [0, \varepsilon_0] \), one has

\[
Z_\varepsilon := \{ x \in \partial D : \|g(x)\| \leq \varepsilon \} \subset N_\delta(A).
\]

Indeed, if it is not the case, there exists a sequence \( (x_n) \) in \( \partial D \) such that \( \|g(x_n)\| \leq 1/n \) and \( x_n \notin N_\delta(A) \). Going if necessary to a subsequence, we can assume that \( (x_n) \) converges to some \( x^* \notin \text{int}(N_\delta(A)), \) and hence \( x^* \notin A \) and \( g(x^*) = 0 \), a contradiction. Consequently, for any \( \varepsilon \in [0, \varepsilon_0] \), one has \( A \subset Z_\varepsilon \subset N_\delta(A) \), and hence, from Propositions 20.6.3 and 20.6.7, \( \gamma(A) \leq \gamma(Z_\varepsilon) \leq \gamma(N_\delta(A)) = \gamma(A) \), and hence

\[
\gamma(Z_\varepsilon) = \gamma(A) \quad (\varepsilon \in [0, \varepsilon_0]).
\]

For any \( \eta > 0 \), let \( Y_\eta := \{ x \in \partial D : \|g(x)\| \geq \eta \} \), and let \( p : \mathbb{R}^{m+1} \to S^n \) be the projection defined by \( p(x) = x/\|x\| \). Then \( p \circ g \in C(Y_\eta, S^m) \) is odd, and hence, from Proposition 20.6.2 \( \gamma(Y_\eta) \leq \gamma(S^m) = m + 1 \). Therefore, using Proposition 20.6.1 and Corollary 20.6.1, we deduce that

\[
\gamma(\partial D \setminus Y_\eta) \geq \gamma(\partial D) - \gamma(Y_\eta) \geq n + 1 - (m + 1) = n - m.
\]

But \( Z_\eta = \partial D \setminus Y_\eta \), and hence, for \( \varepsilon \in [0, \varepsilon_0] \), we get

\[
\gamma(A) = \gamma(Z_\varepsilon) = \gamma(\partial D \setminus Y_\varepsilon) \geq n - m.
\]

Theorem 20.6.1 implies a strengthening of Borsuk-Ulam’s theorem 20.2.1.

**Corollary 20.6.2** If \( D \subset \mathbb{R}^{n+1} \) is an open, bounded symmetric neighborhood of 0, and \( f \in C(\partial D, \mathbb{R}^{m+1}) \) with \( n > m \), then \( \gamma(\{ x \in \partial D : f(x) = f(-x) \}) \geq n - m \).

**Proof.** Take \( g(x) = f(x) - f(-x) \) in Theorem 20.6.1.

Finally, the following lemma will allow to prove a characterization of sets of genus \( n \) through sets of genus one.

**Lemma 20.6.1** There exists \( n + 1 \) closed antipodal sets \( B_i = C_i \cup (-C_i) \subset S^n \) such that \( C_i \cap (-C_i) = \emptyset \) (1 \( \leq i \leq n + 1 \)) and \( S^n = \bigcup_{i=1}^{n+1} B_i \).

**Proof.** We proceed by recurrence over \( n \). The case where \( n = 0 \) is trivial, because \( S^0 = \{1\} \cup \{-1\} \). If \( n = 1 \), one has \( S^1 = B_1 \cup B_2 \) with

\[
B_1 = \{(x, y) \in \mathbb{R}^2 : x > 1/2\} \cup \{(x, y) \in \mathbb{R}^2 : x < -1/2\} = C_1 \cup (-C_1)
\]

\[
B_2 = \{(x, y) \in \mathbb{R}^2 : y > 1/2\} \cup \{(x, y) \in \mathbb{R}^2 : y < -1/2\} = C_2 \cup -C_2.
\]
Let us assume that the result is true till order $n - 1$, so $S^{n-1} = \bigcup_{i=1}^{n} B'_i$, where the $B'_i$ are $n$ closed antipodal sets $B'_i = C'_i \cup (-C'_i)$ and $C'_i \cap (-C'_i) = \emptyset$. Denote by $(x', x_{n+1})$ the point $(x_1, \ldots, x_n, x_{n+1})$ of $\mathbb{R}^{n+1}$ and identify $\mathbb{R}^n$ with the hyperplane $\{ x \in \mathbb{R}^{n+1} : x_{n+1} = 0 \}$. Let $C_{n+1} := \{(x', x_{n+1}) \in S^n : x_{n+1} \geq 1/4 \}$, and denote by $C_i$ the sets

$$C_i = \left\{ (x', x_{n+1}) \in S^n : x_{n+1} \leq 1/2, \quad \frac{x'}{\sqrt{1 - x_{n+1}^2}} \in C'_i \right\} \quad (i = 1, 2, \ldots, n).$$

Each set $C_i \subset S^n$ is closed and $C_i \cap (-C_i) = \emptyset$. Set $B_i = C_i \cup (-C_i)$ ($i = 1, 2, \ldots, n+1$). The family $\{ B_i : i = 1, 2, \ldots, n+1 \}$ is a covering of $S^n$ through closed antipodal sets.

**Lemma 20.6.2** If $A \in \Sigma(E)$ and $\gamma(A) = j$ for some integer $j \geq 1$, there exists $D_1, \ldots, D_j \in \Sigma(E)$ with $\gamma(D_i) = 1$ ($i = 1, 2, \ldots, j$) such that $A \subset \bigcup_{i=1}^{j} D_i$.

**Proof.** As $\gamma(A) = j$, there exists an odd mapping $g \in C(A, \mathbb{R}^j \setminus \{0\})$. From Lemma 20.6.1, $S^{j-1} = \bigcup_{i=1}^{j} B_i$, where the $B_i$ are closed antipodal sets $B_i = C_i \cup (-C_i)$ and $C_i \cap (-C_i) = \emptyset$ ($i = 1, 2, \ldots, j$). Let $p : \mathbb{R}^j \setminus \{0\} \to S^{j-1}$ be the radial projection defined by $p(x) = x/\|x\|$, and let

$$D_i := g^{-1} \circ p^{-1}(B_i) = g^{-1} \circ p^{-1}(C_i) \cup g^{-1} \circ p^{-1}(-C_i) \quad (i = 1, 2, \ldots, j).$$

Each $D_i \in \Sigma(E)$ and $A \subset \bigcup_{i=1}^{j} D_i$. Each $D_i$ is made of two closed disjoint sets, and hence $\gamma(D_i) \leq 1$ ($i = 1, 2, \ldots, j$). But, from $A \subset \bigcup D_i$ and the subadditive property 20.6.5, we get $\gamma(A) = j \leq \sum_{i=1}^{j} \gamma(D_i)$, and hence $\gamma(D_i) = 1$ ($i = 1, 2, \ldots, j$).

**Theorem 20.6.2** If $A \in \Sigma(E)$, then $\gamma(A) = n$ if and only if $n$ is the smallest integer such that one can find $n$ sets $A_1, \ldots, A_n \in \Sigma(E)$ such that $\gamma(A_j) = 1$ ($j = 1, 2, \ldots, n$) and $A \subset \bigcup_{j=1}^{n} A_j$.

**Proof.** **Sufficiency.** If $A \subset \bigcup_{j=1}^{n} A_j$, then, by the subadditive property 20.6.5, one gets $\gamma(A) \leq n$. If we had $\gamma(A) = j < n$, then Lemma 20.6.2 would contradict the minimal character of $n$.

**Necessity.** If $\gamma(A) = n$, Lemma 20.6.2 implies the existence of $A_1, \ldots, A_n \in \Sigma(E)$ such that $\gamma(A_j) = 1$ ($i = 1, 2, \ldots, n$) and $A \subset \bigcup_{j=1}^{n} A_j$. Now, we cannot have $A \subset \bigcup_{j=1}^{m} B_j$, with $B_j \in \Sigma(E)$ and $\gamma(B_i) = 1$ for all $i = 1, 2, \ldots, m$ and some $m < n$, because then, by the subadditive property 20.6.5, we would have $\gamma(A) \leq m < n$.}

### 20.7 Isometric representations of $S^1$ over $\mathbb{R}^n$

Let $G$ be a topological group. Recall that a representation of $G$ over $\mathbb{R}^n$ is a family $\{T(g)\}_{g \in G}$ of linear operators $T(g) : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T(0) = I, \quad T(g_1 + g_2) = T(g_1)T(g_2), \quad (g, u) \to T(g)u \quad \text{is continuous.}$$
A subset \( E \subset \mathbb{R}^n \) is invariant (under the representation) if \( T(g)E = E \) for all \( g \in G \). A representation \( \{T(g)\}_{g \in G} \) is isometric if \( \|T(g)x\| = \|x\| \) for all \( g \in G \) and all \( x \in \mathbb{R}^n \).

For example, an isometric representation of \( G = \mathbb{Z}_2 \) is given by \( T(e) = I, T(-e) = -I \), where \( e \) is the unit element in \( \mathbb{Z}_2 \) and \( I \) the identity matrix in \( \mathbb{R}^n \).

The invariant sets in \( \mathbb{R}^n \) are the sets which are symmetric with respect to the origin.

We are interested in the group \( S^1 \) of rotations on the unit circle. Here we consider \( S^1 \) as the complex numbers of modulus one \( \xi = e^{i\theta} \).

The following result is classical (see e.g. [278]).

**Lemma 20.7.1** Let \( \{T(\theta)\}_{\theta \in S^1} \) be an isometric representation of \( S^1 \) over \( \mathbb{R}^n \). Then \( T(\theta) \) has the matrix representation

\[
\text{diag}\{M_1, \ldots, M_k\}
\]

(20.5)

where \( M_j \) is either of order 1 and \( M_j = 1 \), or is of order 2 and, for some \( n \in \mathbb{N} \setminus \{0\} \) such that

\[
\left( \begin{array}{cc}
\cos n\theta & -\sin n\theta \\
\sin n\theta & \cos n\theta \\
\end{array} \right).
\]

(20.6)

Define

\[ \text{Fix } T := \{ u \in \mathbb{R}^n : T(\theta)u = u \text{ for all } \theta \in S^1 \}. \]

### 20.8 \( S^1 \)-equivariant mappings

Let \( \{T(\theta)\}_{\theta \in S^1} \) be the isometric representation of \( S^1 \) over \( \mathbb{R}^n \) given by

\[
T(\theta)x = T(\theta)(y, z_1, \ldots, z_k) = T(\theta)(y, z) = (y, e^{im_1\theta}z_1, \ldots, e^{im_k\theta}z_k),
\]

(20.7)

where we identify \( \mathbb{R}^2 \) with \( \mathbb{C} \), \( m_1, \ldots, m_k \in \mathbb{N} \setminus \{0\} \) and where \( y \in \mathbb{R}^m \) \((m \geq 0)\).

Thus, \( S^1 \) acts on \( \mathbb{R}^n \) via

\[
e^{i\theta} \cdot (y, z) = (y, e^{im_1\theta}z_1, \ldots, e^{im_k\theta}z_k).
\]

Notice that \( x \in \text{Fix } T \) if and only if \( x = (y, 0) \), so that \( \dim \text{Fix } T = m \). Let \( \{U(\theta)\}_{\theta \in S^1} \) be the isometric representation of \( S^1 \) over \( \mathbb{R}^n \) given by

\[
U(\theta)x = (y, e^{in_1\theta}z_1, \ldots, e^{in_k\theta}z_k),
\]

(20.8)

where \( n_1, \ldots, n_k \in \mathbb{N} \setminus \{0\} \) and where \( y \in \mathbb{R}^m \). Let \( D \subset \mathbb{R}^n \) be an open bounded invariant set, let \( f \in C(D, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n) \) be such that

\[
0 \notin f(\partial D),
\]

(20.9)

and, for all \( \theta \in S^1 \) and \( x \in \partial D \),

\[
[f \circ T(\theta)](x) = [U(\theta) \circ f](x).
\]

(20.10)
To help in the notations, let us write, for \( r \in \mathbb{N} \setminus \{0\} \), \( V_r \cong \mathbb{C} \) for the representation of \( S^1 \) given by \( T(\theta)w = e^{ir\theta}w \), so that we can write

\[
f : \overline{D} \subset \mathbb{R}^m \times V_{m_1} \times \ldots \times V_{m_k} \to \mathbb{R}^m \times V_{n_1} \times \ldots \times V_{n_k}.
\]  

(20.11)

We need a few preliminary lemmas.

**Lemma 20.8.1** Let \( m_j \in \mathbb{N} \setminus \{0\} \) (1 \( \leq j \leq k \)) and let \( p : \mathbb{R}^m \times \mathbb{C}^k \to \mathbb{R}^m \times \mathbb{C}^k \) be defined by

\[
p(y, z_1, \ldots, z_k) = (y, z_1^{m_1}, \ldots, z_k^{m_k}).
\]

Then

\[
d_B[p, B(r), 0] = m_1 \cdots m_k.
\]  

(20.12)

**Proof.** Using excision, the cartesian product property 19.1.1 and formula (17.25), we get, with \( B(R) \subset \mathbb{R}^m \), \( B_j(r) \subset \mathbb{C} \), \( (j = 1, \ldots, k) \),

\[
d_B[p, B(r), 0] = d_B[p, B(R) \times B_1(r) \times \ldots \times B_k(r), 0]
\]

\[
= d_B[I, B(R), 0] \prod_{j=1}^k d_B[z^{m_j}, B_j(r), 0] = \prod_{j=1}^k m_j.
\]

The following result is a special case of the **parametrized Sard theorem**.

**Lemma 20.8.2** Let \( \Omega \subset \mathbb{R}^p \), \( M \subset \mathbb{R}^s \) be open, and let \( G \in C^1(\Omega \times M, \mathbb{R}^p) \). If \( 0 \in \mathbb{R}^p \) is a regular value of \( G \) (i.e. \( G'(x, v) \) is onto for each \( (z, v) \in G^{-1}(0) \)), then, for almost every \( v \in M \), 0 is a regular value of \( G(\cdot, v) \).

Given \( n \in \mathbb{N} \setminus \{0\} \), and \( g \in C(\overline{D}, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n) \), let us define the mapping \( G : \overline{D} \times \mathbb{C}^{k \times k} \to \mathbb{R}^m \times \mathbb{C}^k \) by

\[
G(y, z, v) := g(x) - \sum_{j=1}^k z_j^n v_j.
\]  

(20.13)

**Lemma 20.8.3** Under the assumptions above for \( g \), \( 0 \in \mathbb{R}^m \times \mathbb{C}^k \) is a regular value of the restriction of \( G \) to \( D_0 := \{(y, z) \in D : z \neq 0\} \) for almost every \( v \in \mathbb{C}^{k \times k} \).

**Proof.** Set \( M = \mathbb{C}^{k \times k} \), and apply Lemma 20.8.2 to \( G \) with \( p = n \), and \( s = k^2 \). It suffices to check that \( F'_v(x, v) \) is onto for each \( (x, v) \in D_0 \times M \), i.e. to be able to solve the equation in \( (a, b) \in \mathbb{R}^n \times \mathbb{C}^{k \times k} \)

\[
F'_x(x, v)a + F'_v(x, v)b = c
\]

for each \( c \in \mathbb{R}^n \). Taking \( a = 0 \), it remains to solve the equation

\[
z_1^n b_1 + \cdots + z_k^n b_n = -c,
\]

which is always possible as \( z \neq 0 \).
Let \( f^D : \overline{D} \cap \text{Fix } T \to \mathbb{R}^n \) denote the restriction of \( f \) to the fixed point set \( \text{Fix } T \) of \( T \). By the equivariance property, we have, if we write \( f = (f_0, f_1, \cdots, f_k) \), with \( f_0 \in \mathbb{R}^m, f_j \in \mathbb{C} \) \((1 \leq j \leq k)\),
\[
(f_0(y, 0), f_1(y, 0), \cdots, f_k(y, 0)) = (f_0(y, 0), e^{in_1\theta} f_1(y, 0), \cdots, e^{im_k\theta} f_k(y, 0))
\]
for all \( \theta \in \mathbb{R} \), so that, taking successively \( \theta = \pi/n_j \), we get \( f_j(y, 0) = 0 \) \((1 \leq j \leq k)\), and hence \( f^V(y) = f(y, 0) = (f_0(y, 0), 0, \cdots, 0) \), i.e.
\[
f^V : \overline{D} \cap \text{Fix } T \to \text{Fix } T.
\]

**Theorem 20.8.1** Let \( D \subset \mathbb{R}^m \times V_{m_1} \times \cdots \times V_{m_k} \) be an open, bounded invariant subset and \( f : \overline{D} \to \mathbb{R}^m \times V_{n_1} \times \cdots \times V_{n_k} \) be continuous, equivariant and such that \( 0 \not\in f(\partial D) \). Then
\[
d_B[f, D, 0] : m_1 \cdots m_k = d_B[f^T, D \cap \text{Fix } T, 0] : n_1 \cdots n_k.
\]

**Proof.** We reduce the proof to a simpler situation. For
\[
n = n_1 \cdots n_k,
\]
define the representations of \( S^1 \) in \( \mathbb{R}^n \)
\[
V(\theta)(y, z) = (y, e^{i\theta} z_1, \cdots, e^{i\theta} z_k),
\]
\[
W(\theta)(y, z) = (y, e^{in_1\theta} z_1, \cdots, e^{in_k\theta} z_k).
\]

Consider the maps
\[
\varphi : \mathbb{R}^m \times V_1^k \to \mathbb{R}^m \times V_{n_1} \times \cdots \times V_{n_k}, \quad (y, z_1, \cdots, z_k) \mapsto (y, z_1^{n_1}, \cdots, z_k^{n_k})
\]
\[
\psi : \mathbb{R}^m \times V_{n_1} \times \cdots \times V_{n_k} \to \mathbb{R}^m \times V_k, \quad (y, z_1, \cdots, z_k) \mapsto (y, z_1, \cdots, z_k).
\]

We have, for all \( \theta \in S^1 \),
\[
[\varphi \circ V(\theta)](x) = (y, e^{in_1\theta} z_1, \cdots, e^{im_k\theta} z_k) = [T(\theta) \circ \varphi](x),
\]
\[
[\psi \circ U(\theta)](x) = (y, (e^{in_1\theta} z_1)^{n_1}, \cdots, (e^{im_k\theta} z_k)^{n_k})
\]
\[
= (y, e^{in_1\theta} z_1^{n_1}, \cdots, e^{im_k\theta} z_k^{n_k}) = [W(\theta) \circ \psi](x),
\]

so that \( \varphi \) and \( \psi \) are equivariant. Hence,
\[
\tilde{f} := \psi \circ f \circ \varphi : \overline{D} \to \mathbb{R}^m \times V_k
\]
is equivariant, where
\[
\tilde{D} = \varphi^{-1}(D) \subset \mathbb{R}^m \times V_1^k.
\]

Now, \( \varphi^{-1}(0) = \{0\} \), \( \psi^{-1}(0) = \{0\} \), so that the product formula (21.1) and formula (20.12) imply that
\[
d_B[\tilde{f}, \tilde{D}, 0] = i_B[\psi, 0] \cdot d_B[f, D, 0] \cdot i_B[\varphi, 0] = \frac{n_k}{n} \cdot d_B[f, D, 0] \cdot m_1 \cdots m_k
\]
\[
= n^{k-1} \cdot d_B[f, D, 0] \cdot m_1 \cdots m_k.
\]
Furthermore, as \( \varphi|_{\text{Fix } V} = I, \psi|_{\text{Fix } U} = I \), we have
\[
d_B[\tilde{f}^V, \tilde{D} \cap \text{Fix } V, 0] = d_B[f^T, D \cap \text{Fix } T, 0].
\] (20.17)
Choose a neighbourhood \( U \) of \( \tilde{f}^{-1}(0) \cap \mathbb{R}^m \times \{0\} \) and \( \varepsilon > 0 \) such that \( U \times B(\varepsilon) \subset \tilde{D} \).
Define \( \hat{f} \) by
\[
\hat{f}(y, z) = \begin{cases} 
(\tilde{f}^V(y), z^1, \ldots, z^n) & \text{if } (y, z) \in U \times B(\varepsilon) \\
\tilde{f}(y, z) & \text{if } (y, z) \in [\tilde{D} \cap \text{Fix } V] \cup \partial \tilde{D}.
\end{cases}
\]
Here \( \tilde{f}^V : \tilde{D} \cap \text{Fix } V \to \text{Fix } V \) denotes the restriction as usual. Observe that for \( (y, z) \in [\tilde{D} \cap \text{Fix } V] \cap U \times B(\varepsilon) \) we have \( z = 0 \) and by equivariance \( f(y, 0) = (\tilde{f}^V(y), 0) \). Thus \( \hat{f} \) is well defined. Then extend \( \hat{f} \) continuously over all of \( \tilde{D} \), keeping the same notation. We have
\[
d_B[\hat{f}, \tilde{D}, 0] = d_B[\hat{f}, \tilde{D}, 0].
\] (20.18)
Let
\[
\mu := \min_{\partial \tilde{D}} \|\hat{f}\| = \min_{\partial \tilde{D}} \|\hat{f}\|
\]
and let \( f_\mu \in C^1(\tilde{D}, \mathbb{R}^n) \) be such that
\[
\max_{\tilde{D}} \|\hat{f} - f_\mu\| < \mu.
\]
Set
\[
\bar{f}(y, z) = \int_{S^1} \xi \ast f_\mu(\xi^{-1} \ast (u, z)) d\xi.
\]
\( \bar{f} \) is equivariant, and, if \( (y, z) \in [\tilde{D} \cap \text{Fix } V] \cup \partial \tilde{D} \), we have
\[
\|\bar{f}(y, z) - \hat{f}(y, z)\| = \|\int_{S^1} \xi \ast [f_\mu(y, z) - \hat{f}(y, z)] d\xi\|
= \|\int_{S^1} \xi \ast [f_\mu(y, z) - \hat{f}(y, z)] d\xi\| < \mu.
\]
Hence
\[
d_B[\bar{f}, \tilde{D}, 0] = d_B[\hat{f}, \tilde{D}, 0].
\] (20.19)
and
\[
d_B[\bar{f}^V, \tilde{D} \cap \text{Fix } V, 0] = d_B[\hat{f}^V, \tilde{D} \cap \text{Fix } V, 0].
\] (20.20)
Now, for \( v = (v_1, \ldots, v_k) \) with \( v_i \in \mathbb{C}^k \subset \mathbb{R}^m \times \mathbb{C}^k \) we define
\[
F(y, z, v) = \bar{f}(y, z) - \sum_{j=1}^k z_j^n v_j.
\]
One has
\[ F(V(\theta)x, v) = [\mathcal{T} \circ V(\theta)](x) - \sum_{j=1}^{k} (e^{i\theta z_j})^n v_j \]

so that that \( F(\cdot, v) : \tilde{D} \rightarrow \mathbb{R}^m \times V_n^r \) is equivariant and, by Lemma 20.8.3, \( F \) has 0 as a regular value. Now the parametrized Sard theorem (Lemma 20.8.2) implies that there exists \( v \in C^{k \times k} \) arbitrarily close to 0 such that 0 is a regular value of \( F(\cdot, v) \). If \( v \) is small enough the degrees of \( F(\cdot, v) \) and \( f \) are the same. Also, restricted to \( \mathbb{R}^m \), \( F(\cdot, v) \) and \( \mathcal{T} \) coincide. But if 0 is a regular value of \( F(\cdot, v) \), then \( F(\cdot, v)^{-1}(0) \subset \mathbb{R}^m \). This is obvious since any \((y, z) \in F(\cdot, v)^{-1}(0) \) with \( z \neq 0 \) yields a one-dimensional manifold \( \{ \xi \ast (y, z) : \xi \in S^1 \} \) of zeroes contradicting the regularity condition. So \( F(\cdot, v)^{-1}(0) = \mathcal{T}^{-1}(0) \cap \mathbb{R}^m \). Now, in the neighbourhood \( \mathcal{U} \times B(\varepsilon) \) of \( F(\cdot, v)^{-1}(0) \) we have by construction

\[ F(y, z, v) = [\mathcal{T}^\nu y, (z_1^n, \ldots, z_k^n) - \sum_{j=1}^{k} z_j^n v_j]. \]

Applying excision, homotopy invariance and the product formula once more we obtain
\[
\begin{align*}
\partial_B[\mathcal{T}, \tilde{D}, 0] &= \partial_B[F(\cdot, v), \tilde{D}, 0] \\
&= \partial_B[\mathcal{T}^\nu, \tilde{D} \cap \text{Fix } V, 0] \cdot i_B[(z_1^n, \ldots, z_k^n) - \sum_{j=1}^{k} z_j^n v_j, 0] \\
&= \partial_B[\mathcal{T}^\nu, \tilde{D} \cap \text{Fix } V, 0] \cdot n^k.
\end{align*}
\]

Consequently, using (20.16), (20.19) and (20.20),
\[
\partial_B[\mathcal{T}, D \cap \text{Fix } T, 0] \cdot n^k = n^{k-1} \cdot \partial_B[f, D, 0] \cdot m_1 \cdots m_k,
\]
which immediately gives formula (20.15). \( \blacksquare \)
Chapter 21

Leray’s product formula

21.1 The product formula

Let $D \subset \mathbb{R}^n$ be open and bounded, and $f : \overline{D} \to \mathbb{R}^n$ continuous. Since $f(\partial D)$ is compact, $\mathbb{R}^n \setminus f(\partial D)$ has one unbounded connected component $K_\infty$ if $n \geq 2$, and two unbounded connected components if $n = 1$, whose union will still be denoted by $K_\infty$. As $f(\partial D)$ is bounded as well, $K_\infty$ contains points $y \notin f(\partial D)$, and hence $d_B[f, D, K_\infty] = 0$. Let now $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let $y \notin g(f(\partial D))$. Hence $d_B[g \circ f, D, y]$ is well defined, and the aim is to express it in terms of some degrees of $f$ and of $g$. The following result is due to Leray [250].

**Theorem 21.1.1** Let $y \notin g(f(\partial D))$ and let $K_i$ be the bounded connected components of $\mathbb{R}^n \setminus f(\partial D)$. Then

$$d_B[g \circ f, D, y] = \sum_i d_B[f, D, K_i] d_B[g, K_i, y],$$

(21.1)

where only finitely many terms in the right-hand side are different from zero.

**Proof.** Let us first show that finitely many terms only in the sum of (21.1) are non zero. Let $r > 0$ be such that $f(\overline{D}) \subset B(r)$. Since $M := B(R) \cap g^{-1}(y)$ is compact and contained in $\mathbb{R}^n \setminus f(\partial D) = \bigcup K_i$, the family of mutually disjoint open sets $\{K_i (i \neq \infty); K_\infty \cap B(0, r + 1)\}$ covers $M$, and hence finitely many of them, say (after renumbering) $K_1, \ldots, K_p, K_{p+1} = K_\infty \cap B(0, r + 1)$ cover $M$. We only retain $K_j$ if $K_j \cap M \neq \emptyset$. Now, $d_B[f, D, K_{p+1}] = 0$ because $K_{p+1}$ contains points not in $f(\overline{D})$, and $d_B[g, K_j, y] = 0$ for $j \geq p + 2$ because, for these $j$, $K_j \subset B(r)$, but $g^{-1}(y) \cap K_j = \emptyset$. Thus the summation in (21.1) is finite. The remaining of the proof will be divided into the parts, namely $f$ and $g$ smooth, and then $f$ and $g$ continuous.
21.2 Proof in the smooth case

Let \( f \in C^1(D) \cap C^1(\mathbb{R}^n) \), \( g \in C^1(\mathbb{R}^n) \), and assume first that \( y \) is a regular value of \( g \circ f \). We have, using Corollary 5.2.1,

\[
\begin{align*}
d_B[g \circ f, D, y] &= \sum_{x \in (g \circ f)^{-1}(y)} \text{sign } J_{g \circ f}(x) = \sum_{x \in (g \circ f)^{-1}(y)} \text{sign } J_g(f(x)) \text{ sign } J_f(x) \\
&= \sum_{x \in f^{-1}(z), z \in g^{-1}(y)} \text{sign } J_g(z) \text{ sign } J_f(x) \\
&= \sum_{z \in g^{-1}(y) \cap f(D)} \text{sign } J_g(z) \left[ \sum_{x \in f^{-1}(z)} \text{sign } J_f(x) \right] \\
&= \sum_{z \in f(D) \cap g^{-1}(y)} \text{sign } J_g(z) \cdot d_B[f, D, z].
\end{align*}
\]

Now, in the last sum, we may replace \( z \in f(D) \) by \( z \in \mathbb{R}(R) \setminus f(\partial D) \), since \( d_B[f, D, z] = 0 \) whenever \( f \not\in f(D) \). Consequently, as the \( K_i \) are mutually disjoint, we obtain

\[
\begin{align*}
d_B[g \circ f, D, y] &= \sum_{i=1}^{p} \sum_{z \in K_i \cap g^{-1}(y)} \text{sign } J_g(z) d_B[f, D, z] \\
&= \sum_{i=1}^{p} d_B[f, D, K_i] \left[ \sum_{z \in f^{-1}(K_i)} \text{sign } J_g(z) \right] \\
&= \sum_{i} d_B[f, D, K_i] d_B[g, K_i, y].
\end{align*}
\]

21.3 Proof in the continuous case

To deduce the continuous case from the smooth one, it is convenient to rearrange the right-hand member of (21.1) as follows. Let, for each integer \( m \),

\[
\begin{align*}
S_m &:= \{ z \in B(0, r + 1) \setminus f(\partial D) : d_B[f, D, z] = m \}, \\
N_m &:= \{ i \in \mathbb{N} : d_B[f, D, K_i] = m \}.
\end{align*}
\]

Clearly, \( S_m = \bigcup_{i \in N_m} K_i \), and hence, by the additivity of degree,

\[
\begin{align*}
\sum_{i} d_B[f, D, K_i] d_B[g, K_i, y] &= \sum_{m} m \left[ \sum_{i \in N_m} d_B[g, K_i, y] \right] = \sum_{m} m \cdot d_B[g, S_m, y].
\end{align*}
\]
21.3. PROOF IN THE CONTINUOUS CASE

It remains to show that
\[ d_B[g \circ f, D, y] = \sum_m m \cdot d_B[g, S_m, y]. \] (21.2)

Since \( \partial S_m \subset f(\partial D) \), and \( \|g(f(x)) - y\| \geq \mu > 0 \) for all \( x \in \partial D \), we can find \( g_0 \in C^1(\mathbb{R}^n) \) such that
\[ d_B[g_0 \circ f, D, y] = d_B[g \circ f, D, y], \quad d_B[g_0, S_m, y] = d_B[g, S_m, y] \quad (m \in \mathbb{N}). \] (21.3)

Furthermore, we may assume that \( M_0 := B[0, r + 1] \cap g_0^{-1}(y) \neq \emptyset \), because, if it were empty, (21.3) implies that (21.2) is reduced to \( 0 = 0 \). As \( M_0 \) is compact and \( y \notin g_0(f(\partial D)) \), we have \( \text{dist}[M_0, f(\partial D)] > 0 \). Let us now choose \( f_0 \in C(\overline{T}) \cap C^1(D) \) such that
\[ \max_{\overline{T}} ||f - f_0|| < \text{dist}[M_0, f(\partial D)], \quad f_0(\overline{T}) \subset B(0, r + 1), \]
and let us define
\[ \tilde{S}_m := \{ z \in B(0, r + 1) \setminus f_0(\partial D) : d_B[f_0, D, z] = m \}. \]

We have \( S_m \cap M_0 = \tilde{S}_m \cap M_0 \), since \( z \in M_0 \) implies that
\[ \text{dist}[z, f(\partial D)] \geq \text{dist}[M_0, f(\partial D)] > \max ||f - f_0||, \]
and hence \( d_B[f_0, D, z] = d_B[f, D, z] \) by Rouche’s theorem. Consequently, both sets \( S_m \cap M_0 \) and \( \tilde{S}_m \cap M_0 \) are contained in \( S_m \cap \tilde{S}_m \), and hence, by excision,
\[ d_B[g_0, S_m, y] = d_B[g_0, S_m \cap \tilde{S}_m, y] = d_B[g_0, \tilde{S}_m, y]. \] (21.4)

Using the result of the previous section, (21.3) and (21.4), we obtain
\[ d_B[g_0 \circ f_0, D, y] = \sum_m m \cdot d_B[g_0, \tilde{S}_m, y] = \sum_m m \cdot d_B[g, S_m, y], \]
so that it remains to show that \( d_B[g_0 \circ f_0, D, y] = d_B[g_0 \circ f, D, y] \). Introducing the homotopy
\[ h(x, \lambda) = g_0 \circ [f + \lambda(f_0 - f)], \quad \lambda \in [0, 1], \]
we observe that, if \( y \in h(\partial D \times [0, 1]) \), then \( f(x) + \lambda[f_0(x) - f(x)] \in M_0 \) for some \( (x, \lambda) \in \partial D \times [0, 1] \), but we have, for all \( z \in M_0 \),
\[ ||z - f(x) - \lambda[f_0(x) - f(x)]|| \geq \text{dist}[M_0, f(\partial D)] - \max_{\overline{T}} ||f - f_0|| > 0. \]
21.4 Jordan-Brouwer’s separation theorem

C. Jordan’s famous theorem [187] says that a simple closed curve $\Gamma$ in the plane divides it into two regions $G_1$ and $G_2$ such that $\Gamma = \partial G_1 = \partial G_2$ and $G_2 : \mathbb{R}^2 \setminus G_1$. Since $\Gamma$ is homeomorphic to $\partial B(1)$, and $B(1)$ and $\mathbb{R}^2 \setminus B(1)$ are the components of $\mathbb{R}^2 \setminus \partial B(1)$, Jordan’s separation theorem may also be formulated as follows: if $\Gamma \subset \mathbb{R}^2$ is homeomorphic to $\partial B(1)$, then $\mathbb{R}^2 \setminus \Gamma$ has precisely two components, a bounded one and an unbounded one.

The extension to $\mathbb{R}^n$ of Jordan’s result is due to Brouwer [40] and called the Jordan-Brouwer theorem. The following proof is due to Leray [250].

**Theorem 21.4.1** Let $n \geq 2$ and $C_1 \subset \mathbb{R}^n$, $C_2 \subset \mathbb{R}^n$ be compact sets homeomorphic to each other. Then $\mathbb{R}^n \setminus C_1$ and $\mathbb{R}^n \setminus C_2$ has the same number of connected components, and both numbers are either finite or countably infinite.

*Proof.* Let $h : C_1 \rightarrow C_2$ be a homeomorphism. According to Dugundji’s theorem, $h$ has a continuous extension $\tilde{h}$ over $\mathbb{R}^n$, and $h^{-1}$ has a continuous extension $\tilde{h}^{-1}$ over $\mathbb{R}^n$. Let $K_j$ denote the bounded components of $\mathbb{R}^n \setminus C_1$ and $L_i$ the bounded components of $\mathbb{R}^n \setminus C_2$. They are open in $\mathbb{R}^n$ and such that

$$
\partial K_j \subset C_1,
\tilde{h} (\partial K_j) = h (\partial K_j) \subset h (C_1) = C_2
\mathbb{R}^n \setminus C_2 \subset \mathbb{R}^n \setminus \tilde{h} (\partial K_j) = \mathbb{R}^n \setminus h (\partial K_j),
$$

and similar relations for the $L_i$. Let $j$ be fixed and let us denote by $G_q$ the components of $\mathbb{R}^n \setminus h (\partial K_j)$. Clearly,

$$
\bigcup_i L_i = \mathbb{R}^n \setminus C_2 \subset \mathbb{R}^n \setminus h (\partial K_j) = \bigcup_q G_q.
$$

Now (21.6) implies that, for any $i$ there is some $q$ such that $L_i \subset G_q$. Indeed, let $x \in L_i$; it follows from (21.6) that $x \in G_q$ for some $q$. Now, $L_i$ is a connected component of $\mathbb{R}^n \setminus C_2 \subset \mathbb{R}^n \setminus h (\partial K_j)$, so that $L_i$ is connected in $\mathbb{R}^n \setminus h (\partial K_j)$. On the other hand, $G_q$ is a connected component of $\mathbb{R}^n \setminus h (\partial K_j)$, and $x \in L_i \cap G_q$. We deduce that $L_i \cap G_q$ is connected in $\mathbb{R}^n \setminus h (\partial K_j)$, and the maximality of $G_q$ implies that $L_i \subset G_q$. In particular, the unique unbounded component $L_\infty$ of $\mathbb{R}^n \setminus C_2$ is included in the unique unbounded component $G_\infty$ of $\mathbb{R}^n \setminus h (\partial K_j)$. Let $z \in K_j$ be arbitrarily chosen. Since $K_j$ is open in $\mathbb{R}^n$, $\tilde{h}^{-1} h |_{\partial K_j} = I_{\partial K_j}$ and $z \in K_j$, the Brouwer degree $d_B [\tilde{h}^{-1}, K_j, z]$ is well defined and, using Corollary 3.4.3, we get

$$
$$

On the other hand, using Theorem 21.1.1, one has

$$
1 = d_B [\tilde{h}^{-1} h, K_j, z] = \sum_q d_B [\tilde{h}, K_j, G_q] \cdot d_B [h^{-1}, G_q, z].
$$

(21.7)
Let us fix \( q \) arbitrarily and estimate \( d_B[\tilde{h}^{-1}, G_q, z] \). Let \( x \in G_q \) be a possible solution of equation \( \tilde{h}^{-1}(x) = z \). There exists some \( i \) such that \( x \in L_i \) and then \( L_i \subset G_q \). Such an \( x \) cannot be in \( C_2 \), because, if it would be the case, then
\[
K_j \ni z = \tilde{h}^{-1}(x) = h^{-1}(x) \in C_1,
\]
a contradiction. \( K_j \) being a connected component in \( \mathbb{R}^n \setminus C_1 \), we conclude that \( x \in \mathbb{R}^n \setminus C_2 = \bigcup_i L_i \), so that, for some \( i, x \in L_i \subset G_q \). Let \( N_q := \{i : L_i \subset G_q \} \). Since the sets \( L_i \) are open and pairwise disjoint, Theorem 3.4.1 (excision property) and Theorem 3.4.3 (additivity property) give
\[
d_B[\tilde{h}^{-1}, G_q, z] = \sum_{i \in N_q} d_B[\tilde{h}^{-1}, L_i, z],
\]
which, together with (21.7), give
\[
1 = \sum_q d_B[\tilde{h}, K_j, G_q] \cdot d_B[\tilde{h}^{-1}, L_i, z].
\]
(21.8)
Now, for any \( i \in N_q, L_i \) is connected and included in \( G_q \). Consequently,
\[
d_B[\tilde{h}, K_j, G_q] = d_B[\tilde{h}, K_j, L_i],
\]
and (21.8) can be rewritten as
\[
1 = \sum_q \sum_{i \in N_q} d_B[\tilde{h}, K_j, L_i] \cdot d_B[\tilde{h}^{-1}, L_i, z]
= \sum_q \sum_{i \in N_q, L_i \subset C_q} d_B[\tilde{h}, K_j, L_i] \cdot d_B[\tilde{h}^{-1}, L_i, K_j],
\]
(21.9)
since \( z \in K_j = \mathbb{R}^n \setminus C_1 = \mathbb{R}^n \setminus h^{-1}(C_2) \subset \mathbb{R}^n \setminus h^{-1}(\partial L_i) \). Now, since the \( L_i \) are pairwise disjoint, the same is true for the \( G_q \), and since, to any \( L_i \) there exists a \( G_q \) such that \( L_i \subset G_q \), (21.9) can be rewritten as
\[
1 = \sum_i d_B[\tilde{h}, K_j, L_i] \cdot d_B[\tilde{h}^{-1}, L_i, K_j]
\]
(21.10)
for every \( j \). Repeating the above reasoning with fixed \( L_i \) instead of \( K_j \), we obtain
\[
1 = \sum_j d_B[\tilde{h}, K_j, L_i] \cdot d_B[\tilde{h}^{-1}, L_i, K_j]
\]
(21.11)
for every \( i \). Assume now that there are \( m \) components \( L_i \). Then
\[
m = \sum_{i=1}^m 1 = \sum_{i=1}^m \left[ \sum_j d_B[\tilde{h}, K_j, L_i] \cdot d_B[\tilde{h}^{-1}, L_i, K_j] \right] = \sum_j \left[ \sum_{i=1}^m d_B[\tilde{h}, K_j, L_i] \cdot d_B[\tilde{h}^{-1}, L_i, K_j] \right] = \sum_j 1,
\]
which shows that there are also \( m \) components \( K_j \). The same reasoning can be made assuming that there are a finite number of components \( K_j \). We conclude that \( \mathbb{R}^n \setminus C_1 \) and \( \mathbb{R}^n \setminus C_2 \) either have the same finite number of connected components, or both have countably infinitely many connected components. ■
Chapter 22

Invariance of domain and applications

22.1 Invariance of domain

Let \( D \subset \mathbb{R}^n \) be open and bounded, \( f : D \to \mathbb{R}^n \) continuous, and \( z \in \mathbb{R}^n \setminus f(\partial D) \). Corollary 3.4.7 has interesting consequences for one-to-one continuous mappings.

We first prove a preliminary result.

**Lemma 22.1.1** Let \( \Omega \subset \mathbb{R}^n \) be an open neighborhood of 0 and \( f : \Omega \to \mathbb{R}^n \) be continuous and such that \( f(0) = 0 \). If \( f \) is one-to-one on some closed ball \( B(R) \subset \Omega \), then \( f(\Omega) \) is a neighborhood of 0 = \( f(0) \).

**Proof.** The injectivity of \( f \) on \( B(R) \) implies that \( d_{\mathbb{B}}[f, B(R), 0] \) is well defined because \( f(x) \neq 0 = f(0) \) for each \( x \in \partial B(R) \). In order to apply Lemma 3.4.7, let us show that \( d_{\mathbb{B}}[f, B(R), 0] \neq 0 \).

Define the homotopy \( H : \mathbb{B}(R) \times [0, 1] \to \mathbb{R}^n \) by
\[
H(x, \lambda) = f \left( \frac{1}{1 + \lambda} x \right) - f \left( \frac{-\lambda}{1 + \lambda} x \right) \quad (x \in \mathbb{B}(R), \ \lambda \in [0, 1]).
\]

Notice that \( H(x, 0) = f(x) \) and \( H(x, 1) = f \left( \frac{x}{\lambda} \right) - f \left( -\frac{x}{\lambda} \right) \) is odd. Now, \( H(x, \lambda) \neq 0 \) for any \((x, \lambda) \in \partial B(R) \times [0, 1]\), because, if \( H(x, \lambda) = 0 \) for such a \((x, \lambda)\), then
\[
f \left( \frac{1}{1 + \lambda} x \right) = f \left( \frac{-\lambda x}{1 + \lambda} \right),
\]
and, \( f \) being one-to-one on \( \mathbb{B}(R) \), this gives
\[
\frac{1}{1 + \lambda} x = \frac{-\lambda x}{1 + \lambda}
\]
and hence \( x = 0 \), a contradiction. Using the homotopy invariance property and Borsuk’s theorem 20.1.1, we get
\[
d_B[f, B(R), 0] = d_B[H(\cdot, 0), B(R), 0] = d_B[H(\cdot, 1), B(R), 0] = 1 \quad (mod \ 2).
\]
Thus, from Lemma 3.4.7, \( f(\Omega) \supset f(B(R)) \) is a neighborhood of 0.

We can now prove the important **invariance of domain theorem**.

**Theorem 22.1.1** If \( \Omega \subset \mathbb{R}^n \) is open, and \( f : \Omega \to \mathbb{R}^n \) is continuous and one-to-one, then \( f(\Omega) \) is open.

**Proof.** We show that \( f(\Omega) \) is a neighborhood of each of its points. If \( y_0 \in f(\Omega) \) and \( x_0 \in \Omega \) is such that \( f(x_0) = y_0 \), let us define \( \tilde{\Omega} = \Omega - x_0 \) and \( \tilde{f} : \tilde{\Omega} \to \mathbb{R}^n \) by

\[
\tilde{f}(x) = f(x) - f(x_0).
\]

The assumptions of Lemma 22.1.1 hold for \( \tilde{f} \) on \( \tilde{\Omega} \), and \( f(\tilde{\Omega}) \) is a neighborhood of 0. Hence,

\[
f(\Omega) = f(\tilde{\Omega} + x_0) = \tilde{f}(\tilde{\Omega}) + f(x_0) = \tilde{f}(\tilde{\Omega}) + y_0
\]

is a neighborhood of \( y_0 \).

The invariance of domain theorem implies the **invariance of dimension**.

**Corollary 22.1.1** If \( m < n \), there is no continuous and locally one-to-one mapping \( f : \mathbb{R}^n \to \mathbb{R}^m \).

**Proof.** Suppose such a map \( f \) exists, and define \( g : \mathbb{R}^m \to \mathbb{R}^n \) by

\[
g(x) = (f(x), 0_{\mathbb{R}^{n-m}}).
\]

Theorem 22.1.1 implies that \( g(\mathbb{R}^m) \subset \mathbb{R}^m \times \{0_{\mathbb{R}^{n-m}}\} \) is open in \( \mathbb{R}^n \), a contradiction.

This last result can be completed by some information about the range of a one-to-one continuous mapping \( f : \mathbb{R}^n \to \mathbb{R}^m \) when \( m < n \).

**Proposition 22.1.1** Let \( n < m \) be positive integers and \( f \in C(\mathbb{R}^n, \mathbb{R}^m) \) be one-to-one. Then \( \mathbb{R}^m \setminus f(\mathbb{R}^n) \) is dense in \( \mathbb{R}^m \).

**Proof.** It is sufficient to prove that, for any positive \( k \in \mathbb{N} \), \( \mathbb{R}^m \setminus f(I_k^m) \) is dense in \( \mathbb{R}^m \), where \( I_k = [-k, k] \). Indeed, assuming that \( \mathbb{R}^m \setminus f(I_k^m) \) is dense in \( \mathbb{R}^m \), it follows from \( \mathbb{R}^m = \bigcup_{k=1}^{\infty} I_k^m \) that \( \mathbb{R}^m \setminus f(\mathbb{R}^n) = \bigcap_{k=1}^{\infty} \mathbb{R}^m \setminus f(I_k^m) \). As each \( f(I_k^m) \) is compact, \( \mathbb{R}^m \setminus f(I_k^m) \) is open and, according to our assumption, dense in \( \mathbb{R}^m \). From Baire’s theorem, \( \bigcap_{k=1}^{\infty} (\mathbb{R}^m \setminus f(I_k^m)) \) is dense in \( \mathbb{R}^m \).

Let \( k \in \mathbb{N} \) be positive and assume that \( \mathbb{R}^m \setminus f(I_k^m) \) is not dense in \( \mathbb{R}^m \). Then there exists \( x_0 \in \mathbb{R}^m \) and \( r_0 > 0 \) such that \( \overline{B}_{x_0}(r_0) \subset f(I_k^m) \). Setting \( K := f^{-1}(\overline{B}_{x_0}(r_0)) \) and \( g = f|_K \), we see that \( g : K \to \overline{B}_{x_0}(r_0) \) is a homeomorphism, as well of course as \( g^{-1} : \overline{B}_{x_0}(r_0) \to K \). Now, \( h : \overline{B}(1) \subset \mathbb{R}^m \to \overline{B}_{x_0}(r_0) \) defined by \( h(u) = ru + x_0 \) is a homeomorphism, and the same is true for \( g^{-1} \circ h : \overline{B}(1) \subset \mathbb{R}^m \to K \subset \mathbb{R}^n \). This contradicts Borsuk-Ulam’s theorem 20.2.1 according to there exists \( x^* \in \partial B(1) \subset \mathbb{R}^m \) such that \( g^{-1} \circ h(x^*) = g^{-1} \circ h(-x^*) \).
22.2 A surjectivity result for one-to-one coercive mappings

Another consequence of the invariance of domain theorem is a surjectivity result for some one-to-one continuous coercive mappings.

Proposition 22.2.1 Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be continuous, locally one-to-one and coercive, i.e. such that
\[
\|f(x)\| \to +\infty \quad \text{as} \quad \|x\| \to +\infty. \tag{22.1}
\]
Then \( f \) is onto.

Proof. From Theorem 22.1.1, \( f(\mathbb{R}^n) \) is open. We now show that \( f(\mathbb{R}^n) \) is closed. Let \( (y_n) \) be a sequence in \( f(\mathbb{R}^n) \) converging to \( y_0 \). If \( x_n \) is such that \( f(x_n) = y_n \) \((n \in \mathbb{N})\), then (22.1) implies that \( (x_n) \) is bounded. Going if necessary to a subsequence, we can assume that \( x_n \to x_0 \), which implies that \( f(x_n) \to f(x_0) \), so that \( f(x_0) = y_0 \). Since \( \mathbb{R}^n \) is connected and \( f(\mathbb{R}^n) \) is both open and closed, \( f(\mathbb{R}^n) = \mathbb{R}^n \). \( \blacksquare \)

We can compare Corollary 22.2.1 with Corollary 8.4.1: no injectivity condition is required in Corollary 8.4.1, but the strong coercivity condition (8.6) is stronger than the coercivity condition (22.1) in Corollary 22.2.1.

22.3 Degree of one-to-one mappings

We now prove that the Brouwer degree of a one-to-one continuous map on an open bounded open set with respect to any point of its image has absolute value one.

Theorem 22.3.1 Let \( D \subset \mathbb{R}^n \) be an open bounded set and \( f : D \to \mathbb{R}^n \) be continuous and one-to-one. Then, for any \( z \in f(D) \), one has \( d_B[f, D, z] = \pm 1 \).

Proof. First, the one-to-one character of \( f \) and condition \( z \in f(D) \) imply \( z \notin f(\partial D) \), so that \( d_B[f, D, z] \) is well defined. By the invariance of domain theorem 22.1.1, \( f(D) \) is open and \( f^{-1} : f(D) \to D \) is a homeomorphism. Let \( r > 0 \) be such that \( \overline{B}_z(r) \subset f(D) \). We denote by \( \tilde{f} \) the continuous extension of \( f \) to \( \mathbb{R}^n \). Since \( \tilde{f} \circ f^{-1} = I \) on \( \partial B_z(r) \), we have
\[
d_B[\tilde{f} \circ f^{-1}, B_z(r), z] = d_B[I, B_z(r), z] = 1. \tag{22.2}
\]
Since \( \partial B_z(r) \) and \( f^{-1}(\partial B_z(r)) \) are homeomorphic, it follows from Jordan-Brouwer theorem 21.4.1 that \( \mathbb{R}^n \setminus f^{-1}(\partial B_z(r)) \) has two open connected components \( U_1 \) and \( U_2 \), with, say, \( U_1 \) bounded and \( U_2 \) unbounded. By Leray’s product theorem 21.1.1, it follows from (22.2) that
\[
1 = \sum_{i=1}^{2} d_B[f^{-1}, B_z(r), U_i] \cdot d_B[\tilde{f}, U_i, z]. \tag{22.3}
\]
As \( f^{-1}(B_z(r)) \) is bounded, \( U_2 \) contains points \( p \notin f^{-1}(B_z(r)) \), and hence
\[
d_B[f^{-1}, B_z(r), U_2] = d_B[f^{-1}, B_z(r), p] = 0.
\]
Consequently, (22.3) can be rewritten as
\[
1 = d_B[f^{-1}, B_z(r), U_1] \cdot d_B[\tilde{f}, U_1, z],
\]
which implies in particular that \( d_B[f^{-1}, B_z(r), U_1] \neq 0 \). Consequently,
\[
U_1 \subset f^{-1}(B_z(r)) \subset D \setminus f^{-1}(\partial B_z(r)) \subset \mathbb{R}^n \setminus f^{-1}(\partial B_z(r)).
\]
Since \( B_z(r) \) is connected and \( f^{-1} \) continuous, it follows from (22.5) that
\[
U_1 = f^{-1}(B_z(r)) \subset D. \text{ Consequently, (22.4) becomes}
\]
\[
1 = d_B[f^{-1}, B, f^{-1}(B_z(r))] \cdot d_B[f, f^{-1}(B_z(r)), z].
\]
From this last equality, we deduce that
\[
d_B[f, f^{-1}(B_z(r)), z] = \pm 1.
\]
Finally, we conclude the proof by observing that
\[
d_B[f, f^{-1}(B_z(r)), z] = d_B[f, D, z],
\]
a consequence of the excision theorem 3.4.1, taking in account that
\[
z \in f(f^{-1}(B_z(r))) = B_z(r),
\]
and, from the one-to-one character of \( f, z \notin f(\overline{D} \setminus f^{-1}(B_z(r))) \).

**22.4 Deformation of an elastic body**

We now give, following Ph. Ciarlet [62], an application of Brouwer degree, and in particular of the invariance of domain theorem 22.1.1 to mathematical elasticity. We first recall some used fundamental analytical tools.

**Proposition 22.4.1** Let \( X \) and \( Y \) be topological spaces such that \( X \) is compact and \( Y \) is separated. If \( f : X \to Y \) is bijective and continuous, then \( f \) is a homeomorphism.

If \( X \) and \( Y \) are two normed spaces, let \( \text{Isom}(X,Y) \), or simply \( \text{Isom}(X) \) when \( X = Y \), denote the set of all continuous linear bijective mappings \( A : X \to Y \) with continuous inverse \( A^{-1} : Y \to X \). The following result is a classical consequence of the implicit function theorem.

**Proposition 22.4.2** Let \( X,Y \) be Banach space, \( \Omega \subset X \) open and \( f \in C^1(\Omega, Y) \) a mapping such that \( f'(x) \in \text{Isom}(X,Y) \) for each \( x \in \Omega \). Then \( f(\Omega) \) is open in \( Y \). If, in addition, \( f : \Omega \to Y \) is one-to-one, then \( f : \Omega \to f(\Omega) \) is a \( C^1 \)-diffeomorphism.
We also need the following simple consequences of the invariance of domain theorem 22.1.1.

**Proposition 22.4.3** Let $A \subset \mathbb{R}^n$ have nonempty interior $\text{int } A$ and $f : A \to \mathbb{R}^n$ be continuous. Then

(i) if $f$ is locally one-to-one, $f(\text{int } A) \subset \text{int } f(A)$;

(ii) if $f$ is one-to-one, $f(\text{int } A) = \text{int } f(A)$.

**Proof.** (i) Let $y \in f(\text{int } A)$, so that $y = f(x)$ for some $x \in \text{int } A$, and let $r > 0$ be such that $B_x(r) \subset A$. It follows that $y = f(x) \in f[B_x(r)] \subset f(A)$. Since $f$ is locally one-to-one, we may assume that $f$ is one-to-one on $B_x(r)$. By the invariance of domain theorem 22.1.1, $f(B_x(r))$ is open and, consequently, $y \in \text{int } f(A)$.

(ii) it suffices to prove that $\text{int } f(A) \subset f(\text{int } A)$. Let $y \in \text{int } f(A)$ and $r > 0$ such that $B_y(r) \subset f(A)$. Let $x \in A$ be such that $y = f(x) \in B_y(r) \subset f(A)$. Then, $x \in f^{-1}[B_y(r)] \subset f^{-1}[f(A)] = A$, since $f$ is one-to-one. By the continuity of $f$, $f^{-1}[B_y(r)]$ is open. Consequently, $x \in \text{int } A$ and $y = f(x) \in f(\text{int } A)$.

Let us recall some elementary facts concerning the deformation of an elastic body. We assume that an origin 0 and an orthonormal basis $\{e^1, e^2, e^3\}$ have been chosen in the three-dimensional Euclidian space, which is therefore identified with the space $\mathbb{R}^3$.

Let $\Omega \subset \mathbb{R}^3$ be a domain (i.e. a bounded, open, connected subset) with a sufficiently smooth boundary. We shall think of its closure $\overline{\Omega}$ as representing the volume occupied by a body ‘before it is deformed’, and therefore called the **reference configuration**.

**Definition 22.4.1** A deformation of the reference configuration $\overline{\Omega}$ is a vector field $\varphi : \overline{\Omega} \to \mathbb{R}^3$, where $\varphi$ is sufficiently smooth, one-to-one (except possibly at $\partial \Omega$) and orientation-preserving.

The **orientation-preserving** character, i.e.

$$\varphi \in C^1(\overline{\Omega}, \mathbb{R}^3) \text{ and } J_\varphi(x) > 0 \text{ for all } x \in \overline{\Omega}$$

and the interior injectivity are two properties that the deformation $\varphi$ must possess in order to be physically acceptable as a deformation of an elastic body (see [62], section 5.4). The expression ‘smooth enough’ means that, in a given definition, theorem, proof, etc., the smoothness of the involved deformations is such that all used arguments make sense. The reason why a deformation may lose the injectivity on the boundary $\partial \Omega$ of $\Omega$ is that ‘self-contact’ must be allowed (see [62], Chapter 5, for more details).

Let us remark that, from the implicit function theorem, an orientation-preserving mapping is locally invertible, but local invertibility does not entail, in general, injectivity. As an example, consider a rectangular rod $\overline{\Omega} \subset \mathbb{R}^3$ of length $2\pi$ contained
in \( \{ x \in \mathbb{R}^3 : x_1 > 0 \} \), with sides of the basis parallel to \( e^1 \) and \( e^2 \), symmetrical with respect to the \((e^1, e^2)\)-plane, and the mapping \( \varphi : \overline{\Omega} \rightarrow \mathbb{R}^3 \) defined by

\[
\varphi(x_1, x_2, x_3) = (x_1 \cos x_2, x_1 \sin x_2, x_3).
\]

Then \( J_\varphi(x) = x_1 > 0 \) for all \( x \in \overline{\Omega} \), yet \( \varphi \) is not one-to-one, since \( \varphi(x_1, \pi, x_3) = \varphi(x_1, -\pi, x_3) \), i.e. the injectivity is lost on the boundary. The deformation \( \varphi \) has transformed the rectilinear rod into an annular one. This example shows that it is not unnecessary to discuss the injectiveness of orientation-preserving mappings.

### 22.5 Injectivity of some orientation-preserving mappings

The following result provides a class of one-to-one orientation-preserving mappings.

**Theorem 22.5.1** Let \( \Omega \subset \mathbb{R}^n \) be bounded, open, connected and such that \( \operatorname{int} \overline{\Omega} = \Omega \). Let \( \varphi_0 : \overline{\Omega} \rightarrow \mathbb{R}^n \) be continuous and one-to-one, and let \( \varphi \in C(\overline{\Omega}, \mathbb{R}^n) \cap C^1(\Omega, \mathbb{R}^n) \) be such that \( J_\varphi(x) > 0 \) for all \( x \in \Omega \), \( \varphi(x) = \varphi_0(x) \) for all \( x \in \partial \Omega \).

Then

- **a)** \( \varphi(\Omega) = \varphi_0(\Omega) \), \( \varphi(\overline{\Omega}) = \varphi_0(\overline{\Omega}) \);
- **b)** \( \varphi : \overline{\Omega} \rightarrow \varphi(\overline{\Omega}) \) is a homeomorphism (in particular \( \varphi \) is one-to-one on \( \overline{\Omega} \));
- **c)** \( \varphi : \Omega \rightarrow \varphi(\Omega) \) is a \( C^1 \)-diffeomorphism.

**Proof.**

1. **Step 1:** \( \varphi_0(\Omega) \subset \varphi(\Omega) \), and, for any \( p \in \varphi_0(\Omega) \), there is a unique \( x \in \Omega \) such that \( p = \varphi(x) \).

   Let \( p \in \varphi_0(\Omega) \). By Theorem 22.3.1, \( d_B[\varphi_0, \Omega, p] = \pm 1 \), and, since \( \varphi|_{\partial \Omega} = \varphi_0|_{\partial \Omega} \), it follows from Corollary 3.4.3 that

\[
d_B[\varphi, \Omega, p] = d_B[\varphi_0, \Omega, p] = \pm 1 \tag{22.6}
\]

and hence, by the existence theorem 3.4.1, \( \varphi^{-1}(p) \neq \emptyset \). Now, condition \( J_\varphi(x) > 0 \) implies that

\[
d_B[\varphi, \Omega, p] = \sum_{x \in \varphi^{-1}(p)} \operatorname{sign} J_\varphi(x) = \# \varphi^{-1}(p) \geq 1. \tag{22.7}
\]

By comparing (22.6) and (22.7), we obtain

\[
d_B[\varphi, \Omega, p] = 1 = \# \varphi^{-1}(p).
\]

Consequently, \( p \in \varphi(\Omega) \) and equation \( \varphi(x) = p \) has a unique solution.

2. **Step 2:** \( \varphi(\Omega) \subset \varphi_0(\Omega) \).
We prove, equivalently, that \( \mathbb{R}^n \setminus \varphi_0(\overline{\Omega}) \subset \mathbb{R}^n \setminus \varphi(\overline{\Omega}) \). Let \( p \in \mathbb{R}^n \setminus \varphi_0(\overline{\Omega}) \). Hence \( d_B[\varphi_0, \Omega, p] = d_B[\varphi, \Omega, p] = 0 \), and hence, from the fact that \( d_B[\varphi, \Omega, p] = \# \varphi^{-1}(p) \) if \( p \in \varphi(\Omega) \), we have \( p \notin \varphi(\Omega) \). Now it follows from \( p \notin \varphi_0(\overline{\Omega}) \) that \( p \notin \varphi(\partial \Omega) = \varphi(\partial \Omega) \), and hence that \( p \notin \varphi(\Omega) \cup \varphi(\partial \Omega) = \varphi(\Omega) \).

**Step 3 :** \( \varphi_0(\overline{\Omega}) = \varphi_0(\Omega) \).

Indeed, since \( \varphi_0(\Omega) \subset \varphi_0(\overline{\Omega}) \), and \( \varphi_0(\overline{\Omega}) \) is compact, it follows that \( \overline{\varphi_0(\Omega)} \subset \varphi_0(\Omega) \). Conversely, let \( y \in \varphi_0(\overline{\Omega}) \) and \( x \in \Omega \) such that \( y = \varphi_0(x) \), and let \( (x_n) \) in \( \Omega \) such that \( x_n \to x \). Then \( \varphi_0(x_n) \to \varphi_0(x) = y \), and hence \( y \in \varphi_0(\Omega) \).

To sum up, we have established the following inclusions

\[
\varphi_0(\Omega) \subset \varphi(\Omega) \subset \varphi(\overline{\Omega}) \subset \varphi_0(\Omega),
\]

which gives, by taking the closures,

\[
\overline{\varphi_0(\Omega)} = \varphi_0(\overline{\Omega}) = \varphi(\overline{\Omega}) = \varphi(\Omega),
\]

and, using Proposition 22.4.3,

\[
\text{int } \varphi(\Omega) = \text{int } \varphi_0(\overline{\Omega}) = \varphi_0(\text{int } \overline{\Omega}) = \varphi_0(\Omega).
\]

On the other hand, Proposition 22.4.2 implies that \( \varphi(\Omega) \) is open. Since \( \text{int } \overline{\varphi(\Omega)} \) is the largest open set contained in \( \overline{\varphi(\Omega)} \), it follows that \( \varphi(\Omega) \subset \text{int } \overline{\varphi(\Omega)} \subset \varphi_0(\Omega) \).

Together with the result in Step 1, we obtain \( \varphi_0(\Omega) = \varphi(\Omega) \).

b) It suffices to prove that \( \varphi \) is one-to-one on \( \overline{\Omega} \), and then Theorem 22.1.1 applies. Indeed, according to the result established in Step 1, for any \( p \in \varphi_0(\Omega) \), equation \( \varphi_0(x) = p \) has a unique solution. Since \( \varphi_0(\Omega) = \varphi(\Omega) \), and \( \varphi \) is one-to-one on \( \Omega \), the same is true for equation \( \varphi(x) = p \), and \( \varphi \) is one-to-one on \( \Omega \). Since \( \varphi = \varphi_0 \) on \( \partial \Omega \) and \( \varphi_0 \) is one-to-one on \( \overline{\Omega} \), \( \varphi \) is one-to-one on \( \partial \Omega \). Finally, let \( x \in \partial \Omega \) and \( y \in \Omega \) and let us show that \( \varphi(x) \neq \varphi(y) \). Indeed, \( \varphi(x) = \varphi_0(x) \), \( \varphi(y) \in \varphi(\Omega) = \varphi_0(\Omega) \), and therefore there exists \( z \in \Omega \) such that \( \varphi(y) = \varphi_0(z) \). Consequently \( x \neq z \), so that, using the injectivity of \( \varphi_0 \) on \( \overline{\Omega} \), \( \varphi(x) = \varphi_0(x) \neq \varphi_0(z) = \varphi(y) \).

c) The result follows from Proposition 22.4.1. \( \blacksquare \)

**Corollary 22.5.1** Let \( D \subset \mathbb{R}^n \) be open and bounded, let \( u : \overline{D} \to \mathbb{R}^n \) and \( \varphi = I + u \).

(a) If \( u \) is strictly non-expansive, i.e. such that

\[
||u(x) - u(y)|| < ||x - y|| \quad \text{for each } \quad x \neq y \quad \text{in } \overline{D}.
\]

then, \( \varphi \) is one-to-one on \( \overline{D} \), and, for any \( z \in \varphi(D) \),

\[
d_B[\varphi, D, z] = \pm 1.
\]

(b) If \( D \) is convex, \( u \in C^1(\overline{D}, \mathbb{R}^n) \) and

\[
\sup_{x \in \overline{D}} ||u'(x)|| < 1,
\]

then \( u \) is strictly non-expansive.
(c) If \( D \subset \mathbb{R}^n \) is a domain, there exists a constant \( c(D) > 0 \) such that any mapping \( u \in C(\overline{D}, \mathbb{R}^n) \cap C(D, \mathbb{R}^n) \) satisfying condition

\[
\sup_{x \in \overline{D}} \| u'(x) \| < c(D)
\]

is strictly non-expansive.

**Proof.** (a) It suffices to prove that \( \varphi = I + u \) is one-to-one on \( \overline{D} \) and the second part follows from Theorem 22.3.1. Let \( x \neq y \) in \( \overline{D} \), then

\[
\| \varphi(x) - \varphi(y) \| = \|(x - y) + [u(x) - u(y)]\| \geq \| x - y \| - \| u(x) - u(y) \| > 0.
\]

(b) Let \( x \neq y \) in \( \overline{D} \). Using the mean value inequality, we have, for some \( \tau \in [0, 1] \),

\[
\| u(x) - u(y) \| \leq \sup_{z \in \overline{D}} \| u'(z) \| \| x - y \|,
\]

and (22.10) implies that \( u \) is strictly non-expansive.

(c) The proof is essentially based on the remark (see [62], p. 52, exercice 1.9) that a domain \( D \subset \mathbb{R}^n \) has the following geometrical property: there exists a number \( c(D) \) such that, given arbitrary points \( x \neq y \) in \( \overline{D} \), there exists a finite number of points \( y_k (1 \leq k \leq l + 1) \) such that

\[
y_1 = x, \quad y_{k+1} = y, \quad y_k \in D \quad (2 \leq k \leq l),
\]

\[
\| y_k - y_{k+1} \| \leq \frac{1}{c(D)} \| x - y \|. \tag{22.11}
\]

Now, if \( x \neq y \) in \( \overline{D} \) and \( \sup_{z \in \overline{D}} \| u'(z) \| < c(D) \), we have, using the mean value inequality, for some \( \tau_k \in [0, 1] \) \( (1 \leq k \leq l) \),

\[
\| u(x) - u(y) \| = \left\| \sum_{k=1}^{l} [u(y_{k+1}) - u(y_k)] \right\|
\]

\[
\leq \sum_{k=1}^{l} \| y'_k (y_k + \tau_k (y_{k+1} - y_k)) \| \| y_{k+1} - y_k \|
\]

\[
< c(D) \sum_{k=1}^{l} \| y_{k+1} - y_k \| \leq \| x - y \|.
\]

**Remark 22.5.1** Since the hypotheses of points (b) and (c) entail the fulfilment of condition 22.8 (and, consequently the injectivity of \( \varphi \) on \( \overline{D} \)), it follows, using the result given by (a) that formula (22.9) holds in both situations.
Chapter 23

A history of Brouwer fixed point theorem

23.1 The discovery and the publication

The statement of Brouwer’s fixed point theorem concludes an article on continuous mappings between manifolds [39] by Luitzen Egbertus Jan Brouwer, published July 25 1911 and dated from Amsterdam, July 1910. This work developed a previous paper [38], sent by the same author to the same journal one month before, issued February 14 1911, and giving the first correct proof of the non-existence of any homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^p$ when $p > m$ [77, 85, 186, 184]. To this aim, Brouwer implicitly introduced the concept of topological degree $d_B[g, M, N]$ (the future Brouwer degree) of a continuous mapping $g$ between two oriented, compact, boundaryless manifolds $M$ and $N$ having the same finite dimension, and the technique of simplicial approximation.

Brouwer developed this topological degree in the second paper (the statement of the main properties of degree is his Theorem 1), and his first application concerned the study of vector fields on a sphere, and in particular, his Theorem 2, the so-called hairy ball theorem:

Any continuous vector field on a sphere having even dimension has at least one singular point.

Brouwer then computed, in his Theorem 3, the topological degree of mappings without fixed point of a sphere into itself:

The topological degree of a continuous mappings without fixed point of a $n$-dimensional sphere into itself is equal to $(-1)^{n+1}$,

which easily led him to a fixed point theorem on the sphere, already given by him in 1909 for $n = 2$ [37]:

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Any continuous mapping of a $n$-dimensional sphere into itself, having a topological degree different from $(-1)^{n+1}$, has a fixed point.

Brouwer then used a rather complicated argument to deduce, without further comment, his **fixed point theorem for a ball**, whose assertion avoids any explicit reference to topological degree. The ideas of Brouwer’s proof are the following ones.

If $h$ denotes a homeomorphism from the Northern hemisphere $(x_{n+1} \geq 0)$ $S^n$ of the sphere $S^n$, then $f_1 = h^{-1} \circ f \circ h : S^n \to S^n$ is continuous. Brouwer extends $f_1$ to $S^n$ by setting $f_1(s(x)) = f_1(x)$ for each symmetrical image $s(x)$ of $x$ with respect to the equator plane $\{x_{n+1} = 0\}$. If $f_2 : S^n \to S^n$ denotes this extension, $f_2$ has a fixed point fixed point due to Theorem 3, because, $f_2$ which is not onto, has degree zero and hence different from $(-1)^{n+1}$. This fixed point needs to belong to $S^n$ and its image by $h$ is a fixed point of $f$. Brouwer does not give in [39], and will never give, any application of his fixed point theorem for the ball.

Brouwer’s paper [39] contains few references, all dealing with his earlier work, with the exception of the following preliminary note:

During the printing of this article, has appeared in volume two of the ‘Introduction à la théorie des functions d’une variable’ of J. Tannery, a note by J. Hadamard, ‘Sur quelques applications de l’indice de Kronecker’. The theory developed there meets in various ways the following developments.

What is the meaning of this note?

In 1910, Jules Tannery published the second volume of the second edition of his book ‘Introduction à la théorie des fonctions de variables réelles’ [381], for the time a very modern introduction to analysis, having introduced Weierstrass’ rigor in France. This volume two ends with a Note of Jacques Hadamard, connected in the following way to Tannery’s book contents:

The proof, following M. Ames, of Jordan’s theorem on closed curves without double point is based upon the concept of order of a point or, equivalently on the consideration of the variation of the argument. The generalization to the case where the dimension is larger than two is given by Kronecker index. It is a now classical notion, mainly since the publication of the Traité d’Analyse of Mr. Picard (T. I, p. 123; T. II, p. 193). It has received new applications in various recent works. My aim is to present here some of them. All the following reasonings [...] only use the continuity of the considered functions.

Kronecker index was introduced and developed in 1869 by Leopold Kronecker [222], for systems of $n+1$ $C^1$ real functions of $n$ real variables $g_0, g_1, \ldots, g_n$, such that 0 is a regular value of $g_0$, $K := g_0^{-1}([ -\infty, 0])$ is bounded, and the $g_j$ ($j = 1, \ldots, n$) do not vanish together on $\partial K$. If we set $g = (g_1, \ldots, g_n)$, Kronecker shows that the number $\chi[g_0, g]$ defined (in modern notations) by the integral

$$\frac{1}{\text{vol } S^{n-1}} \int_{\partial K} \sum_{j=1}^{n} (-1)^{j-1} \frac{g_{j} d g_1 \wedge \ldots \wedge d g_{j-1} \wedge d g_{j+1} \wedge \ldots \wedge d g_{n}}{(g_1^2 + \ldots + g_n^2)^{n/2}},$$

(23.1)
is equal to \( \sum_{x \in g^{-1}(0)} J_g(x) \), when this sum exists, i.e. when \( g^{-1}(0) \) is finite and the Jacobian \( J_g \) of \( g \) does not vanish on \( g^{-1}(0) \). Consequently, **Kronecker’s integral** (1) provides an ‘algebrical count’ of the number of zeros of \( g \) located in \( K \), named **Kronecker index**. The special case where \( n = 2 \) was already considered by Augustin Cauchy, in terms of the rotation of \( g \) along the close curve \( \partial K \), when he extended to regular planar mappings his index theory for holomorphic functions. Kronecker’s work was motivated by the papers of Sturm, Liouville, Cauchy, Hermite and Sylvester on the number of real roots of algebraic equations [281], and by the work of Gauss on potential theory [146] and the linking of curves [147]. In [162], Hadamard extended Kronecker index to the case of \( g \) continuous on an oriented manifold \( K \). The concept of oriented manifold is defined in a precise way, and the continuous mapping is approximated through a polyhedral approximations. This generalized index \( \chi[\partial K, g] \) is nothing but Brouwer degree \( \deg_{\partial K, \|g\|} [g/\|g\|, \partial K, S^{n-1}] \), defined in a different way, of the mapping \( g/\|g\| : \partial K \to S^{n-1} \). One can find on p. 472 of [162] the following statement:

The previous considerations also allow to prove, for the volume \( V \) of the sphere in any dimensions, **Brouwer’s theorem** : 

\[
\text{Any perfect and continuous mapping of the volume } V \text{ into itself leaves at least one point invariant, either inside, or on the boundary.}
\]

To prove the result, assuming without loss of generality that \( f : V \to V \) has no fixed point on the sphere \( \partial V \), Hadamard noticed that \( \langle f(x) - x, x - f(x) \rangle > 0 \) for any \( x \in \partial V \), and he had shown previously that this implied \( \chi[\partial V, I - f] = \chi[\partial V, I] = 1 \), and hence the existence of a zero of \( I - f \) in \( V \). When he applied Kronecker index to the proof of the unpublished results of Brouwer on vector fields and mappings of a sphere into itself, Hadamard only mentioned Brouwer through his 1909 paper [37],

How can one explain that Brouwer’s fixed point theorem was already published and named more than one year before the publication of Brouwer’s paper? This has been explained by H. Freudenthal [133], using documents discovered when publishing the topological part of Brouwer’s Complete Work. Freudenthal found the copies of two letters, sent at the very beginning of January 1910, respectively to David Hilbert, and to Hadamard, and written in Paris, where Brouwer spent Seasons holidays with his brother, a geologist, living on 6, rue de l’Abbé de l’Epée. In those letters, Brouwer announced the main results of his future paper [39] dealing with fixed points of mappings of a sphere or of a ball into itself:

\[
\text{Any single-valued (not necessarily one-to-one) continuous transformation of the volume of a } m \text{-dimensional sphere into itself has at least one invariant point.}
\]

Hence Brouwer’s fixed point theorem has a famous godfather, Hadamard. We show now that it has also famous ancestors.
23.2 Early equivalent formulations

In 1883 (Brouwer is two years old), Henri Poincaré was studying the existence of symmetric periodic solutions of the three body problem. To do so, he shows that their initial conditions must satisfy some system of $n$ equations in $n$ unknowns. To show that a solution exists, Poincaré generalizes as follows Bolzano’s intermediate value theorem:

Let $\xi_1, \xi_2, \ldots, \xi_n$ be $n$ continuous functions of $n$ variables $x_1, x_2, \ldots, x_n$; the variable $x_i$ is required to vary between the limits $+a_i$ and $-a_i$.

Assume that, for $x_i = a_i$, $\xi_i$ is constantly positive, and for $x_i = -a_i$ constantly negative; I claim that there exist a system of values $x_1, \ldots, x_n$ for which all the $\xi_i$ vanish.

This result is announced in 1883 in a note to the Comptes-Rendus [320] and developed in 1884 in an article published in the first volume of the Bulletin astronomique [321]. In both cases, the proof given by Poincaré was more than sketched; for example, he wrote in [321]:

Mr. Kronecker has given, in the Monatsberichte (1869), a formula providing the number of the solutions of $n$ equations in $n$ unknowns, satisfying some given inequalities. We will make the following application of this formula.

Probably because most readers of [321] are astronomers, very likely unaware of Kronecker’s work, Poincaré added the following commentary:

To help understanding how one can prove this theorem, assume that we only have two variables $x_1$ and $x_2$, seen as the coordinates of a point in a plane. [...] This point is inside some square $ABCD$. [...] The curve $\xi_2 = 0$ joins then any point of the side $AB$ to some point of $CD$; similarly the curve $\xi_1 = 0$, joining a point of $BC$ to some point of $DA$, must meet the first curve inside the square.

Notice that Kronecker index is only defined for functions $\xi_j$ of class $C^1$ and that Poincaré did not explain how the extension to continuous functions was made. Notice that Weierstrass’ approximation theorem of a continuous function through polynomials only dates from 1885!

It is easy to see that Poincaré’s theorem applied to the function $g = I - f$ gives Brouwer’s theorem on the closed $n$-interval $\prod_{i=1}^{n}[-a_i, a_i]$, because Brouwer’s assumptions imply that $x_i - f_i(x) \leq 0$ when $x_i = -a_i$ and $x_i - f_i(x) \geq 0$ when $x_i = a_i$. But the remark is anachronic, Brouwer’s theorem having still to wait some thirty years before being stated!

In his further important work on periodic solutions, and in particular his memoir crowned by King Oscar’s Prize and developed in the book ‘Méthodes nouvelles de la mécanique céleste’, Poincaré replaces the use of his intermediate value theorem by the more classical implicit function theorem, sufficient for the considered problems. That may explain why Poincaré’s theorem has been rapidly forgotten. One of
the few mathematicians to notice and mention it is Hadamard, the great connoisseur and follower of Poincaré’s work. In his analysis of Poincaré’s mathematical contributions published in 1921 [163], Hadamard writes, p. 263:

To obtain the periodic solutions, the memoir of 1883 uses Kronecker’s theorem. This theorem is essentially the natural generalization, to the case of system of equations in several unknowns, of the most elementary existing method to detect the roots of a single equation, the one based upon sign changes of its left-hand member.

Hadamard returns to the subject in 1920, in a conference delivered at the Rice Institute [164], p. 163-164:

The first paper on the subject [of periodic solutions] is devoted by Poincaré to the construction of a simple class of periodic trajectories. He again applies the remarkable theorem of Kronecker on systems of equations already alluded to. As a matter of fact, however, this theorem is not necessary for the conclusions stated in this case: they all derived from the classic theorem on the existence of the solution for a system of $n$ ordinary equations in $n$ unknowns, the Jacobian of which does not vanish. Poincaré’s argument may even be said to imply a new proof of that theorem, which indeed would be of only mediocre interest if Poincaré had not met with a curious intermediate proposition by which the existence of the required solutions can be asserted inside a certain cube whenever we know that each of the left-hand members of the given equations has contrary signs on two of its opposite faces.

It is somewhat surprising that such a penetrating mind as Hadamard, who knew both Poincaré’s and Brouwer’s theorems, did not see their connection, which, as shown later, will only be made in 1941!

In 1904, the Latvian mathematician Piers Bohl published a long paper on the asymptotic behavior of the trajectories of a mechanical system near an equilibrium [31]. To study the behavior of those trajectories, Bohl stated and proved some results on systems of $n$ functions of $n$ variables, defined and continuous on some closed $n$-interval $K = \prod_{i=1}^{n} [-a_i, a_i]$ of $\mathbb{R}^n$:

I. There is no continuous mapping $g : K \to \mathbb{R}^n$ without zeros and equal to identity on $\partial K$.

II. If $g : K \to \mathbb{R}^n$ is continuous and equal to identity on $\partial K$, then, $g(K) \supset K$ and $g(\text{int } K) \supset \text{int } K$.

As a consequence of those results, Bohl proved the following theorem:

III. If $g : K \to \mathbb{R}^n$ is continuous and does not vanish, there exists $u \in \partial K$ and $\mu < 0$ such that $g(u) = \mu u$.

He also proved the corresponding result for a ball $B$ in $\mathbb{R}^n$ centered at the origin:
IV. If \( g : B \to \mathbb{R}^n \) is continuous and does not vanish, there exists \( u \in \partial B \) and \( \mu < 0 \) such that \( g(u) = \mu u \).

Bohl’s proofs are essentially based upon the fact that the integral (23.1) vanishes when the functions \( g_1, \ldots, g_n \) do not vanish simultaneously in \( K \). Bohl does not quote Kronecker, but only the first volume of Émile Picard’s ‘Traité d’analyse’ of [317] mentioned above by Hadamard, and describing Kronecker’s results in dimensions two and three (vol. I, p. 83-87, 123-127, vol. II, p. 193-207). Like in Poincaré’s paper, the used tools require the functions to be of class \( C^1 \) instead of continuous, a fact also ignored by Bohl.

The contraposition of Bohl’s Theorem provides a special case of Poincaré’s theorem, not quoted by Bohl. The same theorem easily implies the non-retraction theorem for a closed \( n \)-interval:

There exists no continuous mapping \( r : K \to \partial K \) such that \( r \) is equal to identity on \( \partial K \).

But this remark is not explicit in Bohl, and the statement of a variant of Theorems I to III with \( K \) replaced by \( B \) is missing as well. In 1931, in his fundamental paper on the theory of retracts [35], Karel Borsuk deduced from Brouwer’s fixed point theorem the following result:

If \( E \subset \mathbb{R}^n \) is compact and \( g : E \to \mathbb{R}^n \) is continuous and such that \( g(x) = x \) for all \( x \in \partial E \), then \( E \subset g(E) \).

In the special case where \( E \) is the closed \( n \)-interval \( K \), this is nothing but Bohl’s theorem II, that Borsuk does not quote.

Recall that if \( F \subset E \) are Hausdorff topological spaces, \( r : E \to F \) is called a retraction of \( E \) onto \( F \) if \( r \) is continuous on \( E \) and equal to the identity on \( F \). Borsuk’s theorem immediately implies the theorem of non-retraction of a ball onto its sphere:

There exists no continuous mapping \( r : B \to \partial B \) such that \( r \) is the identity on \( \partial B \).

Indeed, if we take \( E = B \), such a mapping should be such that \( \partial B = r(B) \supset B \).
Conversely, this non-retraction theorem implies Brouwer’s fixed point theorem on \( B \). A direct proof is given in [7]. Kakutani [191] has shown in 1943 that, in an infinite-dimensional Hilbert space,

The surface \( S = \{ x \| x \| : 1 \} \) of \( B \) is a retract in \( B \), i.e., there exists a continuous mapping \( x' = \psi(x) \) of \( B \) onto \( S \) such that \( \psi(x) \equiv x \) on \( S \).

The contraposition of Bohl’s theorem IV applied to the function \( g = I - f \), with \( f : B \to \mathbb{R}^n \) continuous, immediately gives a fixed point theorem for a ball \( B \) centered at the origin:

If \( f : B \to \mathbb{R}^n \) is continuous and \( f(x) \neq \nu x \) for any \( x \in \partial B \) and any \( \nu > 1 \), \( f \) has a fixed point in \( B \).
23.2. EARLY EQUIVALENT FORMULATIONS

Brouwer’s fixed point theorem is a special case of this result (again not explicited by Bohl who nowhere deals with fixed points), as well as the following extension, usually and erroneously called, as we will see later, Rothe’s fixed point theorem, by reference to its extension to Banach spaces, given in 1937 by Erich Rothe [335]:

If \( f : B \to \mathbb{R}^n \) is continuous and \( f(\partial B) \subset B \), \( f \) has a fixed point in \( B \).

The fixed point version of Bohl’s Theorem IV can also be written, replacing \( \nu \) by \( 1/\lambda \):

If \( f : B \to \mathbb{R}^n \) is continuous and \( x \neq \lambda f(x) \) for any \( x \in \partial B \) and any \( \lambda \in [0,1] \), \( f \) has a fixed point in \( B \).

This is a special case of Leray-Schauder’s fixed point theorem, stated and proved in 1934 by Jean Leray et Jules Schauder [251] (who do not quote Bohl), in the more general frame of compact perturbations of identity in a Banach space. Brouwer, Rothe and Leray-Schauder’s theorems are indeed equivalent because, as shown essentially by Helmut Schaefer [345] in 1955, Leray-Schauder’s theorem follows from Brouwer’s theorem (see Chapter 8).

If Bohl’s paper did not attract the attention of many mathematicians, he did not escape to the attention of Hadamard, who quotes it in [162], through a result that he named Poincaré-Bohl’s theorem:

Let \( S \) be a compact, boundaryless, oriented manifold of dimension \( n-1 \) and let \( g : S \to \mathbb{R}^n \) and \( h : S \to \mathbb{R}^n \) be continuous mappings which do not vanish. If \( \chi[S,g] \neq \chi[S,h] \), there exists at least one \( x^* \in S \) such that

\[
\frac{g_1}{h_1} = \frac{g_2}{h_2} = \ldots = \frac{g_n}{h_n} < 0.
\]

If \( \chi[S,g]/\chi[S,h] \neq (-1)^n \), there exists at least one \( x^* \in S \) such that

\[
\frac{g_1}{h_1} = \frac{g_2}{h_2} = \ldots = \frac{g_n}{h_n} > 0.
\]

The given reference to Poincaré is [323], and Hadamard notices that

the assertions given by the two quoted authors are different and differ from the one given here, in that they particularize (each in his own way) the choice of the functions \( g \). But their reasonings (which, by the way, are different) easily extend to the case where those functions are arbitrary. The proof we give here is Mr. Poincaré’s one.

The contraposition of this result tells that if \( \langle g(x), h(x) \rangle \geq 0 \) for any \( x \in S \), then \( \chi[S,g] = \chi[S,h] \). Hadamard uses the special case of this result with \( g = I - f \), \( h = I \) and \( S = \partial B \) to prove in [162] Brouwer’s fixed point theorem. It seems that Bohl’s results passed to the posterity only through this ‘Poincaré-Bohl’s theorem’ (for example in the influential book ‘Topologie’ of Alexandroff and Hopf [6]). This is historically questionable, as Bohl never explicitely used any concept of index or degree.
Consequently, Brouwer’s fixed point theorem is an easy, but not explicated, consequence of Poincaré or Bohl’s theorems. We now return to the work following Brouwer’s paper.

23.3 New proofs and infinite-dimensional extensions

In 1922, two new proofs of Brouwer’s fixed point theorem were published in the United States. James W. Alexander’s one [5] only differs from Brouwer’s one by the use of Kronecker index instead of Brouwer degree. The one of George D. Birkhoff and Oliver D. Kellogg [28] is more original and based upon a continuation method applied to a family of equations

\[ x - \lambda p(x) = 0, \quad \lambda \in [0, 1], \]

with \( p : \mathbb{R}^n \to \mathbb{R}^n \) a polynomial, and upon the approximation of the continuous mapping \( f \) through such polynomials. Birkhoff and Kellogg proved the following version of Brouwer’s fixed point theorem :

Let \( R_n \) denote a bounded connected region of real \( n \)-space containing an interior point \( O \) (the origin for a set of rectangular coordinates \( x_1, x_2, \ldots, x_n \)) such that every half-ray originating in \( O \) contains but one boundary point of \( R_n \). Let \( T \) denote a one-valued and continuous transformation \( x'_1 = f_1(x_1, x_2, \ldots, x_n), \ x'_2 = f_2(x_1, x_2, \ldots, x_n), \ldots, \ x'_n = f_n(x_1, x_2, \ldots, x_n) \), or briefly, \( x' = f(x) \), which transforms each point of \( R_n \) into a point of \( R_n \). Then there exists a point \( a \) of \( R_n \) which is invariant under \( f \), i.e. such that \( a = f(a) \).

In a note at the bottom of p. 100, the authors notice that

The theorem is stated and proved in a degree of generality sufficient for our later purposes. It will be seen that the proof holds if merely the boundary points of \( R_n \) are transformed into points of \( R_n \), and the theorem admits the obvious extension to any region susceptible of continuous one to one mapping on a region \( R_n \) of the kind described.

It is, apparently, the first explicit appearance of the statement of the so-called Rothe’s fixed point theorem followed, in 1926, by a note of Rolin Wavre and A. Bruttin [399] (presented par Hadamard), dealing with the special case where \( n = 2 \), which does not mention Birkhoff and Kellogg.

The main aim of Birkhoff and Kellogg’s paper was to extend Brouwer’s fixed point theorem to the function spaces \( C([a, b]) \) and \( L^2(a, b) \), and to apply those extensions to differential equations, and in particular to the existence of a solution to boundary value problems of the form

\[ y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}), \]
23.4. A COMBINATORIAL APPROACH

\[
\int_0^a \sum_{j=0}^{n-1} p_{ij}(x)y^{(j)}(x) \, dx + \sum_{j=0}^{n-1} \sum_{k=1}^m q_{ijk}y^{(j)}(x_k) = c_i,
\]

\((i = 1, 2, \ldots, n; 0 \leq x_1 \leq x_2 \leq \ldots \leq x_m \leq a).\)

Those problems are reduced to a nonlinear integral equation, whose right-hand member defines transformation of \(C([0, a])\) into itself. A similar result was rediscovered in 1930 by Renato Caccioppoli [50], and also applied to some nonlinear differential equations. The extension of Brouwer’s fixed point theorem to a compact convex subset of an arbitrary Banach space, unifying those results, was obtained the same year by Schauder [348], and nowadays called **Schauder’s fixed point theorem**:

*Any continuous mapping \(f : C \to C\) of a compact convex subset \(C\) of a Banach space \(E\) has at least one fixed point.*

In 1927 already, Schauder [346, 347] had given a slightly less general version and had applied it to the Cauchy problem for ordinary differential equations, to some nonlinear Dirichlet problems, and to hyperbolic equations.

The extension of Schauder’s fixed point theorem to locally convex vector spaces \(E\) was made by Andrei N. Tychonoff [391] in 1935. It partly answer problem 54 raised by Schauder in the ‘Scottish book’ [260]. Brouwer’s fixed point theorem remains true in some not locally convex topological vector spaces, but the validity of its extension to an arbitrary topological vector space is still under discussion (see [312]).

In 1943, Shizuo Kakutani [191] gave an example of a fixed-point free continuous mapping of the closed unit ball of an infinite-dimensional Hilbert space into itself. Using his extension theorem, James Dugundji [96] characterized in 1951, the finite-dimensional normed vector spaces, in terms of the fixed point property:

*Let \(L\) be a normed linear space, and \(S = \{x||x|| \leq 1\}\). A necessary and sufficient condition that every continuous \(f : S \to S\) have a fixed point is that \(S\) be compact.*

In 1955, Victor L. Klee [202] answered in the following way Problem 36 raised by Stanislaw Ulam in the ‘Scottish book’ [260]:

*For any non compact, closed convex subset \(K\) of a Banach space, there exists a continuous mapping \(f : K \to K\) without fixed points.*

23.4 A combinatorial approach

In 1928, Emanuel Sperner [371] based a simplified proof of Brouwer’s invariance of dimension theorem upon a combinatorial lemma which is named to-day **Sperner’s lemma** [372]:
Consider a simplicial division of the \( n \)-simplex \( S \) in sub-simplices \( s_1, s_2, \ldots, s_p \). To each vertex of any \( s_k \), let us attach an integer between 0 and \( n \) in such a way that if the vertex belongs to the convex hull of the vertices \( p_{i_0}, \ldots, p_{i_k} \) of \( S \), the integer must belong to \( \{i_0, \ldots, i_k\} \). Then the number of sub-simplices whose vertices are numbered by \( \{0, 1, \ldots, n\} \) is odd.

In the special case where \( S \) is the triangle made of the convex hull of three not colinear points \( A, B, C \), any point \( P \in S \) can be written in a unique way as \( P = \alpha_1 A + \alpha_2 B + \alpha_3 C \), for some \( \alpha_1, \alpha_2, \alpha_3 \geq 0 \) and such that \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \). Thus one can write as well \( P = (\alpha_1, \alpha_2, \alpha_3) \), and the \( \alpha_i \) are the barycentric coordinates of \( P \). In particular, \( A = (1, 0, 0), B = (0, 1, 0) \) and \( C = (0, 0, 1) \). For any non negative integer \( k \), let \( T_k \) be the regular triangulation \( S \) made of the set of points of \( S \) whose barycentric coordinates are all of the type \( \frac{1}{k+1} \), with \( 0 \leq i \leq k-1 \). Attach in an arbitrary way to each of those points one of the three numbers 0, 1 or 2, with the only restriction that \( A \) (resp. \( B, C \)) receives the number 0 (resp. 1, 2) and that any point of \( AB \) (resp. \( BC, CA \)) can only receive one of the numbers 0 or 1 (resp. 1 or 2, 0 or 2). Sperner’s lemma claims that the number of elementary triangles of \( T_k \) with vertices numbered by the three numbers 0, 1 and 2 is odd (and in particular not zero).

Sperner’s lemma was immediately presented by Witold Hurewicz at a meeting of the Polish Mathematical Society, and Bronislaw Knaster raised the question of deducing Brouwer’s fixed point theorem from it. A few weeks later, Stefan Mazurkiewicz gave a positive answer, unfortunately with an uncomplete proof, soon corrected by Kasimir Kuratowski [231]. The famous paper [203] was born [232, 231], deducing Rothe’s fixed point theorem for a \( n \)-simplex from the following intersection theorem, a consequence of Sperner’s lemma:

Let \( S \) be a simplex of dimension \( n \), with vertices \( p_0, p_1, \ldots, p_n \). If the closed sets \( A_0, A_1, \ldots, A_n \) are such that any simplex of dimension \( k \) \((0 \leq k \leq n)\) \( p_0p_1 \ldots p_k \) is contained in \( A_{i_0} \cup A_{i_1} \cup \ldots \cup A_{i_k} \), then
\[
A_0 \cap A_1 \cap \ldots \cap A_n \neq \emptyset.
\]

A variant of this intersection theorem for a cube was given the same year by Hurewicz [176]:

Let us denote by \( H_i \) (resp. \( H^i \)) \((1 \leq i \leq n)\) the opposed faces \( x_i = 0 \), resp. \( x_i = 1 \), of the unit cube \( K = [0, 1]^n \) and assume that the \( n \) closed sets \( B_1, \ldots, B_n \) are such that, for each \( 1 \leq i \leq n \), there exists two sets \( U_i \) and \( U^i \) open in \( K \setminus B_i \) such that
\[
K \setminus B_i = U_i \cup U^i, \quad H_i \subset U_i, \quad H^i \subset U^i, \quad U_i \cap U^i = \emptyset \quad (i \leq k \leq n).
\]

Then \( B_1 \cap \ldots \cap B_n \neq \emptyset \).

It is easy to deduce Poincaré’s theorem from Hurwitz’ one, but Hurwitz did not do it (and did not refer to Poincaré). The theorem is reproduced in [177], p. 40-41.
Brouwer’s fixed point theorem can also be deduced directly from Sperner’s lemma. Let us return to the simplex $S$ made of the convex hull of points $A, B, C$, and let $f : S \rightarrow S$ be continuous. If $P = (\alpha_1, \alpha_2, \alpha_3)$ is a point of the triangulation $T_k$ and if $f(P) = (\alpha^*_1, \alpha^*_2, \alpha^*_3)$, let us attach to $P$ the number $i - 1$, with $i$ the smallest integer such that $\alpha^*_i \leq \alpha_i$. Such an integer necessarily exists because the sum of the $\alpha_i$ and the sum of the $\alpha^*_i$ are equal to one. It is easy to show that this numbering satisfies the conditions of Sperner’s lemma, and there exists at least one elementary triangle of $T_k$ whose vertices $A_k = (\alpha_{k,1}, \alpha_{k,2}, \alpha_{k,3})$, $B_k = (\beta_{k,1}, \beta_{k,2}, \beta_{k,3})$, and $C_k = (\gamma_{k,1}, \gamma_{k,2}, \gamma_{k,3})$, are respectively numbered by 0, 1 and 2 and hence such that $\alpha^*_{k,1} \leq \alpha_{k,1}$, $\beta^*_{k,1} \leq \beta_{k,1}$, $\gamma^*_{k,1} \leq \gamma_{k,1}$. When $k \rightarrow \infty$, the sequence of those points contains a subsequence converging to an element $P = (\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha^*_1 \leq \alpha_1$, $\alpha^*_2 \leq \alpha_2$, $\alpha^*_3 \leq \alpha_3$. Those inequalities are indeed equalities, because the sum of the terms of each member is equal to one, and $P$ is a fixed point of $f$.

There are many generalizations of Sperner’s lemma and of Knaster-Kuratowski-Mazurkiewicz’s theorem in the literature. The most important ones are due to Ky Fan ([110]-[122]). To-day referred as KKM methods, they have been the source of a large number of papers, for which one can consult, for example [313, 314].

23.5 Game theory and mathematical economics

In 1928, John von Neumann [395] stated and proved a fundamental minimax theorem for the theory of zero-sum 2-person-games with mixed strategies:

Investigate continuous functions of two sets of variables $f(\xi, \eta)$ which satisfy the following conditions (K) : If $f(\xi', \eta) \geq A$, $f(\xi, \eta') \geq A$, $f(\xi', \eta') \geq A$, then $f(\xi, \eta) \geq A$ for every $0 \leq \theta \leq 1$, $\xi = \theta \xi' + (1 - \theta) \xi''$ (i.e. $\xi_p = \theta \xi'_p + (1 - \theta) \xi''_p$, $p = 1, 2, \ldots, M$). If $f(\xi, \eta') \leq A$, $f(\xi, \eta'') \leq A$, then $f(\xi, \eta) \leq A$ for every $0 \leq \theta \leq 1$, $\eta = \theta \eta' + (1 - \theta) \eta''$ (i.e. $\eta_q = \theta \eta'_q + (1 - \theta) \eta''_q$, $q = 1, 2, \ldots, N$). For these functions we are going to prove that $\max_{\xi} \min_{\eta} f(\xi, \eta) = \min_{\eta} \max_{\xi} f(\xi, \eta)$, where $\max_{\xi}$ is taken over the range $\xi_1 \geq 0, \ldots, \xi_M \geq 0, \xi_1 + \ldots + \xi_M \leq 1$ and $\min_{\eta}$ is taken over the range $\eta_1 \geq 0, \ldots, \eta_N \geq 0, \eta_1 + \ldots + \eta_N \leq 1$.

He reduces its proof to an extension of Brouwer’s fixed point theorem to a mapping with closed graph from a compact interval of $\mathbb{R}$ into the set of compact intervals of $\mathbb{R}$.

In a seminar devoted to a model of economical equilibrium, given in 1932 at the University of Princeton and published five years later [396], von Neumann is led to the solution of a system of inequalities, and claims that the existence of a solution cannot be proved by any qualitative argument. The mathematical proof is only possible through a generalization of Brouwer’s fixed point theorem, i.e. through the use of completely fundamental topological facts. This generalization of the fixed point theorem is also interesting in itself.

This von Neumann’s topological lemma, which also provides a proof of the minimax theorem, goes as follows:
Let $R_m$ be the $m$-dimensional space of all points $X = (x_1, \ldots, x_n)$, $R_n$ the $n$-dimensional space of all points $Y = (y_1, \ldots, y_n)$, $R_{m+n}$ the $m + n$-dimensional space of all points 

$$(X, Y) = (x_1, \ldots, x_m, y_1, \ldots, y_n).$$

A set (in $R_m$ or $R_n$ or $R_{m+n}$) which is non-empty, convex, closed and bounded we call a set $C$. Let $S^0, T^0$ be sets $C$ in $R_m$ and $R_n$ respectively and let $S^0 \times T^0$ be the set of all $(X, Y)$ (in $R_{m+n}$) where the range of $X$ is $S^0$ and the range of $Y$ is $T^0$. Let $V, W$ be two closed sets of $S^0 \times T^0$. For every $X$ in $S^0$, let the set $Q(X)$ of all $Y$ with $(X, Y)$ in $V$ be a set $C$; for each $Y$ in $T^0$, let the set $P(Y)$ of all $X$ with $(X, Y)$ in $W$ be a set $C$. Under the above assumptions, $V, W$ have (at least) one point in common.

In collaboration with the economist Oskar Morgenstern, von Neumann developed those ideas in a monumental treatise [397], published in 1944, which will inspire generations of experts in game theory and economics. Motivated by a paper of J. Ville [394], the authors prove the minimax theorem without using fixed point theory:

This theorem occurred and was proved first in the original publication of one of the authors on the theory of games (1928). A slightly more general form of this min-max problem arises in another question of mathematical economics in connection with the equations of production (1937). The proof of our theorem, given in the first paper, made a rather involved use of some topology and of functional calculus. The second paper contained a different proof, which was fully topological and connected the theorem with an important device of that discipline: the so-called ‘fixed point theorem’ of L.E.J. Brouwer. This aspect was further clarified and the proof simplified by S. Kakutani (1941). The first elementary proof was given by J. Ville (1938). The proof we are going to give carries the elementarization initiated by J. Ville further, and seems to be particularly simple. The key of the procedure is, of course, the connection with the theory of convexity.

In 1941, Kakutani [190] showed that von Neumann’s results follow very easily from an extension of Brouwer’s fixed point theorem to multivalued applications. Let $K(E)$ denote the family of all closed convex subsets of a normed vector space $E$. A multivalued or point-to-set application $\Phi$ of $E$ in $K$, i.e. an application $\Phi: E \to K(E)$, is called upper semi-continuous if its graph is closed in $E \times E$. Kakutani’s fixed point theorem goes as follows:

If $x \to \Phi(x)$ is an upper semi-continuous point-to-set mapping of a $r$-dimensional closed simplex $S$ into $K(S)$, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$. [...] Theorem 1 is also valid even if $S$ is an arbitrary bounded closed convex set in a Euclidian space.
Kakutani deduced von Neumann’s intersection theorem from this result, and then von Neumann’s minimax theorem. Extensions of Brouwer’s fixed point theorem to multivalued applications with not necessarily convex values have also been given. See [18, 73, 100, 151, 300] and their references. Even if Kakutani’s contribution is finally not necessary to prove von Neumann’s results, it has played and still plays a basic role in other questions of mathematical economics, as emphasized by the Nobel Prize Gérard Debreu in Kakutani’s Selected Works:

von Neumann’s topological lemma, which, through Kakutani’s corollary, had a major influence in particular in economics and in game theory, was not necessary to prove any of the results that von Neumann wanted to establish. [...] However the formulation given by Kakutani is by far the most convenient to use, and its proof is definitely the most attractive one. One of the first, and of the most important applications of Kakutani’s theorem was made by Nash in his proof of the existence of an equilibrium for a finite game. It was followed by several hundreds of applications in game theory and in economics. In this last area, Kakutani’s theorem has been for more than three decades the main tool to prove the existence of an economical equilibrium.

As mentioned by Debreu, Brouwer and Kakutani’s fixed point theorems have been used in 1951 by John Nash, in his PhD thesis at the University of Princeton, to prove a fundamental theorem on the equilibrium of non-cooperative games [294, 295], for which he will share, in 1994, the Nobel Prize of Economics [224]. Such a \textit{n-person game with mixed strategies} is described by \(n\) simplices \(S^1, \ldots, S^n\) with possible different finite dimensions and \(n\) continuous real functions \(f_i : S^1 \times \ldots \times S^n \to \mathbb{R}\) such that \(p \mapsto f_i(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)\) (1 \(\leq i \leq n\)). An equilibrium for such a game is any \((p_1^*, \ldots, p_n^*) \in S^1 \times \ldots \times S^n\) such that, for each \(i = 1, 2, \ldots, n,\)

\[
f_i(p_1^*, \ldots, p_n^*) = \max_{x \in S^i} f_i(p_1^*, \ldots, p_{i-1}^*, x, p_{i+1}^*, \ldots, p_n^*).
\]

\textbf{Nash’s theorem} states that:

\textit{any non-cooperative \(n\)-person-game has an equilibrium.}

Let us notice that Nash has also invented a game, nowadays called \textit{hex game}, which can also be studied using Brouwer’s fixed point theorem [143].

\subsection{23.6 Poincaré’s theorem: topology or not topology} 

In 1932, Adolf Hammerstein studied the two-point boundary value problem

\[
y''(x) = f(x, y(x), y'(x)), \quad y(a) = A, \quad y(b) = B,
\]

(23.2) when \(|f(x, y, z)| \leq c|y|\) for all \(x \in [a, b]\), all \(y, z \in \mathbb{R}\), and some sufficiently small \(c > 0\). Using Galerkin’s method, he reduced the search of approximate solutions to finding the zeros of \(n\) continuous functions \(F_\chi\) in \(n\) unknowns \(\alpha_\chi\) which:
are positive for \( \alpha_\chi = d, |\alpha_\nu| \leq d, \nu \neq \chi \) and negative for \( \alpha_\nu = -d, |\alpha_\nu| \leq d \).

One recognizes the statement of Poincaré’s theorem, who is not quoted. **Brouwer’s displacement lemma** is an intermediate result introduced in 1911 by Brouwer [38], to prove the invariance of dimension theorem:

*When a q-dimensional cube is transformed in a q-dimensional manifold in such a way that the maximum displacement is smaller than half of the side of the cube, there exists a concentric and homothetic cube contained in the range of the transformation.*

Hammerstein did not precise how to go from Brouwer’s displacement lemma to Poincaré’s theorem. In 1935, without quoting [168], Michael Golomb [150] used the same arguments, with almost the same words, to prove the existence of solutions to some systems of nonlinear integral equations.

In 1940, using a shooting method to study a multi-point boundary value problem for a nonlinear differential equation of order larger than two, Silvio Cinquini [63] was led to prove the existence of a solution of a nonlinear system of \( n \) equations in \( n \) unknowns satisfying, on a hypercube, the conditions of Poincaré’s theorem (that he did not quote). He ‘proved’ it by an argument identical to Poincaré’s heuristic remark (see the notes of p. 66-67 in [63]). Namely, in the simplest case of a third order differential equation

\[
y''''(x) = f(x, y(x), y'(x), y''(x)),
\]

with boundary conditions

\[
y(x_1) = y(x_2) = y(x_3) = 0,
\]

with \( x_1 < x_2 < x_3 \), and \( f \) smooth enough and bounded on \([x_1, x_3] \times \mathbb{R}^4\), and denoting by \( y(x) = y(x; y_1', y_1''') \) the solution of (23.3) with initial conditions \( y(x_1) = 0 \), \( y'(x_1) = y_1', y''(x_1) = y_1'' \), Cinquini showed that there exists \( \lambda_1 > 0 \) such that

\[
y(x_2; y_1', y_1''') > 0 \quad \text{if} \quad y_1' > \lambda_1, \quad y(x_2; y_1', y_1''') < 0 \quad \text{if} \quad y_1' < -\lambda_1,
\]

and deduced from Bolzano’s theorem the existence of \( y_1' \) \((y_1''') \in (-\lambda_1, \lambda_1)\) such that \( y(x_2; y_1'(y_1'''), y_1''') = 0 \). He then showed the existence of some \( \lambda_2 > 0 \) such that

\[
y(x_3; y_1'(y_1'''), y_1''') > 0 \quad \text{if} \quad y_1'' > \lambda_2, \quad y(x_3; y_1'(y_1'''), y_1''') < 0 \quad \text{if} \quad y_1'' < -\lambda_2,
\]

and got in the same way \( y_1'' \) \((y_1''') \in (-\lambda_2, \lambda_2)\) such that \( y(x_3; y_1'(y_1'''), y_1''') = 0 \). Introducing this \( y_1'' \) in \( y_1'(y_1''') \), he obtained the initial conditions of a solution of (23.3)-(23.4).

In his review of [63] for Zentralblatt für Mathematik (vol. 22 (1940), p. 339), Carlo Miranda noticed the lack of validity of this proof, due to the non-uniqueness of the function \( y_1'(y_1''') \). To overcome the difficulty, he proved one year later, in two pages, the equivalence between Poincaré’s theorem (whose existence was also unknown to Miranda) and Brouwer’s fixed point theorem. His paper started as follows:
S. Cinquini, in an exchange of ideas that I had with him some time ago, has informed me how the study of existence questions in boundary value problems for nonlinear differential equations of order \( n \), reduced finally to the validity of the following theorem:

I. If \( F_1, F_2, \ldots, F_n \) are \( n \) functions of the variables \((x_1, x_2, \ldots, x_n)\), continuous in the hypercube \(|x_i| \leq L \ (i = 1, 2, \ldots, n)\) and verifying the inequalities

\[
F_i(x_1, \ldots, x_{i-1}, -L, x_{i+1}, \ldots, x_n) \geq 0, \\
F_i(x_1, \ldots, x_{i-1}, L, x_{i+1}, \ldots, x_n) \leq 0,
\]

there exists at least one solution of the system:

\[
F_i(x_1, x_2, \ldots, x_n) = 0, \quad (i = 1, 2, \ldots, n).
\]

In the special case where \( n = 2 \), S. Cinquini has also indicated to me a proof of this theorem; another proof, still for \( n = 2 \), was communicated to me by G. Scorza. As those proofs do not seem to be easily extendible to the case of an arbitrary \( n \), and as it does not seem that other persons have considered those questions, I propose to show how theorem I can be easily deduced from the famous Brouwer’s fixed point theorem [theorem II]. [...] I will further show how theorem II is in turn a consequence of theorem I.

Miranda did not quote the papers of Hammerstein [168] and of Golomb [150].

But Miranda’s result (quoted in a note p. 134 of [65]) did not completely satisfy Cinquini, who still dreamed of a proof independent of topology for his results on differential equations. In a note on p. 168-169 of a paper of 1941 [64], Cinquini (wrongly) claimed that the argument of [63] was valid for polynomial functions, and that it was sufficient, to obtain the general case, to use Weierstrass’ approximation theorem.

Giuseppe Scorza Dragoni, a student of Caccioppoli who had received earlier some undue criticisms from Cinquini, was convinced of the necessity of using topology to obtain the claimed existence results. He started with Cinquini a long quarrel, feeding some Italian mathematical journals for some ten years. In a note on p. 204 of [355], Scorza Dragoni listed the gaps in Cinquini’s ‘proof’ of [63], called ‘trivial’ the remark reducing the continuous case to the polynomial one given in [64], and observed with an undisguised satisfaction that Miranda’s proof to which Cinquini referred in [65], implied that

\[
\text{in this moment, Cinquini renounces also, implicitly, to avoid the use of topology. [...] The point was easily predictable a priori.}
\]

Cinquini’s statement for polynomial mappings was then proved by Luigi Brusotti [49] in 1942, using arguments from algebraic geometry, which should be reconsidered in the light of present rigor criterions. This work allowed Cinquini, in the introduction of [66], to
answer in a definite way to a few claims contained in a recently published note of G. Scorza-Dragoni [355], and to show in this way that, taking in account the proof of prof. Brusotti [49] [...] all the existence theorems for boundary value problems for ordinary differential equations and systems of such equations (of an arbitrary order) can be obtained by completely avoiding the use of topology.

He expressed more clearly his opinion on p. 220, using this opportunity to scratch Caccioppoli as well:

Not only is it neither necessary to use the functional analytical considerations due to Birkhoff and Kellogg (and rediscovered eighth years later by R. Caccioppoli, who was then forced to recognize the priority of the mentioned authors), nor necessary to use the topology of finite-dimensional spaces.

In this politically difficult period, Cinquini added a little bit of nationalism:

But this is a matter of taste: we are pleased that our modest contribution emphasizes the method of the school of Cesare Arzelà; in contrast, of course, Mr. Scorza-Dragoni prefers to insist in valorizing the ideas of Birkhoff and Kellogg!

Cinquini ended his polemical arguments as follows:

We will not loose any time in the future in answering to Scorza-Dragoni, but we want to notice that, taking in account our observation and Miranda’s result, the Note of Brusotti provides an elementary proof of Brouwer’s theorem.

If the Second World War brought a welcome break in the polemics, nothing was forgotten and, in 1946, a memoir of more than seventy pages of Scorza-Dragoni [356] answered point by point to Cinquini’s note [65]. About Cinquini’s conclusion, Scorza-Dragoni pertinently observed that

the fact that the proof of Brouwer’s theorem can be reduced to the case of a polynomial mapping was already observed by Birkhoff and Kellogg. The proof they gave of Brouwer’s theorem is as elementary as the one Brusotti gave of [Miranda’s] theorem I.

The same year, Scorza-Dragoni published three different proofs of Miranda’s theorem [357, 358, 359], respectively based upon Kronecker index, Birkhoff-Kellog’s method and a variant of Sperner’s lemma, and his student Giuseppe Zwirner gave a fourth one based upon Poincaré-Bohl’s theorem [411]. One can consult[227] for more recent versions.

In front of this avalanche, Cinquini renounced in 1947 to his silence and insisted, on page 183 of [67], upon the fact that

Brusotti’s proof can be understood by anybody (including those ones who do not know at all topology), because it neither uses topological concepts (Kronecker index, etc.) nor topological statements (Poincaré-Bohl’s theorem, etc.).
He mentioned, in a note on p. 176 of the same paper, the earlier use by Golomb [150] of Miranda’s theorem I:

Recently, I have been kindly informed that Proposition C [Miranda’s theorem I] had already been used in: M. Golomb, Zur Theorie der nichtlinearen Integralgleichungen etc. (Math Zeitschrift, Bd 39 (1935) pp. 45-75, see p. 55).

There, Proposition C is mentioned as a consequence of Brouwer’s theorem.

Cinquini ended his paper with the unpublished proof communicated in 1940 to Miranda.

Scorza-Dragoni felt forced to answer to [67] in 1949 [360], because

Cinquini has given new evidence of incorrect polemics: he mutilates in fact the sentences and the pages in an essential way and, after having changed the meaning, he triumphantly answers. Cinquini again gives the impression that his competence in each of the arguments in question has doubtful depth.

Making fun of Cinquini’s confusion between Brouwer’s displacement and Brouwer’s fixed point theorems, Scorza Dragoni noticed that the unpublished proof of Miranda’s theorem for \( n = 2 \) given in [67] just reduced this result to a Lebesgue’s topological lemma in dimension theory.

Cinquini’s reaction [68] was just a new promise to stop answering to Scorza-Dragoni, and a ‘Postilla ad una nota di S. Cinquini’ of Scorza Dragoni [361] registered this missed reply, and definitely ends the quarrel.

Miranda has reproduced his result in 1949, without further comments or references, in his interesting monograph [287]. After having developed, in chapter III, following [6], the (local) topological degree \( d_B[g, \Omega, 0] \) of a continuous mapping \( g : \overline{\Omega} \to \mathbb{R}^n \), when \( \Omega \) is open and bounded, and \( 0 \notin g(\partial \Omega) \), Miranda deduces Brouwer’s fixed point theorem by taking \( \overline{\Omega} = B \), \( g = I - f \) and noticing that \( 0 \notin (I - \lambda f)(\partial B) \) for each \( \lambda \in [0, 1] \). The invariance of degree with respect to homotopy implies that \( d_B[I - \lambda f, \text{int } B, 0] = \deg[I, \text{int } B, 0] = 1 \), which gives the existence of a fixed point \( x_\lambda \in \text{int } B \) of \( \lambda f \) for each \( \lambda \in [0, 1] \). A classical compactness argument implies the existence of a fixed point of \( f \) in \( B \). This may be the first appearance of this now classical proof. In the pages of [287] following his equivalence proof, Miranda tells in a sober and diplomatic way the origin of his contribution, mentions the proofs of Cinquini, Brusotti, Scorza Dragoni and Zwirner and the applications to boundary value problems by Cinquini, Zwirner and Stampacchia, but without including, in his rich bibliography, any of the polemical papers related to the question.

The \( n \)-dimensional generalization of Bolzano’s intermediate value theorem has been generally named Miranda’s theorem, until one of the authors of this book called, in 1973, the attention to Poincaré’s forgotten papers, and reestablished the true priority [264, 48]. One usually speaks now of Poincaré-Miranda’s theorem.

The first direct application of Brouwer’s fixed point theorem to nonlinear differential equations appeared in 1943 in the United States, with a paper of Solomon Lefschetz [245], immediately followed by a more general result of Norman Levinson.
Their aim was to prove, when \( e \) has period \( T \), the existence of a solution of period \( T \) for some differential equations of the type
\[
x'' + f(x)x' + g(x) = e(t),
\]
occuring in various questions of mechanics and electronics. The authors showed the existence of a subset \( C \) of the phase plane \((x, x')\), homeomorphic to a closed disc, and such that any solution \((x(t; x_0, y_0), x'(t; x_0, y_0))\) of (23.5) with initial conditions \(x(0) = x_0, \ x'(0) = y_0\) located in \( C \), remains in \( C \) for all \( t \in [0, T] \). Brouwer’s fixed point theorem applied to Poincaré’s operator
\[
P: C \rightarrow C, \quad (x_0, y_0) \mapsto (x(T; x_0, y_0), x'(T; x_0, y_0))
\]
implies the existence of a fixed point \((x_0^*, y_0^*)\) of \( P \), giving the initial conditions of a solution of period \( T \) for (23.5). When describing those results in 1951, Giovanni Sansone [340] wrote:

Brouwer’s theorem has the highest importance in analysis: its proof and informations about the contributions of S. Cinquini, L. Brusotti, C. Miranda, G. Scorza-Dragoni, G. Zwiener and on its applications to the study of boundary value problems can be found in the excellent monograph of C. Miranda: Problemi di esistenza in analisi funzionale.

He reproduced the proof of Miranda’s equivalence result in his influential treatise on nonlinear differential equations [341], written in collaboration with Roberto Conti.

### 23.7 Variational inequalities

Brouwer’s fixed point theorem is closely related to the main result of the theory of variational inequalities stated and proved in 1966 by Philip Hartman and Guido Stampacchia [169] (see also [37]):

Let \( C \) be a compact convex set in \( E^n \) and \( B(u) \) a continuous map of \( C \) into \( E^n \). Then there exists \( u_0 \in C \) such that \( (B(u_0), v - u_0) \geq 0 \) for \( v \in C \), where \((\cdot, \cdot)\) denotes the scalar product in \( E^n \).

Hartman et Stampacchia reduced this result, except for the trivial case where \( C \) is a singleton, to the case where \( 0 \in \text{int} \, C \). For \( \partial C \) regular, they used topological degree, and reduced the general case, to the regular one using an approximation theorem of Minkowski. When \( C \) is a simplex, Hartman-Stampacchia’s theorem was independently proved in 1966 by Stepan Karamardian [193].

Another proof based upon Brouwer’s fixed point theorem has been given by Haïm Brézis (see [327]): the projection mapping \( r : \mathbb{R}^n \rightarrow C \) is characterized by the relation \((x - r(x), y - r(x)) \leq 0 \) for all \( x \in \mathbb{R}^n, y \in C \). Brouwer’s fixed point theorem applied to the continuous mapping \( f = r \circ (I - g) : C \rightarrow C \) gives a fixed point \( u_0 \in C \). Taking \( x = u_0 - g(u_0) \) in the inequality above, one gets
23.8 APPROXIMATION OF FIXED POINTS

(u_0 - g(u_0) - u_0, y - u_0) \leq 0 \text{ for all } y \in C, \text{ which is just Hartman-Stampacchia's statement.}

The extensions of the theory of variational inequalities to infinite-dimensional spaces have important applications to free boundary, obstacle, infiltration and elastoplastic problems [201]. The theory of variational inequalities is also related to the complementarity problem first considered in 1966 by Richard W. Cottle [75]:

Given a continuous mapping \( g : \mathbb{R}^n_+ \to \mathbb{R}^n \), find \( x^* \in \mathbb{R}^n \) such that \( g(x^*) \in \mathbb{R}^n_+ \text{ et } \langle g(x^*), x^* \rangle = 0 \).

This problem unifies various questions coming from mathematical programming, game theory, economical equilibrium theory, mechanics and geometry. Karamardian [194, 195, 196] and Jorge J. Moré [288] have reduced to complementarity problem to a variational inequality. For more details, see [76].

23.8 Approximation of fixed points

For a long time, it was commonly admitted that Brouwer’s fixed point theorem belonged to non constructive mathematics because it did not provide any algorithm to get the fixed point [16, 200]. This dogma was broken by Herbert Scarf in 1967 [342, 343, 344], when, motivated by some applications to mathematical economics, he proposed, for a continuous mapping \( f : S \to S \) from a simplex into itself, a systematic algorithm to obtain sub-simplices \( \sigma(\delta) \) of diameter \( \delta > 0 \), for which, roughly speaking, \( \lim_{\delta \to 0} \sigma(\delta) = x \) and \( x = f(x) \). Scarf’s method is based upon a variant of Sperner’s lemma. The original simplex is decomposed in a simplicial way and the method provides a path along the vertices of the simplicial decomposition, ending to a fixed point of a piece-wise linear approximation of \( f \). One can illustrate Scarf’s algorithm in the case of the triangulation \( T_k \) of the simplex \( S \) made of the convex hull of the three points \( A, B \) et \( C \) considered previously. If \( (\alpha_1, \alpha_2, \alpha_3) \) are the barycentric coordinates of \( P \), let us associate to \( P \) the smallest \( j \in \{1, 2, 3\} \) such that \( \alpha_j = 0 \) if it exists, and, otherwise, the smallest \( i \in \{1, 2, 3\} \) such that \( \alpha_i^* \geq \alpha_i \). In this way, \( A \) is numbered by 2, \( B \) and \( C \) by 1, the other elements of \( AB \) by 3, of \( BC \) by 1 and of \( AC \) by 2. Consider the elementary triangles of \( T_k \) like rooms, any side numbered 12 like a door and any other side like a wall. With the convention that a door can only be crossed once, starting from the only door opened to the outside of \( S \), and going through the possible elementary triangles having two doors, one necessarily reaches a triangle having only one door, and hence numbered by 1, 2 and 3. The remaining of the reasoning is similar to the proof of Brouwer’s fixed point theorem using Sperner’s lemma, except that, numerically, it is sufficient to obtain a fixed point with an error of \( \epsilon > 0 \), provided by taking \( k \) sufficiently large. This work has inspired, in the last thirty years, under the name of simplicial methods, a large number of variants and generalizations. On can consult [8] and its references.

A different algorithm was proposed in 1974 by R. Kellogg, T. Li et J. Yorke [199] (see also [367]), when \( f : C \to C \) and the compact convex set \( C \) are regular.
The idea is linked to that connecting Brouwer’s fixed point theorem to the non-retraction theorem. For any \( x \) not belonging to the set of fixed points of \( f \) in \( C \), denote by \( r(x) \), the intersection with \( \partial C \) of the half-line issued from \( f(x) \) and going through \( x \). If \( x^0 \) is a regular value of \( r \), the component \( \alpha \) of \( r^{-1}(x^0) \) containing \( x^0 \) is a curve \( s \rightarrow x(s) \) which can be parametrized by its length. This curve \( \alpha \) cannot contain any other extremity than \( x_0 \), and hence there exists \( x^* \in \mathbb{R} \setminus \alpha \) for which \( r \) is not defined, which implies that \( f(x^*) = x^* \). As \( r(x(s)) \equiv x^0 \) for all \( x(s) \in \alpha \), \( x(s) \) is solution of the initial value problem

\[
r'(x(s)) \frac{dx(s)}{ds} = 0, \quad x(0) = x^0, \quad \left< \frac{dx}{ds}(s), \frac{dx}{ds}(s) \right> = 1,
\]
on a maximal interval \([0, s^*], s^* \leq \infty \). An algorithm for the fixed point of \( f \) is obtained by solving approximatively the initial value problem, giving approximations for the singular point \( x^* \) of this problem. Several variants of this continuation method for obtaining a fixed point have been developed, for example the method of homotopic continuation [4, 61].

### 23.9 Elementary proofs and equivalent formulations

Brouwer’s fixed point theorem is a result located at the crossing of several important mathematical disciplines, which can and has been proved using techniques from algebraic topology [2], analysis [97, 132, 297, 302, 327], differential topology [156, 182, 282], combinatorics [105, 153, 230, 231], algebraic geometry [49] and even algebra [316]. It has many equivalent formulations, looking quite different, and numerous and diverse applications, going from algebraic topology (proofs of non-retraction theorems, Jordan separation result [226, 257], invariance of domain [228]), to mathematical biology [401], via geometry of convex bodies [208], differential equations [296, 341], control theory [51, 167, 166], game theory [14, 102], nonlinear programming [288], decision theory [364], combinatorics [134, 135] and mathematical economy [14, 103]. Numerical analysis itself, after a long period of stagnation, is now more than interested [9, 350, 387].

The importance and increasing scope of the applications of Brouwer’s fixed point theorem have motivated the search of elementary proofs. Knaster-Kuratowski-Mazurkiewicz’s one surely fills this criterion, but is based upon combinatorial considerations which can look unusual to some users. One already finds other ‘elementary’ proofs in the years 1930, and Cinquini-Scorza Dragoni’s quarrel is akin to this problematics.

A large number of proofs have been proposed until recently, which are based on more or less complicated results of mathematical analysis. Those proofs are very often ‘elementary’ looking translations of arguments based upon the integration of some differential forms, already implicitly contained in the definition of Kronecker index. Some proofs are based upon the comparison of integrals, as done already by Bohl [154, 192, 225, 283, 332, 97], other ones use the change of variables formula
for multiple integrals [15, 242, 246], differential forms [179, 180, 260, 274, 339, 389],
arguments of differential topology [27, 282] or algebraic topology [33], simplicial
approximations [173] (corrected in [188]), polynomial approximations [223, 363], or
continuation methods [255].

We have seen the equivalence of Brouwer’s fixed point theorem with a number
of other statements. This study can be pushed further, and we refer to [159, 313,
314, 327, 390, 407, 408, 409].
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