Shape dependent controllability of a quantum transistor

Florian Méhats¹, Yannick Privat², and Mario Sigalotti³

Abstract—We investigate the controllability of a quantum electron trapped in a two-dimensional device. The problem is modeled by the Schrödinger equation in a bounded domain coupled to the Poisson equation for the electrical potential. The controller acts on the system through the boundary condition on the potential, on a part of the boundary modeling the gate. We prove that, generically with respect to the shape and boundary conditions on the gate, the device is controllable.

Keywords: nonlinear Schrödinger equation, controllability, genericity, shape deformation

AMS classification: 37C20, 47A55, 47A75, 49K20, 93B05

I. MODELING OF THE PROBLEM

As electronic devices shrink to nanometric scales, the numerical simulation of their operation requires multidimensional quantum transport models in the ballistic regime. It is now possible to build nanostructures, like single electron transistors or single electron memories, which involve the transport of only a few electrons ([7], [19]). Schematically, such devices consist in an active region (called the channel or the island) connecting two electrodes, known as the source and the drain, while the electrical potential in this active region can be tuned by a third electrode, the gate. In many applications, the performance of the device will depend on the possibility of controlling the electrons by acting on the gate voltage.

In this work, we analyze the controllability of a simplified mathematical model for a quantum electron trapped in a two-dimensional device. The problem is modeled by the Schrödinger equation in a bounded domain Ω with homogeneous Dirichlet boundary conditions, coupled to the Poisson equation for the electrical potential. The control on this system is done through the boundary condition on the potential, on a part of the boundary corresponding to the gate. The degrees of freedom of the problem are the shape and the Dirichlet boundary condition on the gate of the device: our aim is to prove that, generically, the device is controllable. Controllability of general control-affine systems driven the Schrödinger equation has been widely studied in the recent years (see in particular [2], [5], [6], [15], [13], [14] and references therein). Our analysis is based on the sufficient condition for approximate controllability obtained in [3], which requires a non-resonance condition on the spectrum of the internal Hamiltonian and a coupling property (the connectedness chain property) on the external control field.

Let $L$ be a positive real number. We assume without loss of generality that $\Omega = (0, \pi) \times (0, L)$ so that, with the notations of Figure 1, one has $\Gamma_D^1 = \{0\} \times [0, L]$, $\Gamma_D^2 = \{\pi\} \times [0, L]$, $\Gamma_N^3 = [0, \pi] \times \{0\}$ and $\Gamma_N^1 \cup \Gamma_N^2 \cup \Gamma_N^3 = [0, \pi] \times \{L\}$. The segment $\Gamma_D^2$ is closed and compactly contained in $(0, \pi) \times \{L\}$. We set $\Gamma_D = \Gamma_D^1 \cup \Gamma_D^2$ and $\Gamma_N = \Gamma_N^1 \cup \Gamma_N^2 \cup \Gamma_N^3$.

In the whole paper, the notation $\frac{\partial}{\partial v}$, when it makes sense, denotes the outward normal derivative.

We focus on the control problem

\begin{align}
    & i \partial_t \psi(t, x) = -\Delta \psi(t, x) + V(t, x) \psi(t, x) & x & \in \Omega \\
    & -\Delta V(t, x) = 0 & x & \in \partial \Omega \\
    & \psi(t, x) = 0 & x & \in \partial \Omega \\
    & V(t, x) = u(t, x) & x & \in \Gamma_D^g \\
    & V(t, x) = 0 & x & \in \Gamma_D^d \\
    & \frac{\partial V}{\partial v}(t, x) = 0 & x & \in \Gamma_N.
\end{align}

We assume that the control $u(t, x)$ has the structure

\begin{equation}
    u(t, x) = \chi(x) V_g(t), \ x \in \Gamma_D^g, t \geq 0,
\end{equation}

with $V_g \in L^\infty([0, \infty), [0, 1])$ and $\chi \in C^2(\Gamma_D^g, \mathbb{R})$. In particular, by Hopf’s lemma, the function $x \mapsto V(t, x)$ is uniformly bounded in $L^\infty(\Omega)$ with respect to $t$. 

Fig. 1. Representation of the transistor

\[ V_s = 0 \]
\[ \Gamma^D_s \text{ grid} \]
\[ V = V_g \]
\[ \Gamma^D_N \]
\[ \Gamma^N \]
\[ \Omega \]
\[ \Gamma^D_d \]
\[ \Gamma^N \]
\[ \text{circuit} \]
Let us consider the following harmonic lift of $V$. Denote by $V_0$ the solution of
\begin{align*}
-\Delta V_0(x) &= 0 & x &\in \Omega \\
V_0(x) &= 1 & x &\in \Gamma_0^D \\
V_0(x) &= 0 & x &\in \Gamma_0^I \\
\partial N V_0(x) &= 0 & x &\in \Gamma_N. 
\end{align*}
(3)
Then, clearly,
\[ V(t, x) = V_g(t)V_0(x), \]
and the original equation (1) can be rewritten as
\begin{equation}
\begin{align*}
i\partial_t \psi(t, x) &= -\Delta \psi(t, x) + V(t, x)\psi(t, x) & x &\in \Omega, \\
\psi(t, x) &= 0 & x &\in \partial \Omega. 
\end{align*}
\end{equation}
(4)
\section{Statement of the Main Result}
Our aim is to prove that generically in a class of perturbed version of (4) the system is approximately controllable.

We should consider perturbations of the original system and endow it with the metric structure inherited by that of $C^m$-diffeomorphisms. The family of problems is then
\[ \mathcal{P} = \{(Q(\Omega), Q(\Gamma_0^D), \chi) \mid (Q(\Omega), Q(\Gamma_0^D)) \in \Sigma_m, \chi \in C^2(Q(\Gamma_0^D)) \}, \]
(7)
whose metric, induced by those of $\Sigma_m$ and $C^2(Q(\Gamma_0^D))$, makes it complete ([12]). In particular, it is a Baire space.

Recall that, given a Baire space $X$, a residual set (i.e. the intersection of countably many open and dense subsets) is dense in $X$. A boolean function $P : X \to \{0, 1\}$ is said to be generic in $X$ if there exists a residual set $Y$ such that every $x \in Y$ satisfies property $P$, that is, $P(x) = 1$.

\textbf{Theorem 1:} For a generic element of $\mathcal{P}$ the control problem (5) is approximately controllable.

The rest of paper is devoted to the proof of Theorem 1.

\section{General Controllability Conditions for Bilinear Quantum Systems}
We recall in this section a general approximate controllability result for bilinear quantum systems obtained in [3].

Let $\mathcal{H}$ be an Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and $A, B$ be two linear skew-adjoint operators on $\mathcal{H}$. Let $B$ be bounded and denote by $D(A)$ the domain of $A$. Consider the controlled equation
\[ \frac{d\psi(t)}{dt} = (A + u(t)B)\psi(t), \quad u(t) \in [0, 1]. \]
(8)
We say that $A$ satisfies assumption (9) if there exists an orthonormal basis $(\phi_k)_{k \in \mathbb{N}}$ of $\mathcal{H}$ made of eigenvectors of $A$ such that $A\phi_k$ is a residual set (i.e. the intersection of countably many open and dense subsets) is dense in $\mathcal{H}$.

\textbf{Definition 2:} A subset $S$ of $\mathbb{N}^2$ couples two levels $j, k$ in $S$ if there exists a finite sequence $\{(s_1^1, s_1^2), \ldots, (s_p^1, s_p^2)\}$ in $S$ such that
\begin{align*}
(i) \quad & s_1^1 = j \quad \text{and} \quad s_p^2 = k; \\
(ii) \quad & s_j^1 = s_{j+1}^1 \quad \text{for every} \quad 1 \leq j \leq p - 1.
\end{align*}

$S$ is called a connectedness chain if $S$ couples every pair of levels in $S$.

$S$ is a non-resonant connectedness chain for $(A, B, \Phi)$ if it is a connectedness chain, $\langle \phi_j, B\phi_k \rangle \neq 0$ for every $(j, k) \in S$, and $\lambda_{s_1} - \lambda_{s_2} \neq \lambda_{t_1} - \lambda_{t_2}$ for every $(s_1, s_2) \in S$ and every $(t_1, t_2)$ in $\mathbb{N}^2 \setminus \{(s_1, s_2)\}$ such that $\langle \phi_{t_1}, B\phi_{t_2} \rangle \neq 0$.

\textbf{Theorem 3 ([3])}: Let $A$ satisfy (9) and let $\Phi = (\phi_k)_{k \in \mathbb{N}}$ be the corresponding orthonormal basis. If there exists a non-resonant connectedness chain for $(A, B, \Phi)$ then (8) is approximately controllable.

Remark 1: Notice that the proof of Theorem 3 given in [3] (see also [5]) is constructive, leading to a control design algorithm based on the knowledge of the spectrum of the operator $A$. Stronger conclusions in Theorem 1 such as approximate controllability in $H^s$ for $s < 2$ could be obtained by applying the results in [4].

Remark 2: A similar result based on a stronger requirement has been proposed in [6]. In that paper, the spectrum of the operator $A$ was asked to be non-resonant, in the sense that every nontrivial finite linear combination with rational coefficients of its eigenvalues was asked to be nonzero.
IV. STRUCTURE OF THE PROOF OF THEOREM 1

The proof is based on the idea of propagating sufficient controllability conditions using analytic perturbations ([9], [11], [17]). This is possible since the general controllability criterion for quantum systems seen in the previous section can be seen as a countable set of nonvanishing scalar conditions. More precisely, let us denote by $\Lambda(Q(\Omega))$ the spectrum of the Laplace–Dirichlet operator on $Q(\Omega)$ and, for every $(Q(\Omega), Q(\Gamma_0^q), \chi) \in \mathcal{P}$ such that $\Lambda(Q(\Omega))$ is simple (i.e., each eigenvalue is simple), define

$$S(Q(\Omega), Q(\Gamma_0^q), \chi) = \left\{(k, j) \in \mathbb{N}^2 \mid \int_{Q(\Omega)} V_0^{Q, x}(x) \phi_k(x) \phi_j(x) dx \neq 0 \right\},$$

where $\{\phi_j\}_{j \in \mathbb{N}}$ is a Hilbert basis of eigenfunctions of the Laplace–Dirichlet operator on $Q(\Omega)$, ordered following the growth of the corresponding eigenvalues.

Then we are going to prove that both sets

$$\mathcal{P}_1 = \{(Q(\Omega), Q(\Gamma_0^q), \chi) \in \mathcal{P} \mid \Lambda(Q(\Omega)) \text{ non-resonant}\},$$

where the notion of non-resonant spectrum is the one introduced in Remark 2, and

$$\mathcal{P}_2 = \{(Q(\Omega), Q(\Gamma_0^q), \chi) \in \mathcal{P} \mid \Lambda(Q(\Omega)) \text{ simple, } S(Q(\Omega), Q(\Gamma_0^q), \chi) \text{ connectedness chain} \}$$

are residual in $\mathcal{P}$, and therefore their intersection is residual as well (it is itself the intersection of countably many open dense sets).

The proof is then organized as follows: in Section V, based on general results in [17], we show that $\mathcal{P}_1$ is residual. In Section VI we prove that $V_0^{Q, x}$ depends analytically on $Q$ and $\chi$ in a suitable sense. The main technical difficulties of the proof are contained in Section VII, where we exhibit a domain $Q(\Omega)$ and a boundary condition $\chi$ guaranteeing the existence of a connectedness chain (i.e., we prove that $\mathcal{P}_2$ is nonempty). The analyticity results of Section VI allow then to conclude that $\mathcal{P}_2$ is residual as well.

V. GENERICITY OF THE NON-RESONANCE CONDITION

Let us discuss in this section the genericity of the non-resonance condition on the spectrum of the Laplace–Dirichlet operator. The results are closely related to those obtained in [17] and will be derived from Theorem 2.3 there. Such derivation also introduces in a simple case the method used in the following sections.

Lemma 4: The set $\mathcal{P}_1$ is residual. Hence, for a generic $Q \in \text{Diff}_0^m$ with $C^2(\Gamma_0^q)$, the spectrum $\Lambda(Q(\Omega))$ is non-resonant.

**Proof:** Let $l \in \mathbb{N}$ and $q = (q_1, \ldots, q_l) \in \mathbb{Q}^l \setminus \{0\}$. Let $L > 0$ be such that $\pi^2 l^2 < L^2$ and consider $Q = (0, \pi) \times (0, L)$. Notice that the $l$ smallest eigenvalues of $-\Delta$ on $Q$ with Dirichlet boundary conditions are $\lambda_j = 1 + j^2 \pi^2 / L^2$, which are simple and whose corresponding eigenfunctions are (up to normalization)

$$\phi_j(x_1, x_2) = \frac{2 \sin(x_1) \sin(j \pi x_2 / L)}{\sqrt{\pi L}},$$

Let $X$ be a $C^m$ vector field with compact support intersecting $\{0\} \times (0, \bar{L})$ but not any other side of $\Omega$. For $t_0 > 0$ small enough and $t \in (-t_0, t_0)$, $I + tX$ is a diffeomorphism between $\Omega$ and its image, which we will denote by $\Omega_t$.

Denote by $\{\lambda_j(t)\}_{j \in \mathbb{N}}$ the spectrum of the Laplace–Dirichlet operator on $\Omega_t$. According to Rellich’s theorem (see [10], [18]), each function $\lambda_j(t)$ can be chosen to be analytic on $(-t_0, t_0)$. Moreover, up to reducing $t_0$, we can assume that $\lambda_1(t), \ldots, \lambda_l(t)$ are simple for $t \in (-t_0, t_0)$.

It is well known that

$$\lambda_j(0) = -\int_{\partial \Omega} \left(\frac{\partial \phi_j}{\partial n}\right)^2 (X \cdot v) dx$$

for every $j$ such that $\lambda_j(0)$ is simple (see, for instance, [8]). Notice that

$$\left(\frac{\partial \phi_j}{\partial n}\right)^2 = \frac{4 \sin^2(j \pi x_2 / \bar{L})}{\pi \bar{L}}$$

on $\{0\} \times (0, \bar{L})$ for $j = 1, \ldots, l$.

Henceforth, since $x_2 \mapsto \sin(j \pi x_2 / \bar{L})$, $j = 1, \ldots, l$, are linearly independent functions on $(0, Q)$ (as it follows from the trigonometric identity $\sin^2 \theta = 1 - \cos(2\theta)/2$ and by injectivity of Fourier series), then we can choose the vector field $X$ in such a way that $\sum_{j=1}^l q_j \lambda_j(0) \neq 0$. Hence, there exists $t \in (-t_0, t_0)$ such that $\sum_{j=1}^l q_j \lambda_j(t) \neq 0$. The lemma then follows from [17, Theorem 2.3].

VI. ANALYTIC DEPENDENCE

As recalled in the previous section, the spectrum $\Lambda(Q(\Omega))$ of the Laplace–Dirichlet operator on $Q(\Omega)$ varies analytically if $t \mapsto Q_t$ is an analytic curve in $\text{Diff}_0^m$. It is also well known that for every $t$ it is possible to chose a orthonormal basis $\Phi(Q_t(\Omega)) = (\phi_j(t))_{j \in \mathbb{N}}$ in $L^2(Q_t(\Omega), \mathbb{R})$ of eigenfunctions of the Laplace–Dirichlet operator on $Q_t(\Omega)$ in such a way that $\phi_j(t) \circ Q_t$ is analytic for every $j \in \mathbb{N}$.

Concerning the analytic dependence of $V_0^{Q, x}$ on $Q$ and $\chi$, we have the following result.

**Proposition 5:** Let $I$ be an open interval and $I \ni t \mapsto (Q_t, \varphi_t)$ be an analytic curve in the product space of $\text{Diff}_0^m$ with $C^2(\Gamma_0^q)$. Denote by $\chi_t$ the composition $\varphi_t \circ Q_t^{-1}$ and by $V_{0,t}$ the function $V_{0,2t}^{Q_t, \chi_t}$ defined as in (6). Then $t \mapsto V_{0,t} \circ Q_t$ is an analytic curve in $H^1(\Omega)$.

**Proof:** We start by noticing that, since $t \mapsto \varphi_t$ is an analytic curve in $C^2(\Gamma_0^q)$, it is possible, by interpolation with polynomials of degree 3, to consider $\varphi_t$ as the restriction on $\Gamma_0^q$ of an analytic curve $t \mapsto \varphi_t$ in $C^3([0, \pi] \times \{L\})$ satisfying $\varphi_t(0, L) \equiv \varphi_t(\pi, L) \equiv 0$. We still use the notation $\varphi_t$ to denote its extension on $\bar{\Omega}$ which is constant on every vertical segment. Then $t \mapsto \varphi_t$ is an analytic curve in $C^2(\Omega)$.

Define $\chi_t = \hat{\chi}_t \circ Q_t^{-1}$ and let $V_{0,t} = V_{0,t} - \chi_t$. Notice that $\hat{V}_{0,t}$ is a solution to the problem

$$
\begin{cases}
-\Delta \hat{V}_{0,t}(x) = \Delta \hat{\chi}_t & \text{in } Q_t(\Omega) \\
V_{0,t}(x) = 0 & \text{in } Q_t(\Gamma_D) \\
\frac{\partial V_{0,t}}{\partial n}(x) = 0 & \text{on } Q_t(\Gamma_N)
\end{cases}
$$

(9)
Equivalently,
\[ \int_{Q_t(\Omega)} \nabla \hat{V}_{0,t}(x) \cdot \nabla \phi(x) \, dx = \int_{Q_t(\Omega)} \Delta \hat{x}_t(x) \phi(x) \, dx, \]
for every \( \phi \in H^1_{0,Q_t(\Gamma_D)}(Q_t(\Omega)), \) where
\[ H^1_{0,Q_t(\Gamma_D)}(Q_t(\Omega)) = \{ \phi \in H^1(\Omega(\Omega)) \mid \phi = 0 \text{ on } Q_t(\Gamma_D) \}. \]

Fix \( t_0 \in I \) and notice that, for every \( t \in I, \phi \) belongs to \( H^1_{0,Q_t(\Gamma_D)}(Q_t(\Omega)) \) if and only if \( \phi \circ Q_{t_0} \circ \hat{R}_t^{-1} \) belongs to \( H^1_{0,Q_t(\Gamma_D)}(Q_t(\Omega)) \). Set \( R_t = Q_t \circ Q_{t_0}^{-1} \).

By the standard change of coordinates formula,
\[ \int_{Q_{t_0}(\Omega)} ((DR_t^1)^{-1} \nabla W_t) \cdot ((DR_t^T)^{-1} \nabla \phi) J_t = \int_{Q_{t_0}(\Omega)} ((\Delta \hat{x}_t \circ R_t) \phi J_t \]
for every \( \phi \in H^1_{0,Q_{t_0}(\Gamma_D)}(Q_{t_0}(\Omega)), \) where \( DR_t \) and \( DR_t^T \) are, respectively, the Jacobian matrix of \( R_t \) and its transpose, while \( W_t = V_0 \circ R_t \) and \( J_t = \det(DR_t). \)

In other words, \( (t, W_t) \) is the solution of \( F(t, W_t) = 0 \) in \( H^1_{0,Q_{t_0}(Q_{t_0}(\Omega))), \) where \( H^1_{0,Q_{t_0}(Q_{t_0}(\Omega))) \) stands for the dual space of \( H^1_{0,Q_{t_0}(Q_{t_0}(\Omega))), \) with respect to the pivot space \( L^2(Q_{t_0}(\Omega)), \) with
\[ F(t, W) = -\text{div}(A_t \nabla W) - (\Delta \hat{x}_t \circ R_t) J_t, \]
where \( A_t = J_t(DR_t)^{-1}(DR_t^T)^{-1}. \)

The analyticity of \( t \to W_t \) follows by the implicit function theorem, since \( F \) is analytic from \( I \times H^1_{0,Q_{t_0}(\Gamma_D)}(Q_{t_0}(\Omega)) \) into \( H^1_{0,Q_{t_0}(Q_{t_0}(\Omega))), \) and the operator \( D_W F(t_0, W_{t_0}) \) is an isomorphism from \( H^1_{0,Q_{t_0}(\Gamma_D)}(Q_{t_0}(\Omega))) \) into \( H^1_{0,Q_{t_0}(Q_{t_0}(\Omega))), \) respectively.

Indeed, by linearity of \( F \) with respect to \( W \) and because \( R_{t_0} \) is the identity, \( D_W F(t_0, W_{t_0})Z \) is nothing else than \( -\Delta Z, \) which is an isomorphism from \( H^1_{0,Q_{t_0}(Q_{t_0}(\Omega))), \) to \( H^1_{0,Q_{t_0}(Q_{t_0}(\Omega))), \) by Lax-Milgram’s lemma.

The analyticity properties mentioned above, together with the results of Section 6, lead to the following corollary, which reduces the proof of Theorem 1 to the search of an element of \( P_2. \)

**Corollary 6:** If \( \mathcal{P}_2 \) is nonempty, then \( \mathcal{P}_2 \) is residual.

**Proof:** Let us first prove that the intersection of countably many open sets is simple and there exist \( r \in \mathbb{N} \) and \( r \) other simple eigenvalues of \( \lambda_{k_1}, \ldots, \lambda_{k_r} \) such that the matrix
\[ \left( \int_{Q(\Omega)} \phi_{j_1} \phi_{j} V_0^{Q-x} \right)_{j,j_1 \in \{1, \ldots, n\}, \{1, \ldots, n\}} \]
is connected, where each \( \phi_{j} \) is an eigenfunction corresponding to \( \lambda_{j}. \) It is clear that an element of \( \cap_{n \in \mathbb{N}} A_{n} \)

\[ \lambda_{k} = k^2 + k_2 \pi^2 / L^2, \quad \phi_k(x) = 2 \frac{\sin(k_1 x_1)}{\sqrt{\pi L}}, \]
is connected, since its corresponding spectrum is simple and a connectedness chain is given by the union of the all the connectedness chains for the matrices of the type (10). Conversely, if \( (\xi_1, \xi(\xi_{(j)})) \in \mathcal{P}_2 \), there exists a bijection \( \xi : \mathbb{N} \to \mathbb{N} \) such that each matrix
\[
\left( \int_{Q(\Omega)} \phi_{\xi(j)} (\xi_{(j)}) V_0^{Q-x} \right)_{j,j \in \{1, \ldots, n\}}
\]
is connected (see [11, Remark 4.2]). Given \( n \in \mathbb{N}, \) let \( N \) be such that \( \xi(\{1, \ldots, N\}) \supset \{1, \ldots, n\}. \) Then, taking \( r = N - n \) and \( \{\lambda_1, \ldots, \lambda_k\} = \{\xi_{(1)}, \ldots, \xi_{(N)}\} \}
\[{\lambda_1, \ldots, \lambda_k}, \] we prove that \( (Q, Q(\Gamma_D^g), \lambda_1) \in A_n. \)

Since each \( A_n \) is open (by continuity of the eigenpairs corresponding to simple eigenvalues), we have proved that \( \mathcal{P}_2 \) is the intersection of countably many open sets.

Let us now show that \( \mathcal{P}_2 \) is dense if it is nonempty. Fix \( (Q(\Omega), Q(\Gamma_D^g), \lambda) \in \mathcal{P}_2. \)

Let \( I \ni t \to (Q(t), \phi(t)) \) be an analytic curve in the product space \( \text{Diff}^m_{\Omega} \times C^2(\Gamma_D^g) \) and assume that there exist \( t_0 \) in \( I \) such that \( Q_{t_0} = Q \) and \( \phi_{t_0} \circ Q = \chi. \)

Let \( (\phi_j(t))_{j \in \mathbb{N}} \) be an orthonormal basis of eigenfunctions of the Laplace–Dirichlet operator in \( L^2(Q(\Omega), \mathbb{R}) \) depending analytically on \( t. \)

Proposition 5 implies that for every \( j, k \in \mathbb{N}, \) the function
\[ t \mapsto \int_{Q(\Omega)} \phi_j(t) \phi_k(t) V_0^{Q-x} \]
is analytic on \( I. \) Moreover, the spectrum \( \Lambda(Q(\Omega)) \) can be written as \( (\lambda_j(t))_{j \in \mathbb{N}} \) where each \( \lambda_j(t) \) is analytic. It follows that \( \Lambda(Q(\Omega)) \) is simple for almost every \( t \in I. \)

We can assume that the sequence \( (\lambda_j(t_0))_{j \in \mathbb{N}} \) is (strictly) increasing. For every \( t \in I \) such that \( \Lambda(Q(\Omega)) \) is simple, there exists \( \xi_t : \mathbb{N} \to \mathbb{N} \) bijective such that \( (\lambda_{\xi(t)}(t))_{j \in \mathbb{N}} \) is increasing. By analyticity of \( t \to \int_{Q(\Omega)} \phi_j(t) \phi_k(t) V_0^{Q-x} \)
for every \( j, k \in \mathbb{S}, \) we have that \( \{\xi^{-1}(j), \xi^{-1}(k) \mid (j, k) \in \mathbb{S}\} \) is contained in \( S(Q(\Omega), Q(\Gamma_D^g), \phi_k \circ Q^{-1}) \) for almost every \( t \in I. \) Since for every bijection \( \xi : \mathbb{N} \to \mathbb{N} \) the set \( \{\xi(j), \xi(k) \mid (j, k) \in \mathbb{S}\} \) is a connectedness chain, we conclude that for almost every \( t \in I, S(Q(\Omega), Q(\Gamma_D^g), \phi_k \circ Q^{-1}) \) is a connectedness chain. Therefore, identifying the pairs of the type \( (Q(t), \phi(t)) \) with elements of \( P \) for every analytic curve \( I \ni t \to (Q(t), \phi(t)) \) intersecting \( \mathcal{P}_2, \) almost every element of the curve is in \( \mathcal{P}_2. \) Hence, \( \mathcal{P}_2 \) is dense.

**VII. Existence of a Connectedness Chain**

We prove in this section that for every \( L > 0 \) such that \( \bar{L} \notin \pi^2 \mathbb{Q}, \) there exist a segment \( \Gamma_D^g \) in \( [0, \pi] \times \bar{L} \) and \( \bar{x} \in C^2(\Gamma_D^g) \) such that \( (\Omega, \Gamma_D^g, \bar{x}) \in \mathcal{P}_2, \) where \( \Omega = [0, \pi] \times [0, L]. \)

Fix \( L \) and \( \Omega \) as above. Let \( Q \in \text{Diff}^m_{\Omega} \) map \( \Omega \) into \( \Omega \) and \( \Gamma_D^g \) into \( \Gamma_D^g. \) The eigenpairs of the Laplace–Dirichlet operator on \( \Omega \) are naturally parameterized over \( \mathbb{N}^2 \) as follows: for every \( k = (k_1, k_2) \in \mathbb{N}^2, \)
\[ \lambda_k = k_1^2 + k_2^2 \pi^2 / L^2, \quad \phi_k(x) = 2 \frac{\sin(k_1 x_1)}{\sqrt{\pi L}}, \]
for every \( k = (k_1, k_2) \in \mathbb{N}^2, \) let
It is useful to notice that the notion of connectedness chain introduced in Definition 2 naturally extends to subsets of $(\mathbb{N}^2)^2$. The property of $(\Omega, \Gamma_D, \tilde{\chi})$ being in $P_2$ is then equivalent to say that
\[
\left\{(j, k) \in (\mathbb{N}^2)^2 \mid \int_{\Omega} V_0^Q \tilde{\chi}(x) \phi_{jk}(x) \phi_j(x) dx \neq 0\right\}
\]
is a connectedness chain.

A. A limit case

In this section, we investigate the existence of a connectedness chain for a limit case not belonging to $P$. The proof will then work by approximating such a limit case by elements of $P$.

Consider the case where $\Gamma_D^1 = \Gamma_D^2 = \emptyset$. Let $n$ denote a positive integer and $\chi_\infty \in C^\infty([0, \pi] \times \{\tilde{L}\})$ be defined by
\[
\chi_\infty(x_1, \tilde{L}) = \cosh(n\tilde{L}) \sin(nx_1).
\]
Define $V_0^{\infty}$ as the solution of
\[
\begin{align*}
-\Delta V_0^{\infty}(x) &= 0 & x \in \tilde{\Omega}, \\
V_0^{\infty}(x) &= \chi_\infty(x) & x \in (0, \pi) \times \{\tilde{L}\}, \\
\frac{\partial V_0^{\infty}}{\partial \nu}(x) &= 0 & x \in \Gamma_D^1 \cup \Gamma_D^2,
\end{align*}
\tag{11}
\]
that is,
\[
V_0^{\infty}(x) = \sin(nx_1) \cosh(nx_2), \quad x = (x_1, x_2) \in \tilde{\Omega}.
\]

Proposition 7: The set $S_{\infty} = \{(j, k) \in (\mathbb{N}^2)^2 \mid \int_{\tilde{\Omega}} \phi_j \phi_k V_0^{\infty} \neq 0\}$ is a connectedness chain of $(\mathbb{N}^2)^2$ if and only if $n$ is even.

Proof: To prove that $S_{\infty}$ is a connectedness chain, we are led to compute the quantities
\[
\int_{\tilde{\Omega}} V_0^{\infty}(x) \phi_j(x) \phi_k(x) dx = \frac{4}{L\pi} A_{nk} B_{nkj}, \quad j, k \in (\mathbb{N}^2)^2
\]
with
\[
A_{nkj} = \int_0^\pi \sin(nx_1) \sin(j_1 x_1) \sin(k_1 x_1) dx_1
\]
and
\[
B_{nkj} = \int_0^\tilde{L} \cosh(nx_2) \sin\left( \frac{j_2 \pi x_2}{L} \right) \sin\left( \frac{k_2 \pi x_2}{L} \right) dx_2.
\]

A tedious, but simple computation proves that $A_{nkj} = 0$ if $j_1 + k_1 + n$ is even, and $A_{nkj} = \frac{4j_1 k_1 n}{(n^2 - (j_1 + k_1)^2)(n^2 - (j_1 - k_1)^2)}$ otherwise, whereas $B_{nkj} = \frac{\sinh(nL)}{\sinh(L^2 \pi^2)} \frac{2}{(j_2 k_2 + n^2)(j_2 + k_2)^2 + n^2}$. One immediately sees that $B_{nkj}$ cannot vanish. As for the coefficients $A_{nkj}$, if $n$ is even then $A_{nkj}$ vanishes if and only if $j_1$ and $k_1$ have the same parity. Then $S_{\infty} = \{(j, k) \mid j_1 + k_1 \text{ is odd}\}$ is a connectedness chain: indeed, given $j$ and $k$ in $\mathbb{N}^2$, either $j_1 + k_1$ is odd, and then $(j, k) \in S_{\infty}$, or $j_1 + k_1$ is even and then $(j', j)$ and $(j', k)$ are in $S_{\infty}$ with $j' = (j_1 + 1, j_2)$.

Conversely, if $n$ is odd then $A_{nkj}$ vanishes if and only if $j_1 + k_1$ is odd. Hence, $S_{\infty}$ cannot couple $j$ and $k$ when $j_1 + k_1$ is odd.

B. Convergence to the limit case

We consider the subclass $\bar{P}_\text{tan}$ of $P$ defined by
\[
\bar{P}_\text{tan} = \{(Q(\Omega), Q(\Gamma_D^g), \chi) \in P \mid Q(\Omega) = \tilde{\Omega}, \quad Q(\Gamma_D^g) \subset [0, \pi] \times \{\tilde{L}\}\}
\]
By definition, $(\phi_j)_{j \in \mathbb{N}^2}$ is a $L^2$-orthonormal basis for the Laplace–Dirichlet operator on $Q(\Omega) = \tilde{\Omega}$ for every $(Q(\Omega), Q(\Gamma_D^g), \chi) \in \bar{P}_\text{tan}$. The asymptotic behavior of the criterion
\[
\int_{\tilde{\Omega}} V_0^{Q, \infty} \phi_j \phi_k dx \neq 0
\]
as $Q(\Gamma_D^g)$ converges to $[0, \pi] \times \{\tilde{L}\}$ is then reduced to the study of the asymptotic behavior of $V_0^{Q, \infty}$.

Let us introduce a sequence $(\Gamma_D^g)_{p \in \mathbb{N}}$ of segments included in $(0, \pi) \times \{\tilde{L}\}$ increasing for the inclusion and such that
\[
\bigcup_{p=1}^{+\infty} \Gamma_D^g = [0, \pi] \times \{\tilde{L}\}.
\]
For every $p \in \mathbb{N}$ let $Q_p \in \text{Diff}^{m,n}$ be such that $Q_p(\Omega) = \tilde{\Omega}$ and $Q_p(\Gamma_D^g) = \Gamma_D^g$. Thus, $W_p$ is the solution of the following partial differential equation
\[
-\Delta W_p(x) = n^2 \chi_\infty(x) \quad x \in (0, \pi) \times (0, \tilde{L}),
\]
\[
W_p(x) = 0 \quad x \in \Gamma_D^g \cup \{(0, \pi) \times (0, \tilde{L})\},
\]
whose variational formulation writes: find $W_p$ in
\[
\mathcal{V}_p = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D^g \cup \{(0, \pi) \times (0, \tilde{L})\} \right\}
\]
such that for every $v \in \mathcal{V}_p$, one has
\[
\int_{\tilde{\Omega}} \nabla W_p(x) \cdot \nabla v(x) dx = n^2 \int_{\tilde{\Omega}} v(x) \chi_\infty(x) dx.
\]
It follows from the Hopf maximum principle that
\[
\|W_p\|_{L^2(\tilde{\Omega})} \leq \max_{x \in [0, \pi]} |\chi_\infty(x)| \leq \cosh(nL).
\]
As a consequence, the sequences $(W^p)_{p \in \mathbb{N}}$ and $(W_p)_{p \in \mathbb{N}}$ are uniformly bounded (with respect to $p$) in $L^2(\Omega)$. Taking now $v = W_p$ in (13) yields
\[
\|\nabla W_p\|_{L^2(\tilde{\Omega})} \leq n^2 \|\chi_\infty\|_{L^2(\tilde{\Omega})} \|W_p\|_{L^2(\tilde{\Omega})}.
\]

The sequence $(W_p)_{p \in \mathbb{N}}$ is thus bounded in $H^1(\tilde{\Omega})$ and, from Rellich compactness embedding theorem, converges up to a subsequence weakly in $H^1(\tilde{\Omega})$ and strongly in $L^2(\tilde{\Omega})$. 

to some \( W_\infty \in H^1(\bar{\Omega}) \). Let us still denote by \( W_p \) the considered subsequence. It is standard that \( W_\infty \) satisfies

\[
-\Delta W_\infty = n^2 \chi_\infty
\]

in distributional sense and that \( W_\infty = 0 \) on \( \{0, \pi\} \times [0, \bar{L}] \), by compactness of the trace operator. Since the sequence \( (\Gamma_{p,D}^q)_{p \in \mathbb{N}} \) is increasing for the inclusion and converges to \( (0, \pi) \times \{ \bar{L} \} \), one sees that for any compact \( K \subset (0, \pi) \times \{ \bar{L} \} \) there exists \( p_0 \) such that \( W_p = 0 \) on \( K \) for every \( p \geq p_0 \). Thus, one yields \( W_\infty = 0 \) on \( (0, \pi) \times \{ \bar{L} \} \). Finally, since \( V_p \) is increasing with respect to \( p \), it is obvious that for every \( p \in \mathbb{N} \) and \( v \in V_p \), \( W_\infty \) satisfies

\[
\int_{\bar{\Omega}} \nabla W_\infty(x) \cdot \nabla v(x) dx = n^2 \int_{\Omega} v(x) \chi_\infty(x) dx. \tag{14}
\]

Introduce \( V_\infty = \bigcup_{p=0}^{\infty} V_p \), that is,

\[
V_\infty = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } ((0, \pi) \times \{ \bar{L} \}) \cup ((0, \pi) \times \{ 0, \bar{L} \}) \}.
\]

It is clear that \( W_\infty \) satisfies (14) for every \( v \in V_\infty \). By taking \( v = W_p \) in (13) and since \( (W_p)_{p \in \mathbb{N}} \) converges strongly in \( L^2(\Omega) \) to \( W_\infty \), it follows that \( \| W_p \|_{H^1(\Omega)} \) converges to \( \| W_\infty \|_{H^1(\Omega)} \) as \( p \to +\infty \). Since \( (W_p)_{p \in \mathbb{N}} \) also converges weakly in \( H^1(\Omega) \) to \( W_\infty \), we deduce that this convergence is in fact strong in \( H^1(\Omega) \), whence the result.

C. Conclusion of the proof of Theorem 1

Based on Corollary 6, we are left to prove that \( P_2 \) is nonempty. We deduce in this section from Proposition 8 that \( \overline{P}_{tan} \), defined in the previous section, has nonempty (and even residual) intersection with \( P_2 \). For \( j, k \in \mathbb{N} \), let

\[
O_{jk} = \{ (Q(\Omega), Q(\Gamma_{p,D}^q), \chi) \in \overline{P}_{tan} \mid \int_{\Omega} V_0^{Q(\chi)} C_{\phi_j} \phi_k \neq 0 \}.
\]

Clearly, each \( O_{jk} \) is open in \( \overline{P}_{tan} \).

If \( j, k \) is in \( S_\infty \) (defined in the statement of Proposition 7), then Proposition 8 implies that for \( p \) large enough (depending eventually on \( j \) and \( k \) \( (Q_p(\Omega), Q_p(\Gamma_{p,D}^q), \chi_p) \in O_{jk} \). By considering all analytic deformations of \( (Q_p(\Omega), Q_p(\Gamma_{p,D}^q), \chi_p) \) within the class \( \overline{P}_{tan} \) we easily prove that \( O_{jk} \) is dense in \( \overline{P}_{tan} \) (similarly to what done in the proof of Corollary 6).

Hence

\[
\bigcap_{(j,k) \in S_\infty} O_{jk}
\]

is residual in \( \overline{P}_{tan} \). Equivalently said, for every element in a residual subset of \( \overline{P}_{tan} \), \( S_\infty \) is a connectedness chain. In particular, \( P_2 \) is nonempty.

VIII. Conclusion

In this paper, we studied controllability properties for a transistor at a nanometric scale. The model couples a Schrödinger equation for the wavefunction and a Poisson equation for the electrical potential. The control acts as a boundary condition in the Poisson equation modeling a gate voltage. We validated the model by showing that for generic boundary data and shape of the device, the system is approximately controllable. Further analysis could be devoted to more complex and relevant physical models introducing nonlinearities in the dependence of the solution of the Poisson equation on the wavefunction. One could also consider more restrictive perturbations on the shape of the device such as perturbations preserving \( \Omega \) but not the length and the location of the gate \( \Gamma_{p,D}^q \).

References