Hard-Constrained Inconsistent Signal Feasibility Problems

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***Abstract*— We consider the problem of synthesizing feasible signals in a Hilbert space in the presence of inconsistent convex constraints, some of which must imperatively be satisfied. This problem is formalized as that of minimizing a convex objective measuring the amount of violation of the soft constraints over the intersection of the sets associated with the hard ones. The resulting convex optimization problem is analyzed, and numerical solution schemes are presented along with convergence results. The proposed formalism and its algorithmic framework unify and extend existing approaches to inconsistent signal feasibility problems. An application to signal synthesis is demonstrated.**

***Index Terms*—Convex feasibility problem, fixed point, Hilbert space, inconsistent constraints, monotone operator, optimization, signal synthesis.**

I. Problem Statement

Throughout the signal space is a real Hilbert space $\mathcal{H}$, with scalar product $\langle \cdot \mid \cdot \rangle$, norm $\|\cdot\|$ and distance d. The distance from a signal $x \in \mathcal{H}$ to a nonempty set $A \subset \mathcal{H}$ is defined as $d(x, A) = \inf\{\|x - y\| \mid y \in A\}$. Γ denotes the class of all lower semicontinuous proper convex functions from $\mathcal{H}$ into $]-\infty, +\infty]$ [9]. Given $g \in \Gamma$ and $\alpha \in \mathbb{R}$, the closed and convex set $\text{lev}_{\leq \alpha} g = \{x \in \mathcal{H} \mid g(x) \leq \alpha\}$ is the lower level set of g at height α and the nonempty convex set $\text{dom}\, g = \{x \in \mathcal{H} \mid g(x) < +\infty\}$ its domain.

The goal of a convex set theoretic signal synthesis (design or estimation) problem in $\mathcal{H}$ is to produce a signal $x^\star$ that satisfies convex constraints, say $(\forall i \in I)\ g_i(x^\star) \leq 0$, where I is a finite index set, and $(g_i)_{i \in I} \subset \Gamma$.[1] The problem can simply be stated in the set theoretic format

$$\text{Find } x^\star \in S = \bigcap_{i \in I} S_i, \quad \text{where } (\forall i \in I) \quad S_i = \text{lev}_{\leq 0} g_i. \quad (1)$$

This convex feasibility framework has been applied to numerous signal processing problems, e.g., [5]–[7], [10], [13], [16]–[18]. Of course, in writing (1), it is tacitly assumed that the problem is consistent in the sense that the constraints are compatible so that $S \neq \varnothing$. However, signal feasibility problems may turn out to be inconsistent for a variety of reasons. In design problems, this situation typically results from the incorporation of specifications that are too demanding and therefore conflicting. In estimation problems, it may be due to inaccurate deterministic constraints, to overly aggressive confidence levels on stochastic constraints, or to inadequate data modeling [5]. Specific examples in the areas of signal deconvolution, image recovery, data window design, pulse shape design, and tomography will be found in [5], [6], [10], and [13]. Naturally, when the feasibility problem is inconsistent, $S = \varnothing$, and (1) must be reformulated in a physically meaningful way. Two frameworks emerge from the literature.

- **Framework 1** [10], [20]: Two constraints are present, say $I = \{1, 2\}$. We seek a signal $x^\star$ satisfying the first constraint and closest to the set of signals satisfying the second, i.e., $x^\star \in S_1$ and $d(x^\star, S_2) = \inf_{x \in S_1} d(x, S_2)$.
- **Framework 2** [5]: The number of constraints is arbitrary. We seek a signal $x^\star$ in $\mathcal{H}$ that is closest to all the constraint sets $(S_i)_{i \in I}$ in a weighted least-squares sense, i.e., a minimizer $x^\star$ of the function $x \mapsto \sum_{i \in I} w_i d(x, S_i)^2$, where $(w_i)_{i \in I} \subset]0, 1]$, and $\sum_{i \in I} w_i = 1$.

In this paper, we propose a broad convex programming formulation for inconsistent problems that unifies and extends the above frameworks. Underlying our formulation is the splitting of the collection of constraints into hard and soft constraints. Hard constraints may, for instance, arise from imperative specifications in design problems, e.g., stability in filter design, or from reliable *a priori* information in estimation problems, e.g, non-negativity in image restoration. The problem is then formulated as that of finding a signal $x^\star$, which satisfies the hard constraints and least violates—in some suitable sense—the soft ones.

The remainder of the paper is divided into four sections. The hard-constrained signal feasibility problem is formalized and analyzed in Section II, and its numerical solution is discussed in Section III. Section IV is devoted to an application to pulse shape design, and Section V concludes the paper with some remarks. Technical proofs are relegated to Appendix A.

II. Mathematical Analysis

A. General Formulation

$I^\blacktriangle \subset I$ denotes the possibly empty hard constraints index set, $I^\triangle = I \backslash I^\blacktriangle$ the nonempty soft constraints index set, $S^\blacktriangle = \bigcap_{i \in I^\blacktriangle} S_i$ the hard feasibility set and, by convention,

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[1] As is prevalent in optimization theory [1], [9], [14], functions are allowed to take the value $+\infty$. Thus, a function g defined on $A \subset \mathcal{H}$ can conveniently be extended to the whole space $\mathcal{H}$ by setting $g(x) = +\infty$ if $x \notin A$.

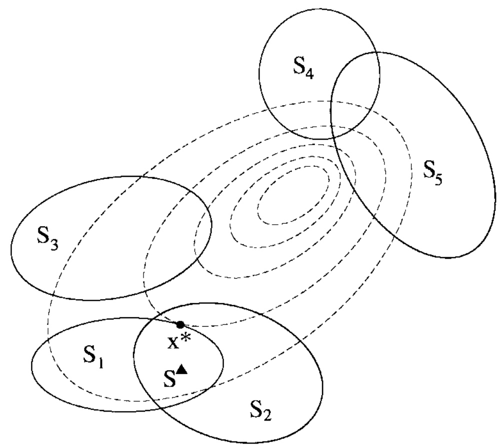

Fig. 1. S_1 and S_2 are the hard constraint sets; S_3, S_4, and S_5 are the soft constraint sets. The dashed lines represent level curves $(\text{lev}_{=\alpha_i}\Phi^{\triangle})_{1\leq i\leq 5}$ of the objective $\Phi^{\triangle}$ and $x^\star$ an optimal signal, i.e., a minimizer of $\Phi^{\triangle}$ over $S^{\blacktriangle}$.

we put $S^{\blacktriangle}=\mathcal{H}$ if $I^{\blacktriangle}=\varnothing$. Moreover, we define $S^{\triangle}=\bigcap_{i\in I^{\triangle}}S_i$, $D^{\triangle}=\bigcap_{i\in I^{\triangle}}\text{dom}\,g_i$, and assume henceforth that $S^{\blacktriangle}\cap D^{\triangle}\neq\varnothing$. $\mathcal{F}$ is the class of all increasing convex functions from $[0,+\infty[$ into $[0,+\infty[$ that vanish (only) at 0; every $f\in\mathcal{F}$ is extended to the argument $+\infty$ by setting $f(+\infty)=+\infty$. For every $g\in\Gamma$, $g^+=\max\{0,g\}$.

The amount of violation of the soft constraints $(g_i(x)\leq 0)_{i\in I^{\triangle}}$ is measured by an objective function $\Phi^{\triangle}:\mathcal{H}\to[0,+\infty]$ taking the general form

$$\Phi^{\triangle}=\sum_{i\in I^{\triangle}}f_i\circ g_i^+,\quad \text{where } (f_i)_{i\in I^{\triangle}}\subset\mathcal{F}. \tag{2}$$

Such functions arise as exterior penalty functions in constrained optimization [14], [15], [21], and they possess all the properties required for our purposes.

Proposition 1: Let $\Phi^{\triangle}$ be any objective constructed via (2), and let x and y be any two points in $D^{\triangle}$. Then, we have the following.

i) $\Phi^{\triangle}\in\Gamma$.
ii) $\Phi^{\triangle}(x)=0\Leftrightarrow x\in S^{\triangle}$.
iii) If $(\exists\, i\in I^{\triangle})\ \ g_i^+(y)>g_i^+(x)$ and $(\forall i\in I^{\triangle})\ g_i^+(y)\geq g_i^+(x)$, then $\Phi^{\triangle}(y)>\Phi^{\triangle}(x)$.

In words, ii) states that $\Phi^{\triangle}(x)$ vanishes only when x satisfies all the soft constraints; iii) states that if y violates a soft constraint more than x does and, at the same time, does not violate any soft constraint less than x does, then y will be more penalized than x.

Mathematically, the hard-constrained signal feasibility problem is to minimize the objective $\Phi^{\triangle}$ of (2) over the hard feasibility set $S^{\blacktriangle}$ (see Fig. 1). If we set $\alpha^\star=\inf_{x\in S^{\blacktriangle}}\Phi^{\triangle}(x)$, the problem reads

$$\text{Find } x^\star\in G=\{x\in S^{\blacktriangle}\mid\Phi^{\triangle}(x)=\alpha^\star\}. \tag{3}$$

The function $\Phi^{\triangle}$ is lower semicontinuous, convex, and proper by Proposition 1(i), whereas the set $S^{\blacktriangle}$ is closed and convex by construction. Hence, (3) is a standard convex optimization problem, and powerful tools are available to analyze and solve it. Thus, as is well known, any minimizer is global, and therefore, we do not have to contend with local minimizers outside G. Furthermore, relatively simple conditions are available for the existence and the uniqueness of solutions, as well as for their characterization.

In order to establish existence and uniqueness conditions, some definitions need to be recalled [15], [21]. Take $g\in\Gamma$ and a convex set $A\subset\mathcal{H}$, and let $\mathring{A}$ be the interior of A. Then, g is strictly convex over A if, for any two distinct points x and y in $A\cap\text{dom}\,g$, $g((x+y)/2)<(g(x)+g(y))/2$; A is strictly convex if, for any two distinct points x and y in A, $(x+y)/2\in\mathring{A}$. Finally, g is weakly coercive over $\mathcal{H}$ if $\lim_{\|x\|\to+\infty}g(x)=+\infty$.

Proposition 2: The solution set G in (3) is closed and convex. It is nonempty if, for some $i\in I$, g_i is weakly coercive over $\mathcal{H}$. Finally, it contains at most one point if one of the conditions below holds.

i) For some $u\in S^{\blacktriangle}\cap D^{\triangle}$ and some $i\in I^{\triangle}$, g_i^+ is strictly convex over $S^{\blacktriangle}\cap\text{lev}_{\leq\Phi^{\triangle}(u)}\Phi^{\triangle}$.
ii) $\Phi^{\triangle}$ has no free minimizer in $S^{\blacktriangle}$ and, for every $i\in I^{\blacktriangle}$, g_i is strictly convex and continuous over $S^{\blacktriangle}$.

Moving on now to the characterization of solutions, fix arbitrarily $x^\star\in S^{\blacktriangle}$, $y\in\mathcal{H}$, and $\gamma\in]0,+\infty[$. Let $P^{\blacktriangle}$ be the projector onto $S^{\blacktriangle}$, i.e., $P^{\blacktriangle}(y)$ is the unique point in $S^{\blacktriangle}$ such that $\|y-P^{\blacktriangle}(y)\|=d(y,S^{\blacktriangle})$. Then [1, Th. 2.3]

$$x^\star=P^{\blacktriangle}(y)\Leftrightarrow(\forall x\in S^{\blacktriangle})\quad\langle x^\star-x\mid x^\star-y\rangle\leq 0. \tag{4}$$

On the other hand, it follows from Proposition 1(i) and [9, Prop. II.2.1] that, if $\Phi^{\triangle}$ is differentiable[2] at $x^\star$, then

$$\begin{aligned}&\Phi^{\triangle}(x^\star)=\alpha^\star\Leftrightarrow\\&(\forall x\in S^{\blacktriangle})\quad\langle x^\star-x\mid\nabla\Phi^{\triangle}(x^\star)\rangle\leq 0\Leftrightarrow\\&(\forall x\in S^{\blacktriangle})\quad\langle x^\star-x\mid x^\star-(x^\star-\gamma\nabla\Phi^{\triangle}(x^\star))\rangle\leq 0.\end{aligned} \tag{5}$$

Upon comparing (4) and (5), we obtain $x^\star\in G\Leftrightarrow x^\star=P^{\blacktriangle}(x^\star-\gamma\nabla\Phi^{\triangle}(x^\star))$. This key fact is recorded below, where $\text{Fix}\,T=\{x\in A\mid x=T(x)\}$ denotes the set of fixed points of an operator $T:A\subset\mathcal{H}\to\mathcal{H}$ and Id the identity operator on $\mathcal{H}$.

Proposition 3: Suppose that $\Phi^{\triangle}$ is differentiable on $S^{\blacktriangle}$ with derivative $\nabla\Phi^{\triangle}$. Then, for any $\gamma\in]0,+\infty[$, $G=\text{Fix}\,P^{\blacktriangle}\circ(\text{Id}-\gamma\nabla\Phi^{\triangle})$.

Finally, let us note that if the problem is consistent, i.e., $S\neq\varnothing$, then $G=S$, and (3) reverts to (1). Indeed, $S\neq\varnothing$ implies $S^{\triangle}\neq\varnothing$, and Proposition 1(ii) then asserts that $S^{\triangle}$ is the set of minimizers of $\Phi^{\triangle}$ over $\mathcal{H}$. Hence, since $S^{\blacktriangle}\cap S^{\triangle}\neq\varnothing$, we obtain $G=S^{\blacktriangle}\cap S^{\triangle}$, i.e., $G=S$.

B. Application to Proximity Functions

An important special instance of (2) is the convex combination of halved squared distances

$$\Phi^{\triangle}=\frac{1}{2}\sum_{i\in I^{\triangle}}w_i d(\cdot,S_i)^2 \tag{6}$$

where $(w_i)_{i\in I^{\triangle}}\subset]0,1]$, and $\sum_{i\in I^{\triangle}}w_i=1$. Such an objective will be called a proximity function.

[2] Differentiability will always be understood in the sense of Fréchet. Indeed, we shall deal only with continuously differentiable functions hereafter and, for such functions, Gâteaux and Fréchet derivatives coincide [21].

It is readily noted that $\Phi^{\triangle}$ has full domain: $D^{\triangle} = \mathcal{H}$. Moreover, for every $i \in I^{\triangle}$, the function $d(\cdot, S_i)^2/2$ is differentiable on $\mathcal{H}$ with derivative $\mathrm{Id} - P_i$, where P_i is the projector onto S_i [1, Th. 5.2]. Consequently, $\Phi^{\triangle}$ is also differentiable on $\mathcal{H}$ and $\nabla\Phi^{\triangle} = \mathrm{Id} - \sum_{i \in I^{\triangle}} w_i P_i$. Upon taking $\gamma = 1$ in Proposition 3, the solution set can therefore be written as

$$G = \mathrm{Fix}\, P^{\blacktriangle} \circ \sum_{i \in I^{\triangle}} w_i P_i. \tag{7}$$

We can further specialize this result to Frameworks 1 and 2 described in the Introduction and recover the characterizations of [10] and [5], respectively. Thus, in Framework 1, $P^{\blacktriangle} = P_1$, $I^{\triangle} = \{2\}$, and (7) therefore yields $G = \mathrm{Fix}\, P_1 \circ P_2$. In Framework 2, $I^{\blacktriangle} = \varnothing$, $S^{\blacktriangle} = \mathcal{H}$, $P^{\blacktriangle} = \mathrm{Id}$, $I^{\triangle} = I$, and (7) therefore yields $G = \mathrm{Fix}\, \sum_{i \in I} w_i P_i$.

The next proposition furnishes existence and uniqueness conditions in terms of properties of the constraint sets $(S_i)_{i \in I}$.

Proposition 4: The solution set G in (7) is not empty if, for some $i \in I$, S_i is bounded. It contains at most one point if, for some $i \in I^{\triangle}$, S_i is strictly convex, and $S_i \cap S^{\blacktriangle} = \varnothing$.

III. Solution Methods

While there exists no universal method to solve the general convex minimization problem (3), various schemes are available that exploit certain properties of its constituents, e.g., [14] and [15]. As it is impossible to attempt a presentation of all pertinent algorithms, we limit ourselves to a fixed-point approach that will be seen to cover the algorithms employed in Frameworks 1 and 2. It is assumed throughout this section that the solution set G is not empty (see Proposition 2) and that $\Phi^{\triangle}$ is differentiable on $S^{\blacktriangle}$.

A. Fixed-Point Iterations

Proposition 3 states that the hard-constrained signal feasibility problem (3) is a fixed-point problem. In connection with the numerical solution of such problems, the following definitions are pertinent [8], [21]. Let β and η be two positive real numbers, let $\varnothing \neq C \subset A \subset \mathcal{H}$, and let x and y be any two points in C. An operator $T: A \to \mathcal{H}$ is β-lipschitzian on C if

$$\|T(x) - T(y)\| \leq \beta \|x - y\|. \tag{8}$$

Furthermore, T is strictly contractive or nonexpansive accordingly as $\beta \in]0,1[$ or $\beta = 1$ in (8). T is η-strongly monotone on C if

$$\langle x - y \mid T(x) - T(y) \rangle \geq \eta \|x - y\|^2. \tag{9}$$

Finally, T is η-cocoercive on C if

$$\langle x - y \mid T(x) - T(y) \rangle \geq \eta \|T(x) - T(y)\|^2 \tag{10}$$

and firmly nonexpansive if $\eta = 1$ in (10).

In the following, given $\gamma \in]0, +\infty[$, we set $T = P^{\blacktriangle} \circ (\mathrm{Id} - \gamma \nabla\Phi^{\triangle})$, and let x and y be any two points in $S^{\blacktriangle}$. Since $P^{\blacktriangle}$ is a projector onto a nonempty closed convex set, it is firmly nonexpansive [1, Prop. 2.7.(i)] and, therefore, nonexpansive. Hence

$$\begin{aligned} &\|T(x) - T(y)\|^2 \\ &\quad = \|P^{\blacktriangle}(x - \gamma\nabla\Phi^{\triangle}(x)) - P^{\blacktriangle}(y - \gamma\nabla\Phi^{\triangle}(y))\|^2 \\ &\quad \leq \|x - y - \gamma(\nabla\Phi^{\triangle}(x) - \nabla\Phi^{\triangle}(y))\|^2 \\ &\quad = \|x - y\|^2 - 2\gamma\langle x - y \mid \nabla\Phi^{\triangle}(x) - \nabla\Phi^{\triangle}(y)\rangle \\ &\qquad + \gamma^2 \|\nabla\Phi^{\triangle}(x) - \nabla\Phi^{\triangle}(y)\|^2. \end{aligned} \tag{11}$$

This inequality plays a central role in analyzing the properties of T. Thus, if we assume that $\nabla\Phi^{\triangle}$ is β-lipschitzian and η-strongly monotone on $S^{\blacktriangle}$, (11) implies

$$\|T(x) - T(y)\|^2 \leq (1 - \gamma(2\eta - \gamma\beta^2))\|x - y\|^2. \tag{12}$$

Therefore, $T: S^{\blacktriangle} \to S^{\blacktriangle}$ is a strict contraction if $\gamma < 2\eta/\beta^2$, and the Banach–Picard contraction theorem yields at once the following result, where $\mathbb{N}$ denotes the set of nonnegative integers (see also [21, Th. 46.C] for a more general perspective).

Proposition 5: Suppose that $\nabla\Phi^{\triangle}$ is β-lipschitzian and η-strongly monotone on $S^{\blacktriangle}$. Take $\gamma \in]0, 2\eta/\beta^2[$, $x_0 \in S^{\blacktriangle}$, and let

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = P^{\blacktriangle}(x_n - \gamma\nabla\Phi^{\triangle}(x_n)). \tag{13}$$

Then, $(x_n)_{n \geq 0}$ converges strongly to the unique point $x^\star$ in G. More specifically, the convergence is linear with rate $\kappa = \sqrt{1 - 2\gamma\eta + \beta^2\gamma^2}$, i.e.,

$$(\forall n \in \mathbb{N}) \quad \|x_n - x^\star\| \leq \kappa^n (1 - \kappa)^{-1} \|x_1 - x_0\|. \tag{14}$$

Although algorithm (13) displays nice convergence properties, its scope is limited by the stringent requirement that $\nabla\Phi^{\triangle}$ be η-strongly monotone on $S^{\blacktriangle}$. To shed more light on this strong convexity property of $\Phi^{\triangle}$, let us describe a typical situation in which it is fulfilled.

Proposition 6: Suppose that for some $i \in I^{\triangle}$, $f_i: t \mapsto t$, and $g_i: x \mapsto \|L(x) - r\|^2 - \xi$, where $L: \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, $r \in \mathcal{H}$, and $\xi \in]0, +\infty[$. Given $\chi \in]0, +\infty[$, suppose that

$$(\forall x \in S^{\blacktriangle}) \quad \|L(x)\| \geq \chi\|x\| \quad \text{and} \quad \|L(x) - r\|^2 > \xi. \tag{15}$$

Then, $\nabla\Phi^{\triangle}$ is η-strongly monotone on $S^{\blacktriangle}$ with $\eta = 2\chi^2$.

The type of constraint function g_i described above is not uncommon in signal synthesis problems, e.g., [6, Sec. IV.B] and [18]. However, (15) may be difficult to fulfill in practice.

An inspection of (8)–(10) shows that lipschitzian strongly monotone operators are cocoercive. Assuming that $\nabla\Phi^{\triangle}$ belongs to this larger class of operators will lead us to a more widely applicable algorithm. Indeed, if $\nabla\Phi^{\triangle}$ is η-cocoercive on $S^{\blacktriangle}$, we derive from (11) the inequality

$$\begin{aligned} &\|T(x) - T(y)\|^2 \\ &\quad \leq \|x - y\|^2 - \gamma(2\eta - \gamma)\|\nabla\Phi^{\triangle}(x) - \nabla\Phi^{\triangle}(y)\|^2 \end{aligned} \tag{16}$$

which shows that T is nonexpansive if $\gamma \leq 2\eta$. The convergence result stated below can then be established.

Proposition 7: Suppose that $\nabla\Phi^{\triangle}$ is η-cocoercive on $S^{\blacktriangle}$, and take $\gamma \in]0, 2\eta]$, $x_0 \in S^{\blacktriangle}$, and $(\lambda_n)_{n\geq 0} \subset [0,1]$ such that $\sum_{n\geq 0}\lambda_n(1-\lambda_n) = +\infty$. Let

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = (1-\lambda_n)x_n + \lambda_n P^{\blacktriangle}(x_n - \gamma\nabla\Phi^{\triangle}(x_n)). \tag{17}$$

Then, $(x_n)_{n\geq 0}$ converges weakly to a point in G. The convergence is strong if $S^{\blacktriangle}$ is boundedly compact.[3]

A few remarks are in order concerning Algorithm (17). First, we observe that it is well defined since $S^{\blacktriangle}$ is convex, which forces $(x_n)_{n\geq 0} \subset S^{\blacktriangle}$. Second, if $S^{\blacktriangle}$ is contained in a finite-dimensional affine subspace, it is boundedly compact, and we obtain a simple instance of strong convergence. Third, the algorithm allows for variable relaxation parameters $(\lambda_n)_{n\geq 0}$ over the course of the iterations. Several theoretical and numerical studies have shown that this flexibility could be effectively exploited to enhance the progression of such algorithms toward a solution, e.g., [5]–[7] and the references therein.

The signals generated by Algorithm (17) are arbitrary points in the solution set G. In some problems, it may be desirable to obtain the signal in G, which is the closest to some reference signal r [4]. The next result describes a simple scheme for generating such a solution.

Proposition 8: Let P_G be the projector onto G and, given $r \in \mathcal{H}$, let C be the convex hull of $\{r\} \cup S^{\blacktriangle}$. Suppose that $\Phi^{\triangle}$ is differentiable on C and that $\nabla\Phi^{\triangle}$ is η-cocoercive on C. Take $\gamma \in]0, 2\eta]$ and $(\lambda_n)_{n\geq 0} \subset [0,1]$ such that

$$\begin{cases} \lim_{n\to+\infty}\lambda_n = 1 \\ \sum_{n\geq 0}(1-\lambda_n) = +\infty \\ \sum_{n\geq 0}|\lambda_{n+1}-\lambda_n| < +\infty. \end{cases} \tag{18}$$

Let $x_0 = r$ and

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = (1-\lambda_n)r + \lambda_n P^{\blacktriangle}(x_n - \gamma\nabla\Phi^{\triangle}(x_n)). \tag{19}$$

Then, $(x_n)_{n\geq 0}$ converges strongly to $P_G(r)$.

An important example of sequence satisfying (18) is given by $(\forall n \in \mathbb{N})\ \lambda_n = n/(n+1)$.

Unlike Proposition 7, Proposition 8 offers strong convergence without additional conditions on $S^{\blacktriangle}$. Moreover, the limit point is a specific signal, namely, the best approximation to the reference signal r from G. However, algorithm (18)–(19) is less flexible than (17), and it requires differentiability on a larger set. The second limitation actually vanishes when $S^{\blacktriangle}$ is an affine subspace. Indeed, the orthogonality property of projections ensures $P_G(r) = P_G \circ P^{\blacktriangle}(r)$ and, therefore, $P^{\blacktriangle}(r) \in S^{\blacktriangle}$ can be used in lieu of r in Proposition 8.

B. Application to Proximity Functions

We now focus on the objective (6). According to Proposition 4, $G \neq \varnothing$ if S_i is bounded for some $i \in I$. As noted in Section II-B, $\nabla\Phi^{\triangle} = \sum_{i\in I^{\triangle}} w_i(\mathrm{Id}-P_i)$. Consequently, since the operators $(\mathrm{Id}-P_i)_{i\in I^{\triangle}}$ are firmly nonexpansive [1, Prop. 2.7(ii)], so is the convex combination $\nabla\Phi^{\triangle}$ by convexity of $\|\cdot\|^2$ [see (A4)]. We derive immediately from Proposition 7 the following result.

Proposition 9: Suppose that $\Phi^{\triangle}$ is as in (6), and take $x_0 \in S^{\blacktriangle}$, $(\lambda_n)_{n\geq 0} \subset [0,1]$ such that $\sum_{n\geq 0}\lambda_n(1-\lambda_n) = +\infty$, and $\gamma \in]0,2]$. Let

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = (1-\lambda_n)x_n + \lambda_n P^{\blacktriangle}\left((1-\gamma)x_n + \gamma\sum_{i\in I^{\triangle}} w_i P_i(x_n)\right). \tag{20}$$

Then, $(x_n)_{n\geq 0}$ converges weakly to a point in G. The convergence is strong if $S^{\blacktriangle}$ is boundedly compact.

It is noteworthy that the algorithms used in Frameworks 1 [10] and 2 [5], as well as their weak convergence results, are encompassed by Proposition 9. In Framework 1, $P^{\blacktriangle} = P_1$, and $I^{\triangle} = \{2\}$. If we further specialize (9) by imposing $(\forall n \in \mathbb{N})$ $\lambda_n = \lambda \in]0,1[$ and $\gamma = 1$, Proposition 9 secures the weak convergence to a solution of the under-relaxed alternating projection method

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = (1-\lambda)x_n + \lambda P_1 \circ P_2(x_n). \tag{21}$$

This is precisely the result given in [10]. On the other hand, in Framework 2, $S^{\blacktriangle} = \mathcal{H}$, $P^{\blacktriangle} = \mathrm{Id}$, and $I^{\triangle} = I$. It follows from Proposition 9 with these parameters and $\gamma = 2$ that the parallel projection method

$$(\forall n \in \mathbb{N}) \quad \begin{cases} x_{n+1} = (1-\lambda_n)x_n + \lambda_n \sum_{i\in I} w_i P_i(x_n) \\ 0 < \varepsilon \leq \lambda_n \leq 2-\varepsilon \end{cases} \tag{22}$$

converges weakly to a solution. This result is given in [5].

With regard to the alternating projection method, let us remark that the unrelaxed scheme $(\forall n \in \mathbb{N})\quad x_{n+1} = P_1 \circ P_2(x_n)$ also converges weakly to a fixed point of $P_1 \circ P_2$ [12, Th. 2]. However, although the m-set extension of this scheme, i.e., the so-called (unrelaxed) POCS algorithm

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = P_1 \circ P_2 \circ \cdots \circ P_m(x_n) \tag{23}$$

converges weakly to a point in S_1 [12, Th. 2], this point fails in general to exhibit any degree of proximity with respect to the other sets [2].

In the present context, Proposition 8 extends [5, Th. 5] and [6, Th. 5.6]. We wind up this section by applying it to the problem of synthesizing the $\Phi^{\triangle}$-optimal hard-constrained signal $x^{\star}$ of minimum energy, i.e, $x^{\star} = P_G(0)$.

Proposition 10: Suppose that $\Phi^{\triangle}$ is as in (6), and take $\gamma \in]0,2]$. Let $x_0 = 0$, and

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = \frac{n}{n+1}P^{\blacktriangle}\left((1-\gamma)x_n + \gamma\sum_{i\in I^{\triangle}} w_i P_i(x_n)\right). \tag{24}$$

Then, $(x_n)_{n\geq 0}$ converges strongly to $P_G(0)$.

Let us add that Algorithm (24) is of interest even in the consistent case. We will then put $P^{\blacktriangle} = \mathrm{Id}$ and $I^{\triangle} = I$ in (24) and obtain strong convergence to the feasible signal of minimum energy. This algorithm is easier to implement than those presented in [4], which require the storage of outward normals to the sets at each iteration.

[3] Its intersection with every closed ball in $\mathcal{H}$ is compact, e.g., [6].

IV. Numerical Example: Pulse Shape Design

We revisit a design problem presented in [5], whose goal is to synthesize a pulse shape for digital communications over European power lines under specifications that give rise to four incompatible constraints:

- C_1: The lines have a bandwidth of 300 Hz and are contaminated by a DC component and the harmonic noise of the 50 Hz power distribution system. To avoid DC and harmonic noise and be compatible with the available bandwidth, the Fourier transform of the pulse should vanish at the zero frequency, at integer multiples of 50 Hz, and beyond 300 Hz.
- C_2: The pulse is symmetric about its midpoint, and its main lobe has amplitude 1.
- C_3: The energy of the pulse does not exceed a prescribed bound in order not to interfere with other systems.
- C_4: The duration of the pulse is 50 ms, and it has periodic zero crossings every 3.125 ms to avoid intersymbol interference.

Numerically, the problem is discretized with an underlying sampling rate of 2560 Hz, and the parent Hilbert space $\mathcal{H}$ is the euclidean space of N-point ($N = 512$) signals $x : \{0, \ldots, N-1\} \to \mathbb{R}$, with norm $\|x\| = (\sum_{l=0}^{N-1} |x(l)|^2)^{1/2}$. The associated sets and projections are defined as follows, where $\complement Q$ denotes the complement of a set Q and 1_Q its characteristic function, i.e., $1_Q(l) = 1$ if $l \in Q$ and $1_Q(l) = 0$ if $l \in \complement Q$ (see [5] and [6] for details).

- C_1 is associated with the vector subspace

$$S_1 = \{x \in \mathcal{H} \mid \hat{x} 1_{\mathbb{K}} = 0\} \tag{25}$$

where $\hat{x}$ is the N-point discrete Fourier transform (DFT) of x and $\mathbb{K}$ the set of frequencies at which $\hat{x}$ must vanish. The projection of $x \in \mathcal{H}$ onto S_1 is the inverse DFT of $\widehat{P_1(x)} = \hat{x} 1_{\complement\mathbb{K}}$.

- C_2 is associated with the affine subspace

$$S_2 = \{x \in \mathcal{H} \mid x(N/2) = 1 \text{ and } x = \breve{x}\} \tag{26}$$

where $\breve{x} : l \mapsto x(N-1-l)$. Now, let $\mathbb{A} = \{N/2 - 1, N/2\}$. Then, the projection of $x \in \mathcal{H}$ onto S_2 is $P_2(x) = 1_{\mathbb{A}} + (1/2)(x + \breve{x})1_{\complement\mathbb{A}}$.

- C_3 is associated with the closed ball

$$S_3 = \{x \in \mathcal{H} \mid \|x\|^2 \le \xi\}. \tag{27}$$

The projection of $x \in \mathcal{H}$ onto S_3 is

$$P_3(x) = \begin{cases} \sqrt{\xi}x/\|x\| & \text{if } \|x\|^2 > \xi \\ x & \text{otherwise.} \end{cases} \tag{28}$$

- C_4 is associated with the vector subspace

$$S_4 = \{x \in \mathcal{H} \mid x 1_{\mathbb{L}} = 0\} \tag{29}$$

where $\mathbb{L}$ is the set of time indices in the zero areas. The projection of $x \in \mathcal{H}$ onto S_4 is $P_4(x) = x 1_{\complement\mathbb{L}}$.

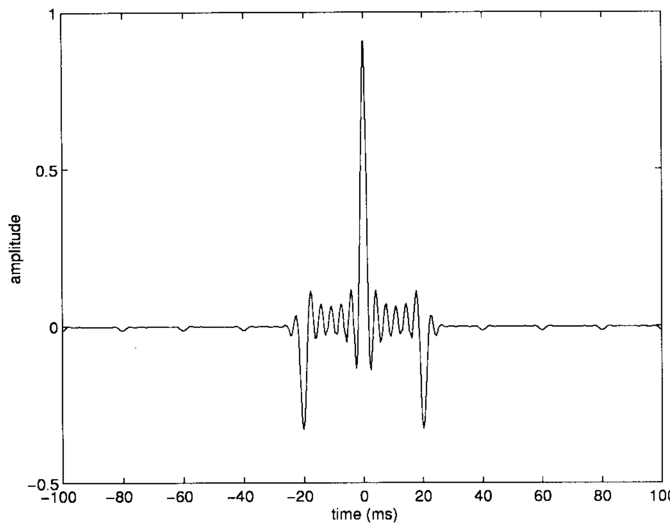

Fig. 2. Pulse generated without hard constraint.

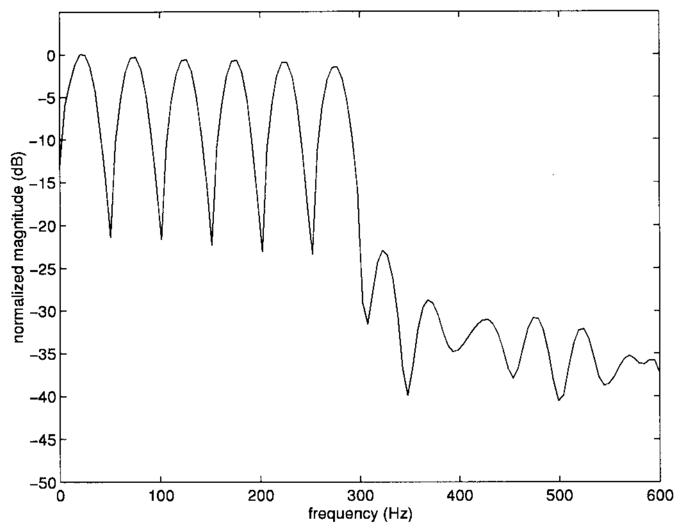

Fig. 3. Normalized spectral density of the pulse of Fig. 2.

Three design scenarios based on the objective (6) and algorithm (20) are considered:

- **Scenario 1:** No hard constraint is imposed. $I^{\triangle} = \{1,2,3,4\}$, $\Phi_1^{\triangle} : x \mapsto (1/8)\sum_{i=1}^{4} d(x, S_i)^2$, and $P^{\blacktriangle} = \text{Id}$ in (20). The results are shown in Figs. 2 and 3.
- **Scenario 2:** C_1 is the hard constraint. $I^{\triangle} = \{2,3,4\}$, $\Phi_2^{\triangle} : x \mapsto (1/6)\sum_{i=2}^{4} d(x, S_i)^2$, and $P^{\blacktriangle} = P_1$ in (20). The results are shown in Figs. 4 and 5.
- **Scenario 3:** C_4 is the hard constraint. $I^{\triangle} = \{1,2,3\}$, $\Phi_3^{\triangle} : x \mapsto (1/6)\sum_{i=1}^{3} d(x, S_i)^2$, and $P^{\blacktriangle} = P_4$ in (20). The results are shown in Figs. 6 and 7.

It is important to observe that since S_3 is bounded, a solution exists in each scenario by Proposition 4. In addition, (strong) convergence of (20) to a solution is guaranteed by Proposition 9. In connection with Scenario 3, let us remark that a pulse satisfying C_4 can also be obtained by implementing POCS (23) in the form

$$(\forall n \in \mathbb{N}) \quad x_{n+1} = P_4 \circ P_1 \circ P_2 \circ P_3(x_n). \tag{30}$$

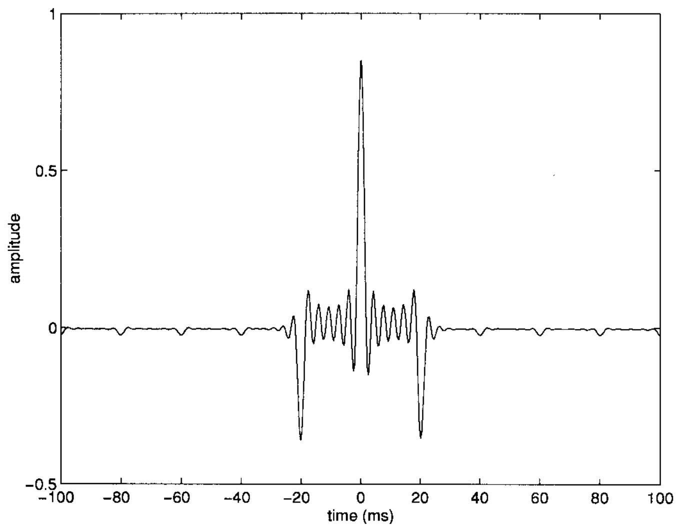

Fig. 4. Pulse generated with C_1 as a hard constraint.

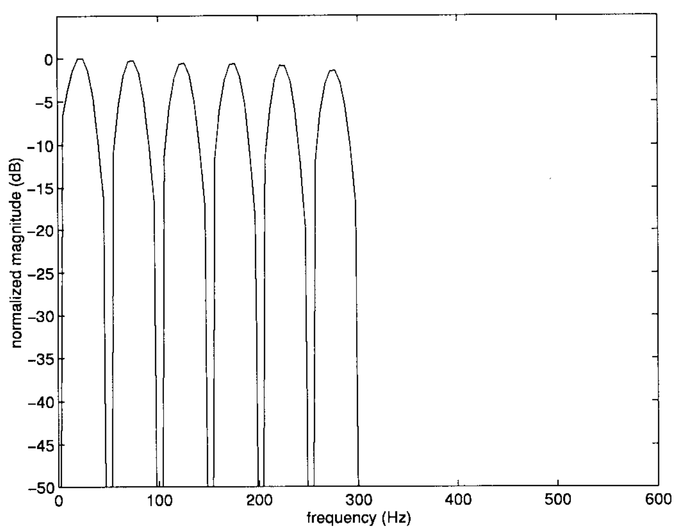

Fig. 5. Normalized spectral density of the pulse of Fig. 4.

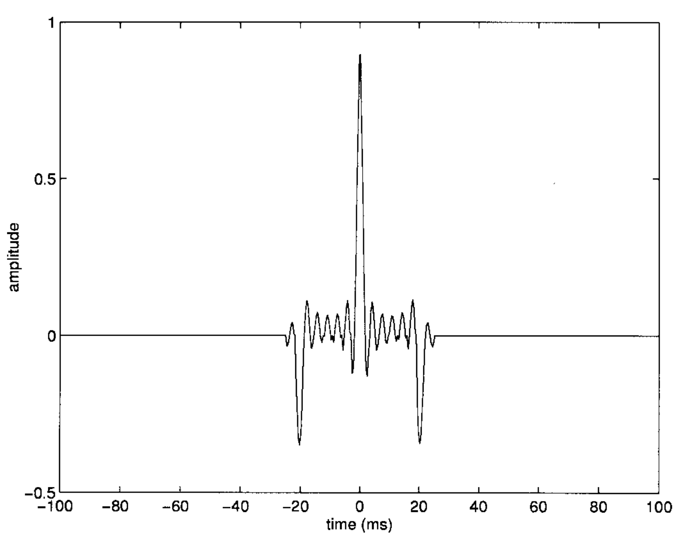

Fig. 6. Pulse generated with C_4 as a hard constraint.

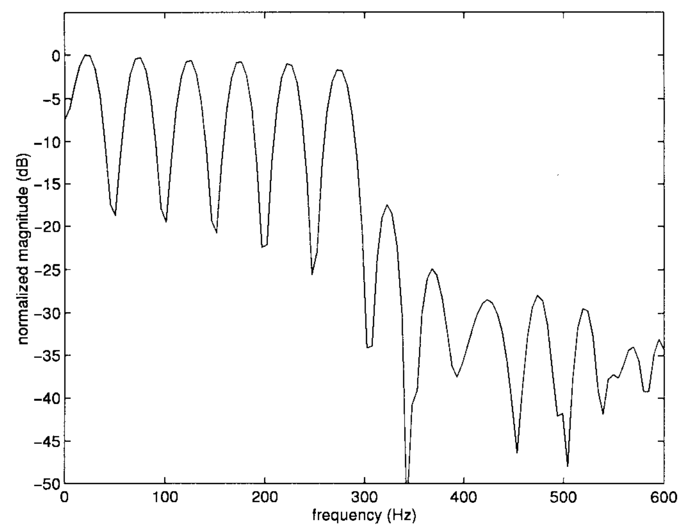

Fig. 7. Normalized spectral density of the pulse of Fig. 6.

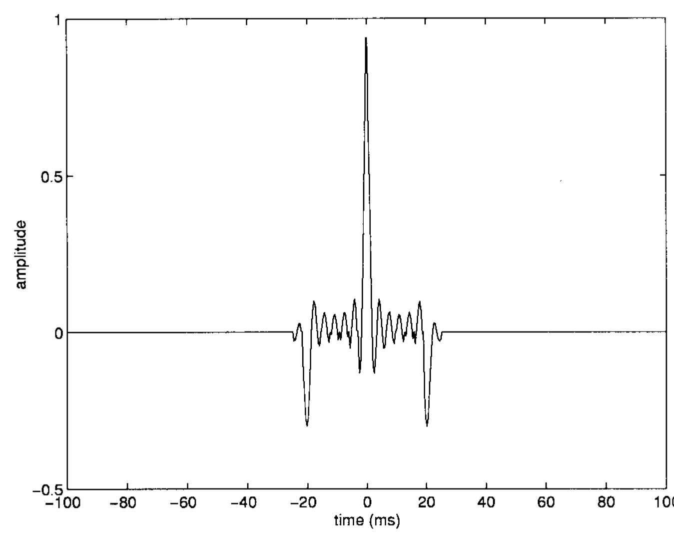

Fig. 8. Pulse generated by POCS.

However, as noted in Section III-B, there is no guarantee that the pulse $y^\star$ thus obtained is close to the other sets in any sense. One can check in Figs. 8 and 9 that $y^\star$ does indeed satisfy C_4 but is worse than the pulse $x^\star$ produced by Scenario 3 and displayed in Figs. 6 and 7 in terms of satisfying the remaining constraints $(C_i)_{1\leq i\leq 3}$. Quantitatively, this is confirmed by the fact that $\Phi_3^\triangle(x^\star)=0.0165$, whereas $\Phi_3^\triangle(y^\star)=0.0206$. We conclude by pointing out that the pulse shape design problem of [16] was treated within Framework 1 and was therefore limited to two constraints.

V. Concluding Remarks

In this paper, we have studied the problem of synthesizing signals in Hilbert spaces subject to inconsistent convex inequality constraints. Our problem formulation, which consists of minimizing an objective function $\Phi^\triangle$ penalizing the violation of the soft constraints over the feasibility set $S^\blacktriangle$

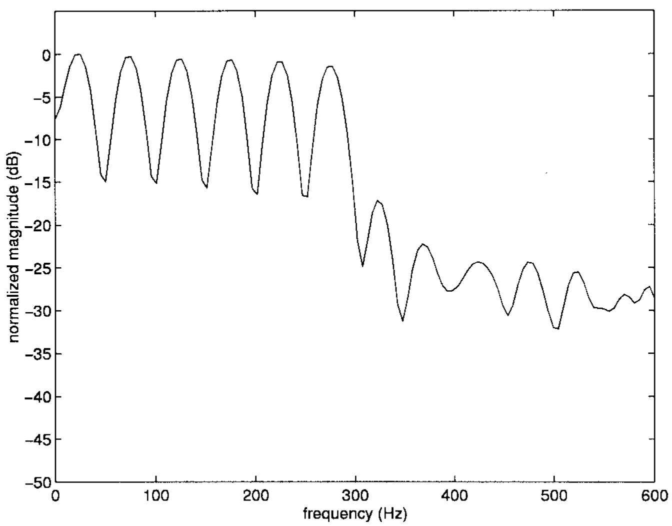

Fig. 9. Normalized spectral density of the pulse of Fig. 8.

induced by the hard constraints, covers and extends existing approaches. General conditions for the existence and the uniqueness of solutions involving only the constraint functions have been established.

In terms of numerical solution schemes, we have adopted a fixed-point approach that led to gradient projection methods for which we have provided convergence conditions under various hypotheses. One such scheme was seen to include as special cases the algorithms used in existing approaches and to be adequate for the proposed pulse shape synthesis problem. Naturally, these methods are by no means universally applicable. Indeed, they impose that $\Phi^{\triangle}$ be differentiable on $S^{\blacktriangle}$, and furthermore, their efficient implementation implicitly requires that $S^{\blacktriangle}$ be geometrically simple so that projections onto this set be easily computable (see [6] and [17] for examples of constraint sets admitting closed-form projectors). Alternative schemes should therefore be investigated. In this regard, let us notice that if a reasonably tight upper bound $\tilde{\alpha}$ is available for $\alpha^{\star}$, then the above restrictions can be lifted to the extent that the exact problem (3) can be approximated by

$$\text{Find } x^{\star} \in \tilde{G} = S^{\blacktriangle} \cap \text{lev}_{\leq \tilde{\alpha}} \Phi^{\triangle} = \left(\bigcap_{i \in I^{\blacktriangle}} \text{lev}_{\leq 0} g_i \right) \cap \text{lev}_{\leq \tilde{\alpha}} \Phi^{\triangle}. \quad (31)$$

This consistent feasibility problem can be solved via the block-iterative extrapolated subgradient projection method of [7], which requires only the ability to compute subgradients of the functions $(g_i)_{i \in I^{\blacktriangle}}$ and $\Phi^{\triangle}$.

Our last remark concerns the fuzzy set theoretic signal estimation framework proposed in [3]. The motivation behind this approach to (1) is to introduce a graded transition between the signals that satisfy a constraint $g_i(x) \leq 0$ and those that violate it when the information inducing this constraint is imprecise. Each constraint is associated with a fuzzy set, i.e., a membership function $\mu_i : \mathcal{H} \to [0, 1]$ taking value 1 on S_i. The fuzzy feasibility problem is then formulated as that of finding a signal $x^{\star}$ that maximizes the membership function μ of the intersection of the fuzzy constraint sets, say $\mu = \prod_{i \in I} \mu_i$. Now, let $I^{\blacktriangle}$ (resp. $I^{\triangle} = I \setminus I^{\blacktriangle}$) be the index set of the constraints based on hard (resp. imprecise) information. Then, for $i \in I^{\blacktriangle}$, $\mu_i = 1_{S_i}$ is simply the characteristic function of S_i, and for $i \in I^{\triangle}$, we can take $\mu_i = \exp(-f_i \circ g_i^{+})$, where $f_i \in \mathcal{F}$. Consequently, (2) gives

$$\sup_{x \in \mathcal{H}} \mu(x) = \sup_{x \in \mathcal{H}} \prod_{i \in I} \mu_i(x) = \sup_{x \in S^{\blacktriangle}} \prod_{i \in I^{\triangle}} \mu_i(x) = \sup_{x \in S^{\blacktriangle}} \exp(-\Phi^{\triangle}(x)) = \inf_{x \in S^{\blacktriangle}} \Phi^{\triangle}(x) \quad (32)$$

which casts this fuzzy signal feasibility problem in the general format (3).

Appendix A
Proofs

Proof of Proposition 1: i) Fix $i \in I^{\triangle}$. Since $\text{dom}\, f_i \circ g_i^{+} = \text{dom}\, g_i$ in (2), $\text{dom}\, \Phi^{\triangle} = D^{\triangle} \neq \varnothing$, and $\Phi^{\triangle}$ is therefore proper. It remains to show that $\Phi^{\triangle}$ is convex and lower semicontinuous, i.e., in light of [1, Props. 1.5(a) and 2.2(a)], that $f_i \circ g_i^{+}$ is convex and lower semicontinuous. The convexity of $f_i \circ g_i^{+}$ follows from [1, Prop. 2.2(d) and (e)]. To establish its lower semicontinuity, it suffices to show that $\text{lev}_{\leq \alpha} f_i \circ g_i^{+}$ is closed for an arbitrary $\alpha \in \mathbb{R}$ [1, Prop. 1.4(c)]. If $\alpha < 0$, $\text{lev}_{\leq \alpha} f_i \circ g_i^{+} = \varnothing$ is certainly closed. Now, suppose $\alpha \geq 0$. Since $f_i \in \mathcal{F}$, it is continuous relative to $[0, +\infty[$ [14, Sec. I.3.1] with $\lim_{t \to +\infty} f_i(t) = +\infty$ [14, Sec. I.2.3]. Hence, f_i is an increasing bijection from $[0, +\infty[$ onto $[0, +\infty[$ and so is its inverse f_i^{-1}. Accordingly, $\text{lev}_{\leq \alpha} f_i \circ g_i^{+} = \text{lev}_{\leq f_i^{-1}(\alpha)} g_i^{+} = \text{lev}_{\leq f_i^{-1}(\alpha)} g_i$. However, since g_i is lower semicontinuous, $\text{lev}_{\leq f_i^{-1}(\alpha)} g_i$ is closed. Assertions ii) and iii) follow at once from (2).

□

Proof of Proposition 2: In view of (3), G can be written as an intersection of closed and convex sets, namely, $G = S^{\blacktriangle} \cap \bigcap_{\alpha > \alpha^{\star}} \text{lev}_{\leq \alpha} \Phi^{\triangle}$, and it is thereby closed and convex. In connection with the existence of solutions, let us recall a fundamental fact [9, Prop. II.1.2]: Any function in Γ achieves its infimum over any nonempty, closed, convex, and bounded set over which it is proper. Now, fix $u \in S^{\blacktriangle} \cap D^{\triangle}$, and set $S_u^{\blacktriangle} = S^{\blacktriangle} \cap \text{lev}_{\leq \Phi^{\triangle}(u)} \Phi^{\triangle}$. Note that $S_u^{\blacktriangle}$ is nonempty, closed, and convex and that minimizing $\Phi^{\triangle}$ over $S^{\blacktriangle}$ is equivalent to minimizing it over $S_u^{\blacktriangle}$. By virtue of Proposition 1(i), it therefore suffices to show that $S_u^{\blacktriangle}$ is bounded if, for some $i \in I$, g_i is weakly coercive over $\mathcal{H}$. Suppose first that $i \in I^{\blacktriangle}$. Then, by the weak coercivity of g_i

$$(\exists \gamma \in]0, +\infty[)(\forall x \in \mathcal{H}) \quad g_i(x) \leq 0 \Rightarrow \|x\| \leq \gamma. \quad \text{(A1)}$$

We thus obtain the boundedness of S_i and, in turn, that of $S_u^{\blacktriangle} \subset S_i$. Suppose next that $i \in I^{\triangle}$. By (2), $f_i \circ g_i^{+} \leq \Phi^{\triangle}$. Hence, since g_i^{+} is weakly coercive and $f_i \in \mathcal{F}$

$$\|x\| \to +\infty \Rightarrow f_i \circ g_i^{+}(x) \to +\infty \Rightarrow \Phi^{\triangle}(x) \to +\infty. \quad \text{(A2)}$$

Therefore

$$(\exists\gamma\in]0,+\infty[)(\forall x\in\mathcal{H})\quad \Phi^{\triangle}(x)\leq\Phi^{\triangle}(u)\Rightarrow\|x\|\leq\gamma. \tag{A3}$$

Hence, $\mathrm{lev}_{\leq\Phi^{\triangle}(u)}\Phi^{\triangle}$ is bounded and so is $S_u^{\blacktriangle}$. We now turn to uniqueness. i) Since g_i^+ is strictly convex over $S_u^{\blacktriangle}=S^{\blacktriangle}\cap\mathrm{lev}_{\leq\Phi^{\triangle}(u)}\Phi^{\triangle}$, so are $f_i\circ g_i^+$ and, by virtue of (2), $\Phi^{\triangle}$. The claim therefore follows from [9, Prop. II.1.2] since minimizing $\Phi^{\triangle}$ over $S^{\blacktriangle}$ is equivalent to minimizing it over $S_u^{\blacktriangle}$. ii) Suppose that G contains two distinct points, say $x^\star$ and $y^\star$, and let $z^\star=(x^\star+y^\star)/2$. As shown above, G is convex, and therefore, $z^\star\in G$, i.e., $z^\star$ is a minimizer of $\Phi^{\triangle}$ over $S^{\blacktriangle}$. Now, fix $i\in I^{\blacktriangle}$. Then, $(x^\star,y^\star)\in G^2\subset {S^{\blacktriangle}}^2\subset S_i^2$, and it follows from the strict convexity of g_i over $S^{\blacktriangle}$ that $g_i(z^\star)<(g_i(x^\star)+g_i(y^\star))/2\leq 0$. The continuity of g_i at $z^\star\in S^{\blacktriangle}$ then yields $z^\star\in\overset{\circ}{S}_i$. Since $I^{\blacktriangle}$ is finite, we obtain $z^\star\in\overset{\circ}{S}{}^{\blacktriangle}$, say $z^\star\in B\subset S^{\blacktriangle}$ for some ball B. However, since $z^\star\in G$, $z^\star$ minimizes $\Phi^{\triangle}$ over B. In other words, $z^\star$ is a local minimizer of $\Phi^{\triangle}$ and, therefore, by convexity of $\Phi^{\triangle}$, a global one [21, Prop. 42.3]. This contradicts the assumption that $\Phi^{\triangle}$ has no free minimizer in $S^{\blacktriangle}$. We conclude that G contains at most one point. □

Subsequently, we shall need the well-known parallelogram identity

$$(\forall(x,y)\in\mathcal{H}^2)\quad \left\|\frac{x+y}{2}\right\|^2+\left\|\frac{x-y}{2}\right\|^2=\frac{\|x\|^2+\|y\|^2}{2}. \tag{A4}$$

Proof of Proposition 4: $G\neq\varnothing$ follows from Proposition 2 and the observation that if S_i is bounded, then $d(\cdot,S_i)^2$ is weakly coercive. Let us now prove that the second assertion follows from Proposition 2(i) by proving that $d(\cdot,S_i)^2$ is strictly convex over $S^{\blacktriangle}$. Let x and y be any two distinct points in $S^{\blacktriangle}$, $z=(x+y)/2$, and $z_i=(P_i(x)+P_i(y))/2$. Since $z_i\in S_i$, (A4) implies

$$\begin{aligned}d(z,S_i)^2&\leq\|z-z_i\|^2\\&=\left\|\frac{x-P_i(x)+y-P_i(y)}{2}\right\|^2 \qquad\qquad (A5)\\&=\frac{\|x-P_i(x)\|^2+\|y-P_i(y)\|^2}{2}\\&\quad-\left\|\frac{x-P_i(x)-y+P_i(y)}{2}\right\|^2\\&\leq\frac{d(x,S_i)^2+d(y,S_i)^2}{2}. \qquad\qquad (A6)\end{aligned}$$

If $P_i(x)\neq P_i(y)$ then, since S_i is strictly convex, $z_i\in\overset{\circ}{S}_i$. However, since $z\in S^{\blacktriangle}$, $z\notin S_i$ and, in turn, $P_i(z)\in S_i\backslash\overset{\circ}{S}_i$. Therefore, the inequality in (A5) is strict. On the other hand, if $P_i(x)=P_i(y)$, then $x-P_i(x)-y+P_i(y)\neq 0$, and we obtain a strict inequality in (A6). In both alternatives, we obtain $d(z,S_i)^2<(d(x,S_i)^2+d(y,S_i)^2)/2$, and the claim is proved. □

Proof of Proposition 6: Let x and y be any two points in $S^{\blacktriangle}$. Then, using (A4) and the properties of L, we obtain

$$\begin{aligned}g_i\left(\frac{x+y}{2}\right)&=\left\|\frac{L(x)-r+L(y)-r}{2}\right\|^2-\xi\\&=\frac{\|L(x)-r\|^2+\|L(y)-r\|^2}{2}\\&\quad-\left\|\frac{L(x-y)}{2}\right\|^2-\xi\\&\leq\frac{g_i(x)+g_i(y)}{2}-\frac{\chi^2}{4}\|x-y\|^2. \qquad (A7)\end{aligned}$$

However, under our hypotheses, $g_i=f_i\circ g_i^+$ on $S^{\blacktriangle}$. It then follows from the convexity of the functions $(f_j\circ g_j^+)_{j\in I^{\triangle}\backslash\{i\}}$ and (2) that

$$\Phi^{\triangle}\left(\frac{x+y}{2}\right)\leq\frac{\Phi^{\triangle}(x)+\Phi^{\triangle}(y)}{2}-\frac{\chi^2}{4}\|x-y\|^2. \tag{A8}$$

This inequality translates the fact that $\Phi^{\triangle}$ is strongly convex with modulus $\eta=2\chi^2$ on $S^{\blacktriangle}$, and it implies that $\nabla\Phi^{\triangle}$ is η-strongly monotone on $S^{\blacktriangle}$ [14], [15]. □

Proof of Proposition 7: The first assertion is a direct application of [8, Prop. 9] with the nonexpansive operator $P^{\blacktriangle}\circ(\mathrm{Id}-\gamma\nabla\Phi^{\triangle})$. This result also asserts that $(x_n)_{n\geq0}$ lies in some closed ball B. Hence, by construction, $(x_n)_{n\geq0}$ lies in $B\cap S^{\blacktriangle}$, which is compact if $S^{\blacktriangle}$ is boundedly compact. The proof is completed by noting that the notions of weak and strong convergence coincide in compact sets [11, Th. 7.69]. □

Proof of Proposition 8: This proof is a direct application of [19, Th. 2] with the nonexpansive operator $P^{\blacktriangle}\circ(\mathrm{Id}-\gamma\nabla\Phi^{\triangle})$. □

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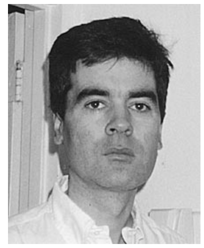

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