Strong Nash Equilibria in Finite Games

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Efficiency and Individual Rationality

In non–cooperative setting:

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Efficiency–versus–Individual rationality

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Efficiency–versus–Individual rationality

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This is the not very well known prisoner dilemma!

Price of stability/anarchy

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This can be defined for a single game, but more interesting for classes of games.

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- $W(s) = \sum_{i \in N} u_i(s)$, (utilitarian objective)
- $W(s) = \min_{i \in N} u_i(s)$, (fairness or egalitarian objective).

When the game represents a lucky situation for the players

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Existence of a strong Nash equilibrium: price of stability $=1$.

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Goal of the paper

Existence of strong Nash equilibria is not guaranteed for standard finite games.

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The set of games with strong Nash equilibria is "small".

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2. U is a $n \times m_1 \times m_2 \times ... \times m_n$ n-matrix. An element of U_i denoted by $U_i(i_1, \dots, i_n)$, $i_1 = a_{j_1}, \dots, i_n = a_{j_n}$

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4. For a strategy profile x, utility of i is

$$v_i(x) = \sum_{i_1, \ldots, i_n} U_i(i_1, \ldots, i_n) \cdot x_{i_1} \cdot \cdots \cdot x_{i_n} := x_i^t U_i \prod_{j \neq i} x_j.$$

Nash equilibrium

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A strategy profile is a Nash equilibrium if it is stable w.r.t. deviations of a single player.

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(1a) is called the Indifference principle.

Pareto Efficiency

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$\bar{x}$ *is* weakly Pareto dominated *if there exists a strategy profile x such that*

$$V(x) \neq V(\bar{x}) \quad \wedge \quad V(x) \in V(\bar{x}) + \mathbb{R}^n_+,$$

strictly Pareto dominated if there exists a strategy profile x such that

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Pareto Efficiency and KKT conditions

Consider the problem (F,G,H):

$$\max F(x) : G(x) \leq 0, H(x) = 0$$

where $F : \mathbb{R}^k \to \mathbb{R}^l, G : \mathbb{R}^n \to \mathbb{R}^j, H : \mathbb{R}^n \to \mathbb{R}^s$, G, H affine and max is intended in weak Pareto sense. Then:

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KKT Conditions: Suppose x is (weakly) efficient for the problem (F,G,H). Then there are vectors λ, μ, ν verifying the following system:

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(x) - \sum_{j=1}^{m} \mu_j \nabla g_j(x) + \sum_{j=1}^{m} \nu_j \nabla h_j(x) = 0, \quad (2a)$$

$$(\lambda, \mu) \geq 0, \quad (2b)$$

$$\mu^\top g(x) = 0, \quad (2c)$$

$$\lambda \neq 0. \quad (2d)$$

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Necessary conditions (for a strong Nash):

- Indifference principle
- KKT (for every subcoalition)

Back to the main question

Let G be the space of all games: $G \simeq \mathbb{R}^{m_1+\dots+m_n}$, let $SN \subset G$ the subset of games having a(t least) strong Nash equilibrium:

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Example above: pure strategy SNE. What about mixed strategy SNE?

The case of a fully mixed SNE (two players)

Let (x, y) be a SNE. We assume $U_1 y = \mathbf{0}$, $x^t U_2 = \mathbf{0}$. **a** is vector, of the right dimension, whose entries are all a's.

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Proposition

Let x be a fully mixed strong Nash equilibrium. Then, for some $\lambda = (\lambda_1, \lambda_2)$ *and* $\nu = (\mu_1, \mu_2)$*:*

$$\lambda_2 x_2^t U_2 - \nu_1 \mathbf{1} = \mathbf{0} \quad (3)$$

$$\lambda_1 x_1^t U_1^- \nu_2 \mathbf{1} = \mathbf{0} \quad (4)$$

$$\lambda > 0 \quad (5)$$

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A further step

In the result above the multiplier λ is non null.

Proposition

Let x be a fully mixed super strong Nash equilibrium. Then it satisfies (3) (4), and also the relations

$$U_1^\top x_1 = \mathbf{0}, \qquad \wedge \qquad U_2^t(x_2) = \mathbf{0}$$

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satisfying the system (3) (4). Then either it satisfies the further conditions

$$U_1^\top x_1 = \mathbf{0}, \qquad \wedge \qquad U_2^t(x_2) = \mathbf{0},$$

, or else all entries of the bimatrix (U_1, U_2) lie on a either vertical or horizontal line through the origin.

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A fully mixed NE is a SNE only when this is trivially true.

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- From above all lines pass through the Nash equilibrium payoff $(0,0)$
- From above the only non trivial case to consider is when a situation of this type occurs:

$$\bar{U} = \begin{pmatrix} U_{1j} & (0,0) \\ (0,0) & U_{2k} \end{pmatrix}$$

Suppose there is $t > 0$ such that

$$tU_{1j} + (1-t)U_{2k} = (2a, 2a)$$

for some $a > 0$. Consider the strategy profile $\bar{x} = [(t, 1-t), (\frac{1}{2}, \frac{1}{2})]$. Then $\bar{x}_1^t \bar{U}_i \bar{x}_1 = a > 0$ for $i = 1, 2$, contradicting the fact that x is a strong Nash equilibrium.

A further step

Theorem

Let x be $s_1 \times s_2$ mixed-strategy SNE. Then in the $s_1 \times s_2$ restriction of the bimatrix U where the outcomes are played with positive probability all the outcomes lie on the same straight line, with non-positive slope.

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Obvious consequence

The set of finite, two player games having a strong Nash equilibrium in mixed strategies is small

An example with more players

Player one chooses a row, Player two a column, and Player three the matrix to play.

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Player one chooses a row, Player two a column, and Player three the matrix to play.

$$\begin{pmatrix} (2,0,0) & (0,2,0) \\ (0,0,2) & (0,0,0) \end{pmatrix}, \begin{pmatrix} (0,0,0) & (0,0,2) \\ (0,2,0) & (2,0,0) \end{pmatrix}$$

Easy to see:

- using equal probabilities for all players is a Nash Equilibrium

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- the outcomes do not lie on a plane

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$$\begin{pmatrix} (2,0,0) & (0,2,0) \\ (0,0,2) & (0,0,0) \end{pmatrix}, \begin{pmatrix} (0,0,0) & (0,0,2) \\ (0,2,0) & (2,0,0) \end{pmatrix}$$

Easy to see:

- using equal probabilities for all players is a Nash Equilibrium
- the outcomes do not lie on a plane
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Less easy to verify that the three players together do not have incentive to deviate, but believe me Observe: a Pareto dominated outcome is played here with positive probability.

And null measure?

Observe, in the two player case the system a SNE must fulfill is linear.

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Definition

A subset A of an Euclidean space is called algebraic *if it can be described as a finite number of polynomial equations. It is called* semialgebraic *if it can be described as a finite number of polynomial equalities and inequalities. A multivalued map between Euclidean spaces is called* algebraic (semialgebraic) *if its graph is an algebraic (semialgebraic) set.*

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Two basic facts on semialgebraic multimaps

- Given an algebraic set A on $X \times Y$ its projection on each space X, Y is semialgebraic
- For any semialgebraic set-valued mapping Φ between two Euclidean spaces $\Phi : \mathbf{E} \rightrightarrows \mathbf{Y}$, if $\dim \Phi(x) \leq k$ for every $x \in \mathbf{E}$, then $\dim \Phi(\mathbf{E}) \leq \dim \mathbf{E} + k$.

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Idea of the proof (three player case)

Consider the coalitions made by two players, apply the KKT conditions and see that the SNE must satisfy:

$$\begin{cases} U_1 x_1 x_2 & = \mathbf{1} \\ U_2 x_1 x_2 & = \mathbf{1} \\ U_3 x_1 x_2 & = \mathbf{1} \end{cases}$$

Proof:continued

Define the map: $\Phi : \triangle^{m_1} \times \triangle^{m_2} \rightrightarrows \left(\mathbf{M}^{m_1 \times m_2}\right)^{3m_3}$ defined by

$$\Phi\left(x_1, x_2\right) = \left\{\left(A_1, A_2, ..., A_{3m_3}\right) : x_1^\top A_i x_2 = b_i \forall i\right\},$$

where A_i are the lines of the equations in the system.

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Make an easy calculation of the dimensions and conclude.

Final remarks

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- When looking for SNE one can limit the search to *pure equilibria*
- a weakening of super strong Nash equilibrium is given by the k-SNE: this allows for deviations of coalitions of size not greater than k: our proof shows that $k = 2$ suffices to have negligibility
- the result on the two player case does not hold in the same way for strong Nash equilibria: a game need not to be strictly competitive to have a SNE. But this happens only in trivial cases.