Multicriteria Optimization
Some continuous and discrete dynamics

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Sestri-Levante: Franco/Italian workshop
8-12 September 2014

Context

- H is an Hilbert space,
- $f_i : H \to \mathbb{R}$ are Lipschitz continuous on bounded sets.
- $K \subset H$ is a closed convex non empty set of constraints,
- One of the objective functions is bounded from below.

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chose $0 \leq \theta_i \leq 1$, $\sum_{i=1}^{q} \theta_i = 1$, and minimize $\sum_{i=1}^{q} \theta_i f_i$.

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We are looking for the **simultaneous** minimization of the f_i's.

Contents

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Directional derivative (of Clarke)

$$df(x;d) := \limsup_{\substack{t \downarrow 0 \\ x' \to x}} \frac{f(x'+td) - f(x')}{t}.$$

Subdifferential (of Clarke)

$$\partial f(x) := \{p \in H \mid \langle p, d \rangle \leq df(x;d) \ \forall d \in H\}.$$

Remark

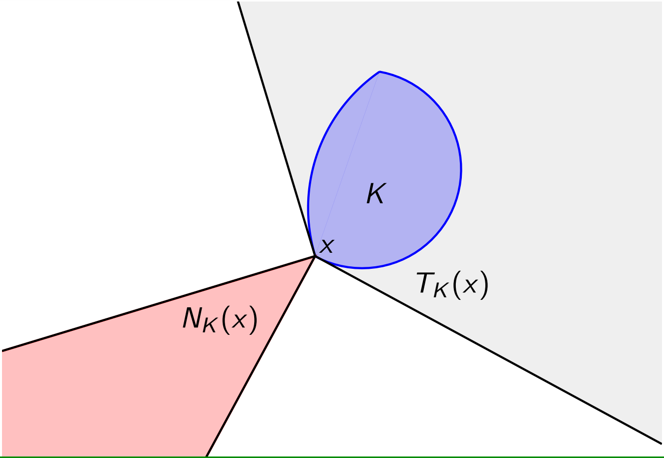
If f is of class C^1, then
$\partial f(x) = \{\nabla f(x)\}$ and $df(x;d) = \langle \nabla f(x), d \rangle$.

Nonsmooth analysis tools

Tangent and normal cones

$$T_K(x) := cl\ \{d \in H \mid \exists \varepsilon > 0, \forall t \in]0, \varepsilon[,\ x + td \in K\}.$$

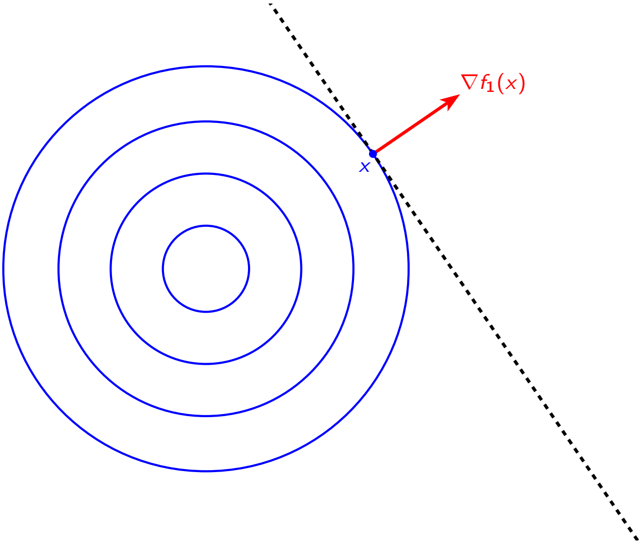
$$N_K(x) := \{p \in H \mid \langle p, d \rangle \leq 0\ \forall d \in T_K(x)\}.$$



Descent direction

We say that $d \in H$ is a *descent direction* at x if $df_i(x; d) < 0$ holds for all $i = 1..q$.
We say that it is an *admissible* descent direction if moreover $d \in T_K(x)$.

Example



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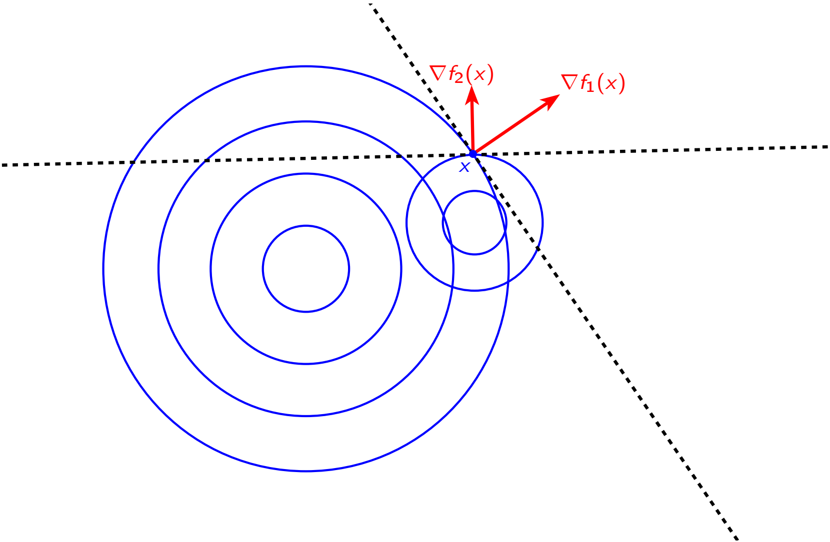

Armijo direction

We say that a descent direction $d \in H$ is an *Armijo direction* if $\exists \varepsilon > 0$ s.t. for all $t \in]0, \varepsilon[$:

$$\forall i, f_i(x + td) < f_i(x) + \frac{t}{2} df_i(x; d).$$

We say that it is an *admissible* Armijo direction if moreover $x + td \in K$.

Pareto equilibrium(s)

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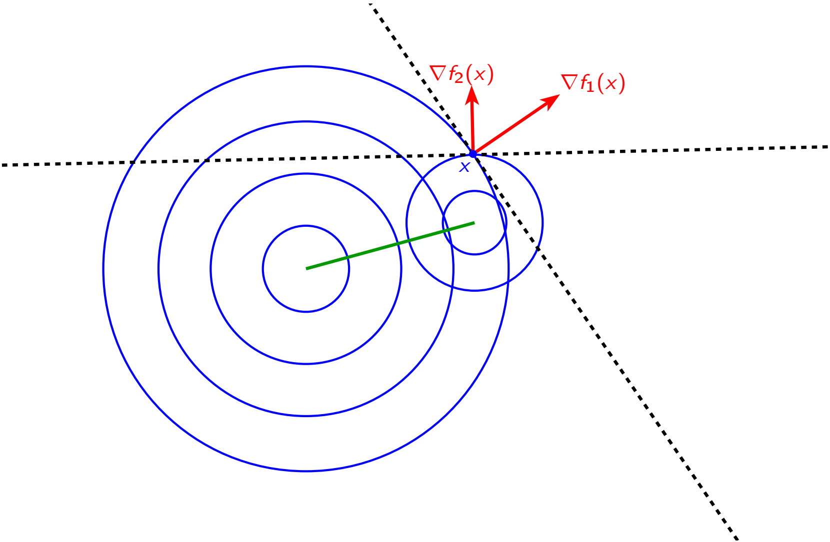

Multicriteria analysis

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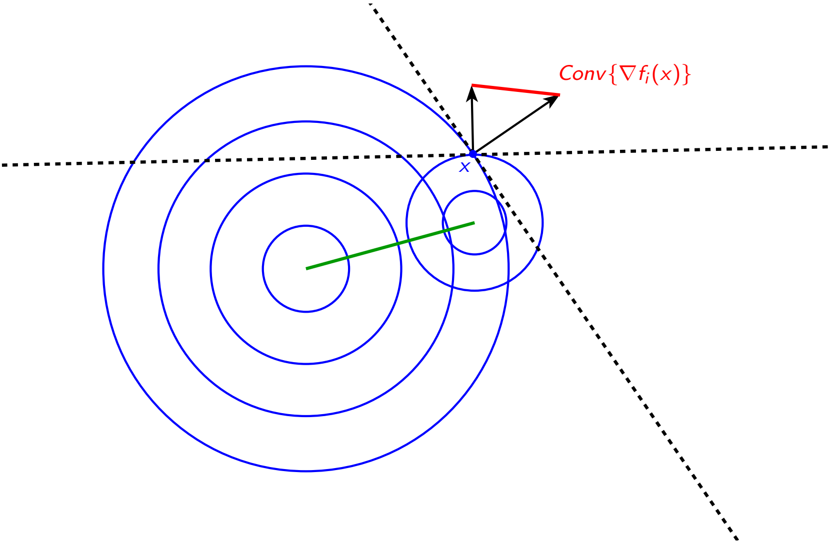

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Properties

- Pareto $\Rightarrow$ weak Pareto $\Rightarrow$ critical Pareto.
- If the f_i are convex, then weak Pareto $\Leftrightarrow$ critical Pareto.
- If the f_i are strictly convex, then the 3 notions both coincide.

Link between descent direction and Pareto equilibrium

Proposition

The following statements are equivalent :

- x is a critical Pareto point,
- There is no admissible descent direction at x,
- There is no admissible Armijo direction at x.

Objectif

We will consider

1. a continuous dynamic $\dot{u}(t) = s(u(t))$, where $s : K \to H$ verify
   - $s(u) = 0$ if u is a critical Pareto point,
   - $s(u)$ is an admissible descent direction else.
2. an algorithm $u_{n+1} = u_n + t_n d_n$ where d_n is an admissible Armijo direction.

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The multiobjective steepest descent direction

Definition

Given $x \in K$, the *multiobjective steepest descent direction* is

$$s(x) := -\left(N_K(x) + Conv\{\partial f_i(x)\}\right)^0 .$$

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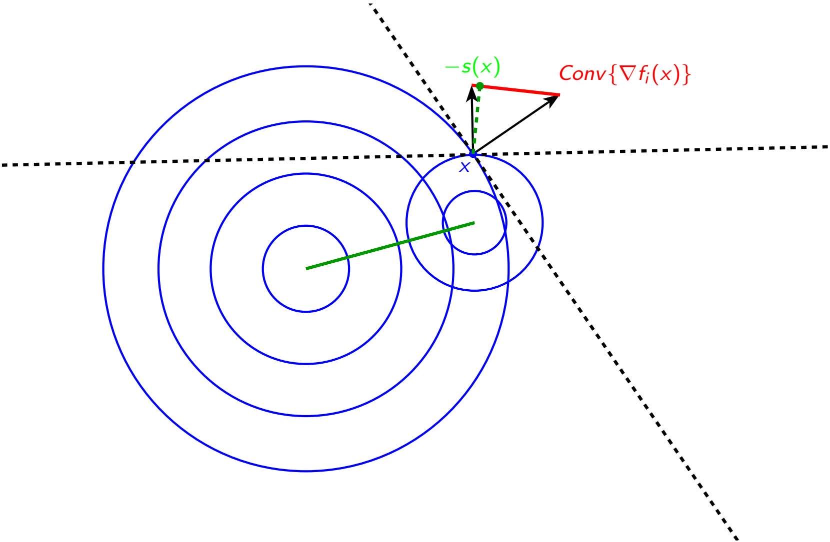

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In a sense, $s(x)$ selects itself a different convex combination of the functions at each x.

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Property

$s(x)$ is an admissible descent direction at x, whenever $s(x) \neq 0$.

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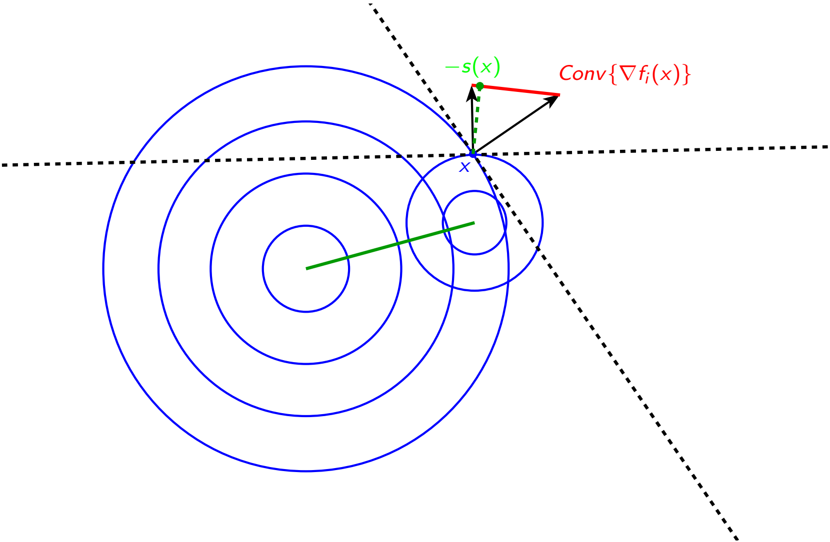

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Recall that (one objective function, no constraint) :

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Recall that (one objective function, no constraint) :

$$\frac{-\nabla f(x)}{\|\nabla f(x)\|} = \underset{\|d\| \leq 1}{\text{argmin}} \;\; df(x, d).$$

The multiobjective steepest descent direction generalizes this fact :

Theorem (Attouch, Garrigos, Goudou, 2014)

$$\frac{s(x)}{\|s(x)\|} = \underset{\|d\| \leq 1, d \in T_K(x)}{\text{argmin}} \;\; \max_i df_i(x, d).$$

A continuous dynamic

The Multi-Objective Gradient dynamic :

(MOG) $\dot{u}(t) = s(u(t))$ i.e $\dot{u}(t) + (N_K(u(t)) + Conv\{\partial f_i(u(t))\})^0 = 0$

A continuous dynamic : example 1

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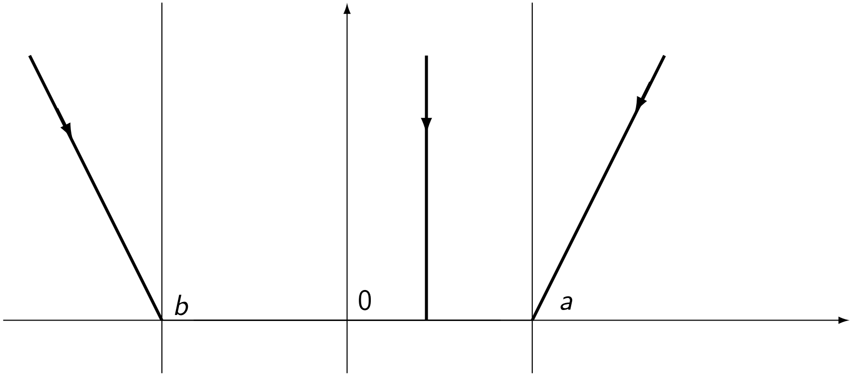

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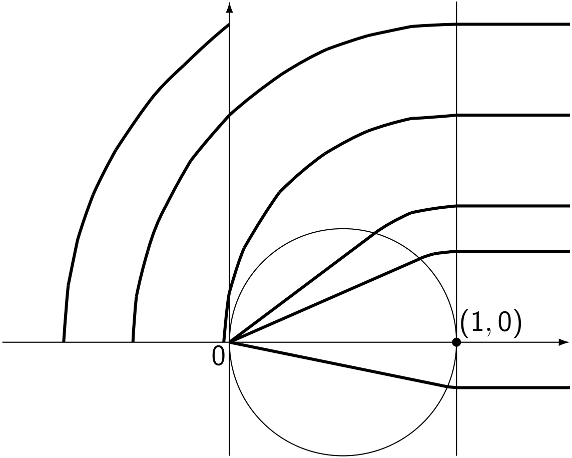

A continuous dynamic : Existence and uniqueness

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Theorem (Attouch, Garrigos, Goudou, 2014)

Suppose that H is finite-dimentional, and that the functions are convex and bounded from below. Then for any $u_0 \in K$, there exists a strong solution $u : [0, +\infty[\to K$ of (MOG), such that $u(0) = u_0$.

Strong solution essentially means an absolutely continuous trajectory u satisfying (MOG) for a.e. $t > 0$.

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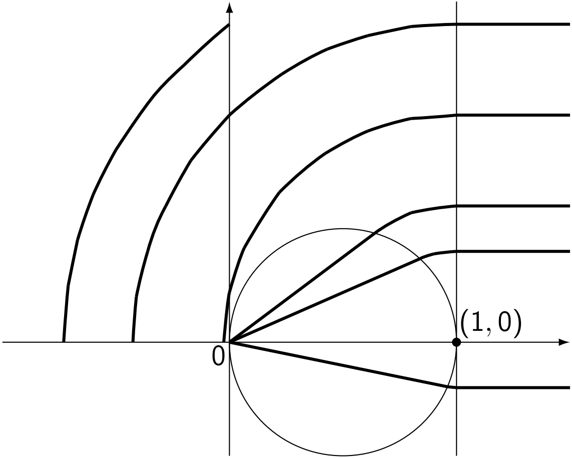

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$$\text{(MOG) } \dot{u}(t) = s(u(t)) \text{ i.e } \dot{u}(t) + \left(N_K(u(t)) + Conv\{\partial f_i(u(t))\}\right)^0 = 0$$

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The proof cannot rely on Cauchy-Lipschitz because of lack of Lipschitz regularity.
$\rightarrow$ Use Morau-Yoshida's regularization onto the f_i's and the indicator function.
$\rightarrow$ Use Peano's existence theorem on the regularized system : it asks only continuity but do not guarantee uniqueness.
$\rightarrow$ Pass to the limit. Hard.

About uniqueness

The problem of uniqueness is still open.
Can we find hypotheses ensuring Lipschitz continuity of $s(u)$?

Local Lipschitz property

Suppose $K = H$, and that the functions are of class $C^{1,1}$.
The vector field s is Lipschitz continuous at u if :

- $q = 2$, and $\nabla f_1(u) \neq \nabla f_2(u)$.
- The vectors $\nabla f_i(u)$ are linearly independent.

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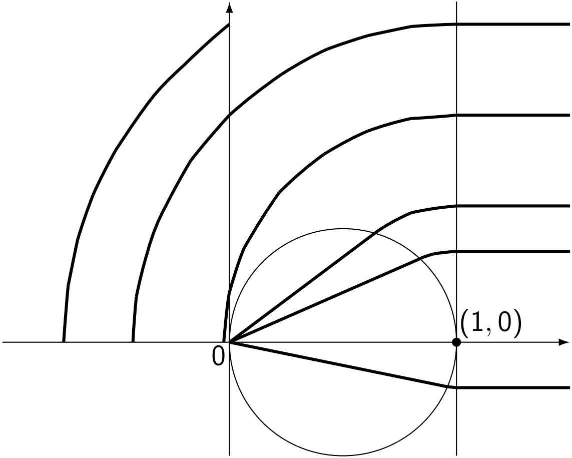

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Suppose that the objective functions are quasi-convex.

- Then any bounded trajectory is weakly convergent.
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- We recover classic results by taking $q = 1$.
- Can we have strong convergence under stronger assumptions ?

Continuous case : What (MOG) is not

- A descent method associated to some scalarization $\sum_{i=1}^{q} \theta_i f_i$. In (MOG) the θ_i are chosen and modified automatically along the time. And ALL the functions decrease.

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- A descent method associated to $\max f_i$.

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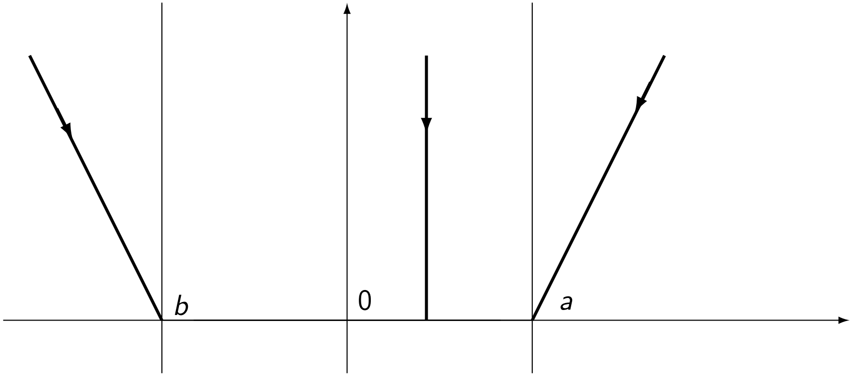

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Discrete descent direction

We want to consider an algorithm $u_{n+1} = u_n + t_n d_n$ where d_n is an admissible Armijo direction.
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An Armijo admissible descent selection

For $\mu > 0$, consider $\sigma_\mu(x) := \underset{d \in \frac{K-x}{\mu}}{\operatorname{argmin}} \ \frac{1}{2}\|d\|^2 + \max_i df_i(x, d)$.

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Remark

- $s(x) = \sigma_\mu(x)$ whenever K is a vectorial subspace.
- In some sense, $\sigma_\mu(x) \to s(x)$ when $\mu \to 0$.

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Proposition

For all $\mu > 0$:

- $\sigma_\mu(x) = 0$ if x is a Pareto critical point.
- $\sigma_\mu(x)$ is an admissible Armijo direction else.

Algorithm : steepest descent method with linesearch

Consider a sequence of parameters $(\mu_n) \subset\subset \mathbb{R}_+$ and $x_0 \in K$.
Compute $d_n = \sigma_{\mu_n}(x_n)$.
Select a stepsize t_n by a common Armijo selection over all the f_i's.
Take $x_{n+1} = x_n + t_n d_n$.

Remark

- The algorithm is well defined.
- At each step, all the objective functions decrease.

Algorithm : steepest descent method with linesearch

Theorem

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Theorem (Drummond, Iusem, 2004) ; (Fliege, Svaiter, 2000)

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Corollary

In finite dimension, if the functions are C^1 quasi-convex, then any bounded sequence converges to a critical Pareto point.

Algorithm : some comments

- Differentiable case, no constraints : the multiobjective descent method is relatively easy to obtain once the gradients $\nabla f_i(x_n)$ are known, mainly if q is small ($q \leq 3\,?$).

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- Even with convexity, explicit schemes are bad designed for non-differentiable functions (recall monocriteria case).

Bibliography

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