Sequential Averaging for Nonexpansive Maps

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T contraction — Banach-Picard iterates

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$$\|x^{n+1} - x^n\| = \|Tx^n - x^n\| \leq \rho^n \|Tx^0 - x^0\| \to 0$$

$$\Downarrow$$

convergence + error estimates + stopping rule

T nonexpansive — Krasnoselski-Mann iterates

$T : C \to C$ non-expansive / C convex bounded in $(X, \|\cdot\|)$

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Equivalent: $\frac{x^{n+1}-x^n}{\alpha_{n+1}} \to 0$?

How is this useful?

If $\|Tx^n - x^n\| \to 0$

- $\Rightarrow$ all strong/weak cluster points are fixed points of T
- $\Rightarrow$ existence: Fixed Point Theorem (Browder-Göhde-Kirk'65)

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$\Rightarrow$ all strong/weak cluster points are fixed points of T

$\Rightarrow$ existence: Fixed Point Theorem (Browder-Göhde-Kirk'65)

and since $\|x^n - \bar{x}\|$ decreases for all $\bar{x} \in \mathrm{Fix}\, T$

$\Rightarrow$ x^n converges strong/weak to a fixed point

$\Rightarrow$ convergence results of Krasnoselski'55, Shaefer'57, Browder-Petryshyn'67, Edelstein'70, Groetsch'72, Ishikawa'76, Edelstein-O'Brien'78, Reich'79... Kohlenbach'03

Baillon-Bruck's conjecture

There exists a universal constant κ such that

$$(BB) \qquad \|Tx^n - x^n\| \leq \kappa \frac{\operatorname{diam}(C)}{\sqrt{\sum_{k=1}^{n} \alpha_k(1-\alpha_k)}}$$

Rescaling the norm we may assume $\operatorname{diam}(C) = 1$

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Proved with $\kappa = 1/\sqrt{\pi} \sim 0.5642$ when $\alpha_n \equiv \alpha$ (Baillon-Bruck'1996).

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- Proved for general α_n with $\kappa = 1/\sqrt{\pi}$ (C-Soto-Vaisman'2013)
- With $\kappa \sim 0.4688 < 1/\sqrt{\pi}$ for affine maps (optimal constant)
- Very recent result... yet to be checked:
  $\kappa = 1/\sqrt{\pi}$ best possible for nonlinear maps (C-Bravo'2014)

Is this bound sharp?

Until yesterday... Worst case known: right shift on $\ell^1(\mathbb{N})$

$$\kappa \geq \gamma \sim 0.4688 \text{ (83\% of upper bound)}$$

Today... We can numerically build examples attaining $\kappa \sim 0.5621$

Some open questions

- Errors and/or stochastic noise: $T(x_n) + \varepsilon_n$?
- Best constant κ_d in dimension d ?
- Hilbert spaces: prove bound with $\kappa = \frac{1}{2}$
- C unbounded: Rate of convergence for Borwein-Reich-Shafrir?

$$\| Tx_n - x_n \| \to r \triangleq \inf_{x \in C} \| Tx - x \|$$

- Other iterations:

$$x_n = \frac{1}{n} \sum_{i=0}^{n-1} T^i(x_0)$$

$$x_{n+1} = (I - \lambda_{n+1} A)^{-1} (I + \lambda_{n+1} B)^{-1} x_n$$

Some ideas behind the proof

Since $Tx^n - x^n = \frac{x^{n+1} - x^n}{\alpha_{n+1}}$ it suffices to bound $\|x^m - x^n\|$

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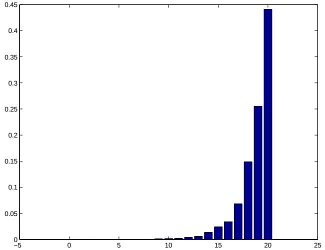
Let $\pi_i^n = \alpha_i \prod_{i+1}^n (1-\alpha_k)$ and adopt the convention $Tx^{-1} = x_0$

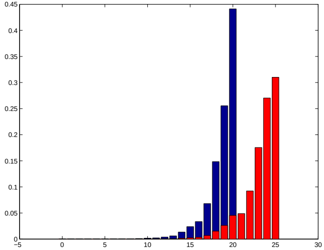
$$\boxed{x^n = \sum_{i=0}^n \pi_i^n Tx^{i-1}}$$



A recursive bound $\|x^m - x^n\| \leq d_{mn}$

Let P_{mn} be the set of transport plans $z \geq 0$ taking π^m to π^n

$$\begin{aligned} \pi_j^m &= \textstyle\sum_{i=0}^n z_{ji} \\ \pi_i^n &= \textstyle\sum_{j=0}^m z_{ji} \end{aligned}$$



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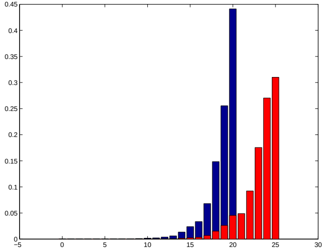

$$x^m - x^n = \sum_{j=0}^{m} \pi_j^m T x^{j-1} - \sum_{i=0}^{n} \pi_i^n T x^{i-1}$$

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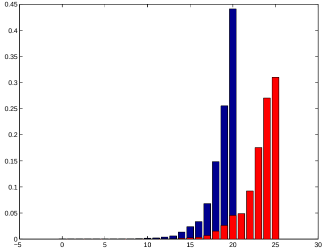

$$x^m - x^n = \sum_{j=0}^{m} \sum_{i=0}^{n} z_{ji}[Tx^{j-1} - Tx^{i-1}]$$

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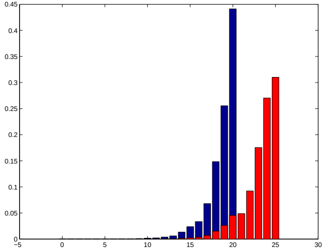

$$\|x^m - x^n\| \le \sum_{j=0}^{m} \sum_{i=0}^{n} z_{ji} \|x^{j-1} - x^{i-1}\|$$

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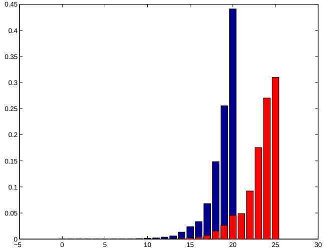

Setting $d_{-1,n} = 1$ we get inductively

$$\|x^m - x^n\| \leq d_{mn} \triangleq \min_{z \in P_{mn}} \sum_{j=0}^{m} \sum_{i=0}^{n} z_{ji} d_{j-1,i-1}$$

A recursive bound $\|x^m - x^n\| \leq d_{mn}$

$$(R) \qquad d_{mn} \triangleq \min_{z \in P_{mn}} \sum_{j=0}^{m} \sum_{i=0}^{n} z_{ji} d_{j-1,i-1}$$

Proposition

The recursion (R) defines a metric on the ground set $\{-1, 0, 1, 2, 3, \ldots\}$

$$\Rightarrow \qquad \|Tx^n - x^n\| \leq \frac{d_{n,n+1}}{\alpha_{n+1}} = \ ?$$

Upper estimate: $d_{mn} \leq c_{mn}$

Consider the particular transport plan

$$z_{ji} = \begin{cases} \pi_j^n & \text{for } i = j \\ \pi_j^m \pi_i^n & \text{for } i = m+1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

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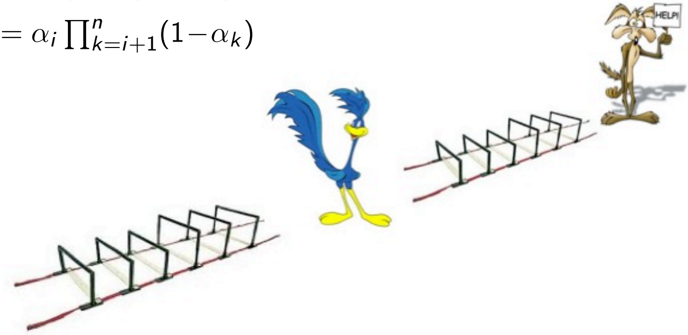
Probabilistic interpretation of the recursion

$\mathbb{P}[C_i = 1] = \mathbb{P}[R_i = 1] = \alpha_i$

$\pi_i^n = \alpha_i \prod_{k=i+1}^{n}(1-\alpha_k)$



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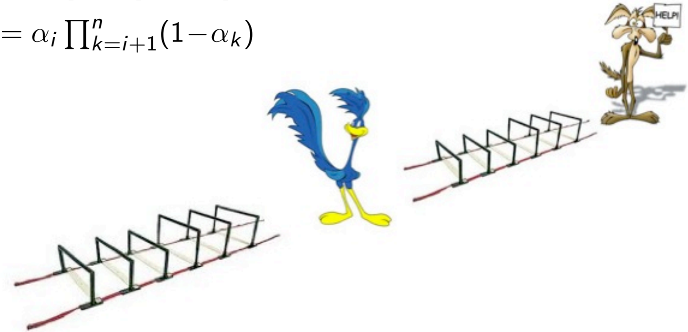

$$c_{mn} = \mathbb{P}[\text{roadrunner escapes}]$$

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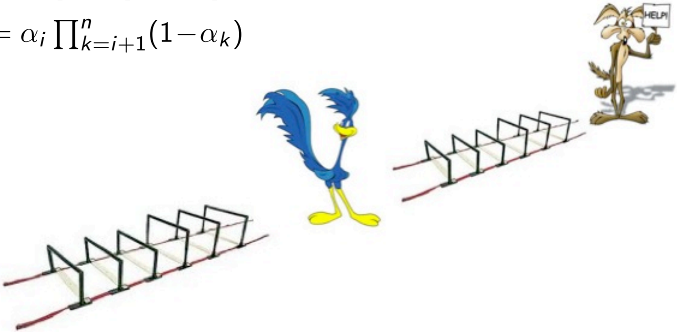

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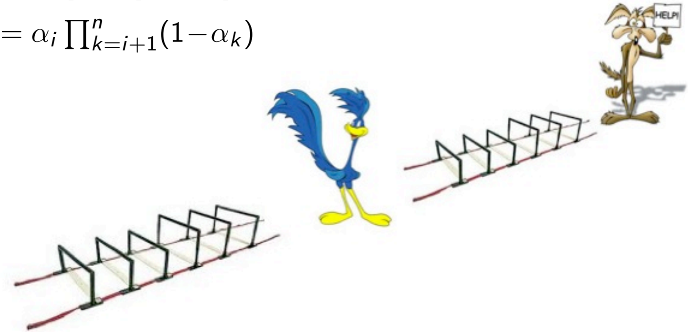

$$c_{mn} = \mathbb{P}[\sum_k^n C_i > \sum_k^m R_i, \forall k = m+1, \dots, 1]$$

Coyote must fall more often than Roadrunner

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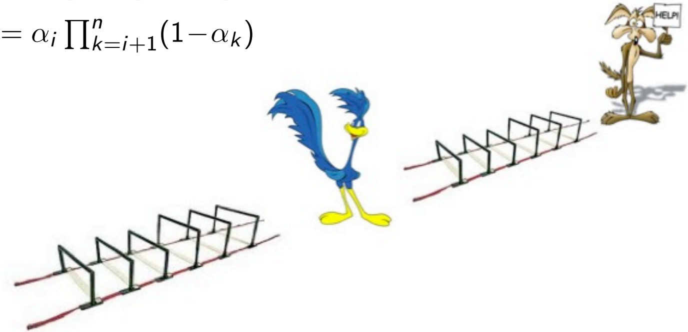

$$\|Tx^n - x^n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} = \mathbb{P}[\textstyle\sum_k^n Z_i \geq 0, \forall k = n, \ldots, 1]$$

$$Z_i = C_i - R_i$$

The random walk and the gambler's ruin appear...

$$Z_i = C_i - R_i = \begin{cases} -1 & \text{pbb} \quad \alpha_i(1-\alpha_i) \\ 0 & \text{pbb} \quad 1-2\alpha_i(1-\alpha_i) \\ 1 & \text{pbb} \quad \alpha_i(1-\alpha_i) \end{cases}$$

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$\Rightarrow$ random walk on $\mathbb{Z}$ that moves with probability $p_i = 2\alpha_i(1-\alpha_i)$ and then tosses a coin to decide whether to go left or right

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$\Rightarrow$ $Z_i = M_i D_i$ with M_i=*move/stay* and D_i=*direction*

$$M_i = \begin{cases} 1 & \text{pbb} & p_i \\ 0 & \text{pbb} & 1-p_i \end{cases} \quad ; \quad D_i = \begin{cases} -1 & \text{pbb} & \frac{1}{2} \\ 1 & \text{pbb} & \frac{1}{2} \end{cases}$$

An explicit formula for the bound

Conditional on the number of moves $M = M_1 + \ldots + M_n = m$, this is a standard random walk on m stages. The probability for the latter to remain non-negative is $F(m) = \binom{m}{\lfloor m/2 \rfloor} 2^{-m}$, therefore

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Thus (BB) has been reduced to

$$\underbrace{\sqrt{\tfrac{\pi}{2}(p_1 + \ldots + p_n)}\,\mathbb{E}[F(M_1 + \ldots + M_n)]}_{R(p)} \leq 1$$

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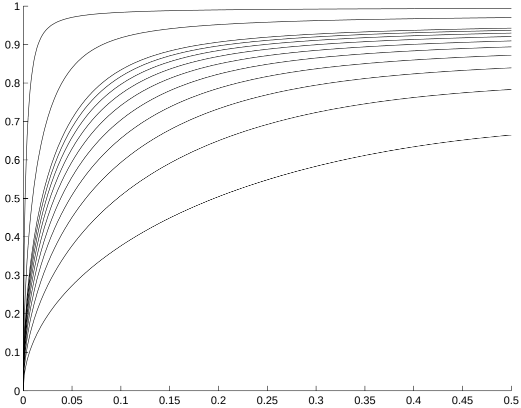
Lemma

$R(p)$ is maximal when $p_i \in \{u, \frac{1}{2}\}$ for some $0 < u < \frac{1}{2}$

Sharp bound: all $p_i = u$

$$R(p) = \sqrt{\tfrac{\pi}{2} nu}\, \mathbb{E}[F(B(n,u))] = \sqrt{\tfrac{\pi}{2} nu}\, {}_2F_1(-n, \tfrac{1}{2}; 2; 2u)$$



Sharp bound: some $p_i = \frac{1}{2}$

Suppose $p_1 = \frac{1}{2}$ and let $S = M_2 + \ldots + M_n$. Conditioning on M_1

$$\mathbb{E}[F(M)] = \mathbb{E}[G(S)]$$

where $G(k) = \frac{1}{2}[F(k) + F(k+1)]$.

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This G is convex so we may use the following Hoeffding-type inequality

Theorem (C-Soto-Vaisman, arXiv'2012)

Let Z be Poisson with $z = \mathbb{E}(Z) = \mathbb{E}(S)$. Then $\mathbb{E}[G(S)] \leq \mathbb{E}[G(Z)]$.

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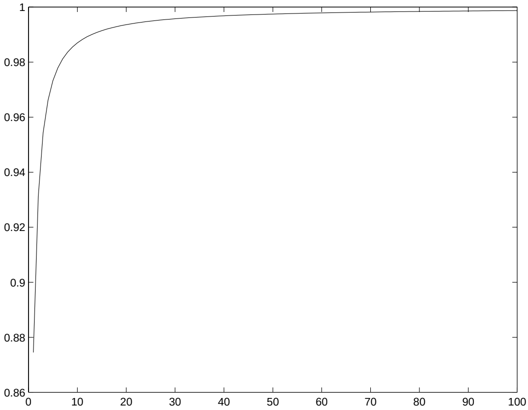
$$\Rightarrow \mathbb{E}[F(M)] \leq \mathbb{E}[G(Z)] = I_0(z) + (1 - \tfrac{1}{2z}) I_1(z)$$

with $I_0(z), I_1(z)$ modified Bessel functions

Sharp explicit bound: some $p_i = \frac{1}{2}$

$$R(p) \leq \sqrt{\tfrac{\pi}{2}(\tfrac{1}{2}+z)}\,[I_0(z) + (1-\tfrac{1}{2z})I_1(z)]$$



Conclusion

Theorem (C-Soto-Vaisman, arXiv'2012)

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with $\kappa = 1/\sqrt{\pi} \sim 0.5642$

$$\|Tx^n - x^n\| \leq \frac{1}{\sqrt{\pi}} \frac{\text{diam}(C)}{\sqrt{\sum_{k=1}^n \alpha_k(1-\alpha_k)}}$$

Grazie!
Merci!
Gracias!

Example: Right Shift on $\ell^1(\mathbb{N})$

$C = \{p \in \ell^1(\mathbb{N}) : p_i \geq 0, \sum_{i=0}^{\infty} p_i = 1\}$ with $\mathrm{diam}(C) = 2$

$S(p_0, p_1, p_2, \ldots) = (0, p_0, p_1, p_2, \ldots)$ is an isometry

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$$
\begin{aligned}
p^0 &= (1, 0, 0, 0, \ldots) \\
p^1 &= (1-\alpha_1, \alpha_1, 0, 0, \ldots) \\
p^2 &= ((1-\alpha_2)(1-\alpha_1), (1-\alpha_2)\alpha_1 + \alpha_2(1-\alpha_1), \alpha_2\alpha_1, 0, \ldots) \\
p^3 &= \ldots
\end{aligned}
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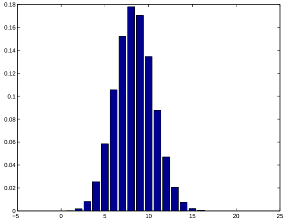

$p_k^n = \mathbb{P}(X_1 + \cdots + X_n = k)$

$X_i \sim$Bernoulli(α_i)

$\|Sp^n - p^n\|_1 = 2 \max_k p_k^n$

Sums of Bernoullis

Theorem (Baillon-C-Vaisman, arXiv'2013)

Let X_i be independent Bernoullis with $\mathbb{P}(X_i\!=\!1) = \alpha_i$. Then

$$p_k^n = \mathbb{P}(X_1 + \ldots + X_n = k) \leq \frac{\eta}{\sqrt{\sum_{i=1}^n \alpha_i(1-\alpha_i)}}$$

where $\eta = \max_{u\geq 0} \sqrt{u}\, e^{-u} I_0(u) \sim 0.4688$ with $I_0(\cdot)$ modified Bessel function. This bound is sharp.

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Corollary

(BB) holds for the right shift in $\ell^1(\mathbb{N})$ with $\kappa=\eta$.

Affine Maps

Let $\bar{x} \in \operatorname{Fix} T$ and $C = B(\bar{x}, r)$ with $r = \|x^0 - \bar{x}\|$ so that $T : C \to C$.

Affine Maps

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Corollary

For affine maps (BB) *holds with* $\kappa = \eta$. *The bound* η *is attained for the right shift so that the optimal* κ *is at least* η.

The random walk and the gambler's ruin appear...

$$c_{n,n+1} \quad = \quad \mathbb{P}[\sum_k^{n+1} C_i > \sum_k^n R_i, \forall k = n+1, \ldots, 1]$$

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$$Z_i = C_i - R_i = \begin{cases} -1 & \text{pbb} \quad \alpha_i(1-\alpha_i) \\ 0 & \text{pbb} \quad 1 - 2\alpha_i(1-\alpha_i) \\ 1 & \text{pbb} \quad \alpha_i(1-\alpha_i) \end{cases}$$

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$\Rightarrow$ random walk on $\mathbb{Z}$ that moves with probability $p_i = 2\alpha_i(1-\alpha_i)$ and then tosses a coin to decide whether to go left or right

$$\|Tx^n - x^n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} = \mathbb{P}[\text{process} \geq 0 \text{ over } n \text{ stages}]$$

An explicit formula for the bound

Rewrite $Z_i = M_i D_i$ with M_i=*move/stay* and D_i=*direction*

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Conditional on the number of moves $M = M_1 + \ldots + M_n = m$, this is a standard random walk on m stages. The probability for the latter to remain non-negative is $F(m) = \binom{m}{\lfloor m/2 \rfloor} 2^{-m}$, therefore

$$\|x_n - Tx_n\| \leq \frac{c_{n,n+1}}{\alpha_{n+1}} = \sum_{m=0}^{n} F(m)\mathbb{P}[M = m] = \mathbb{E}[F(M)]$$

Sharp bound

Thus (BB) has been reduced to

$$\mathbb{E}[F(M)] \leq \frac{1}{\sqrt{\pi \sum_{i=1}^{n} \alpha_i(1-\alpha_i)}}$$

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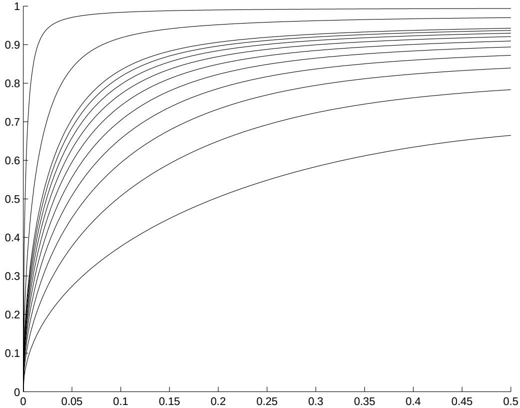
$$\underbrace{\sqrt{\tfrac{\pi}{2}(p_1+\ldots+p_n)}\,\mathbb{E}[F(M_1+\ldots+M_n)]}_{R(p)} \leq 1$$

Lemma

$R(p)$ is maximal when $p_i \in \{u, \frac{1}{2}\}$ for some $0 < u < \frac{1}{2}$

Sharp bound: all $p_i = u$

$$R(p) = \sqrt{\tfrac{\pi}{2} n u}\,\mathbb{E}[F(B(n,u))] = \sqrt{\tfrac{\pi}{2} n u}\;{}_2F_1(-n, \tfrac{1}{2}; 2; 2u)$$



Sharp bound: some $p_i = \frac{1}{2}$

Suppose $p_1 = \frac{1}{2}$ and let $S = M_2 + \ldots + M_n$. Conditioning on M_1

$$\mathbb{E}[F(M)] = \mathbb{E}[G(S)]$$

where $G(k) = \frac{1}{2}[F(k) + F(k+1)]$.

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This G is convex so we may use the following Hoeffding-type inequality

Theorem (C-Soto-Vaisman, arXiv'2012)

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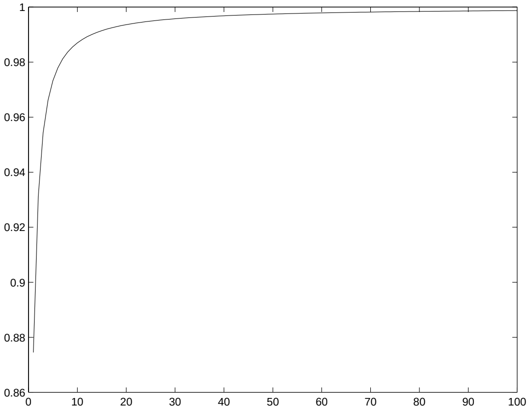
$$\Rightarrow \mathbb{E}[F(M)] \leq \mathbb{E}[G(Z)] = I_0(z) + (1 - \tfrac{1}{2z})I_1(z)$$

with $I_0(z), I_1(z)$ modified Bessel functions

Sharp explicit bound: some $p_i = \frac{1}{2}$

$$R(p) \leq \sqrt{\frac{\pi}{2}\left(\frac{1}{2}+z\right)}\,\left[I_0(z) + \left(1-\frac{1}{2z}\right)I_1(z)\right]$$



Conclusion

Theorem (C-Soto-Vaisman, arXiv'2012)

$$(BB) \qquad \|Tx^n - x^n\| \leq \kappa \frac{diam(C)}{\sqrt{\sum_{k=1}^{n} \alpha_k(1-\alpha_k)}}$$

with $\kappa = 1/\sqrt{\pi} \sim 0.5642$

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Is this bound sharp?

Until yesterday... Worst case known: right shift on $\ell^1(\mathbb{N})$

$$\kappa \geq \gamma \sim 0.4688 \text{ (83\% of upper bound)}$$