Some Aspects of the Interplay Between Convex Analysis and Monotone Operator Theory

Patrick L. Combettes

Laboratoire Jacques-Louis Lions
Faculté de Mathématiques
Université Pierre et Marie Curie – Paris 6
75005 Paris, France

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Overview

Our objective is to discuss certain aspects of the importance of the theory of monotone operators and of nonexpansive operators in the analysis and the numerical solution of problems in inverse problems and learning theory, even when those admit a purely variational formulation.

Special emphasis is placed on the role played by duality.

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Duality-closed classes of objects:

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- Firmly nonexpansive operators: $J \to \mathrm{Id} - J$ $\uparrow$: (Minty; $\updownarrow$ $J = (\mathrm{Id} + A)^{-1}$)
- Nonexpansive operators: $R \to -R$ ($\updownarrow$ $R = 2J - \mathrm{Id}$)

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- **Open problem 1:** (Firmly) nonexpansive operator setting?

Open problem 1: Duality for nonexpansive operators?

Consider the inclusion $0 \in Ax + Bx$, call $\mathscr{P}$ its solution set and $\mathscr{D}$ the dual solution set. Then:

- $\mathscr{P} = \big\{x \in \mathcal{H} \mid (\exists\, u \in \mathscr{D}) \;\; -u \in Ax \;\text{ and }\; u \in Bx\big\}$ and $\mathscr{D} = \big\{u \in \mathcal{H} \mid (\exists\, x \in \mathscr{P}) \;\; x \in A^{-1}(-u) \;\text{ and }\; x \in B^{-1}u\big\}$
- $\mathscr{P} = \operatorname{dom}(A \cap (-B))$ and $\mathscr{D} = \operatorname{dom}(A^{-1} \circ (-\mathrm{Id}) \cap B^{-1})$
- Suppose that A and B are maximally monotone, set $T_{A,B} = J_A(2J_B - \mathrm{Id}) + \mathrm{Id} - J_B$, and let $\gamma > 0$. Then:
  - $T_{A,B}$ is firmly nonexpansive
  - $\mathscr{P} = J_{\gamma B}\big(\operatorname{Fix} T_{\gamma A,\gamma B}\big)$
  - $\mathscr{D} = {}^{\gamma}B\big(\operatorname{Fix} T_{\gamma A,\gamma B}\big)$
  - $\mathscr{P} \neq \varnothing \Leftrightarrow \mathscr{D} \neq \varnothing \Leftrightarrow \operatorname{Fix} T_{\gamma A,\gamma B} \neq \varnothing$

Open problem 2: True Fenchel duality for monotone operators?

- Suppose that $0 \in \operatorname{sri}(\operatorname{dom} f - \operatorname{dom} g)$. Then

$$\inf(f+g)(\mathcal{H}) = -\min(f^{\vee *} + g^{*})(\mathcal{H})$$

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$$\operatorname{zer}(A+B) \neq \varnothing \quad \Leftrightarrow \quad \operatorname{zer}(-A^{-1} \circ (-\mathrm{Id}) + B^{-1}) \neq \varnothing.$$

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- For suitable "closure" or "enlargement" operations, can

$$\operatorname{zer}(A+B)^{\natural} \neq \varnothing \quad \Rightarrow \quad \operatorname{zer}\big(-A^{-1} \circ (-\mathrm{Id}) + B^{-1}\big)^{\flat} \neq \varnothing$$

recover Fenchel duality?

Splitting methods: Quick overview

- Traditional splitting techniques were developed in the late 1970s for inclusions involving the sum of two maximally monotone operators:

$$0 \in Ax + Bx$$

- **Open problem 3:** The continuous-time dynamics leading to FB are reasonably understood ($-x' \in Ax + Bx$); what about FBF and DR?
- Main algorithms:
  - Forward-backward method (Mercier, 1979)
  - Douglas-Rachford method (Lions and Mercier, 1979)
  - Forward-backward-forward algorithm (Tseng, 2000).
- Until recently, general splitting methods lacked for the composite problem

$$0 \in Ax + L^* BLx$$

Splitting methods: Composite problems

- $L\colon \mathcal{H} \to \mathcal{G}$ linear and bounded, $A\colon \mathcal{H} \to 2^{\mathcal{H}}$, $B\colon \mathcal{G} \to 2^{\mathcal{G}}$ maximally monotone
- Solve
$$0 \in Ax + L^*BLx$$
- Main issue: 3 objects (A, B, L) to split... and a binary relation $\in$ binding them
- We need to reduce a problem to a 2-object problem in a larger space
- Key: recast the problem in the primal dual space

Kuhn-Tucker set of a composite inclusion

- Primal solutions: $\mathscr{P} = \{x \in \mathcal{H} \mid 0 \in Ax + L^* BLx\}$
- Dual solutions: $\mathscr{D} = \{v^* \in \mathcal{G} \mid 0 \in -L \circ A^{-1}(-L^* v^*) + B^{-1} v^*\}$

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- Kuhn-Tucker set

$$\boldsymbol{Z} = \big\{ (x, v^*) \in \mathcal{H} \oplus \mathcal{G} \mid -L^* v^* \in Ax \ \text{ and } \ Lx \in B^{-1} v^* \big\}$$

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- $\boldsymbol{Z}$ is a closed convex set
- $\boldsymbol{Z} \neq \varnothing \Leftrightarrow \mathscr{P} \neq \varnothing \Leftrightarrow \mathscr{D} \neq \varnothing$
- $\boldsymbol{Z} \subset \mathscr{P} \times \mathscr{D}$

Kuhn-Tucker set of a composite inclusion

- Strategy: find a Kuhn-Tucker pair (x, v^*) by applying a standard (e.g., Douglas-Rachford or forward-backward-forward) method to a monotone+skew decomposition of the problem in $\mathcal{H} \oplus \mathcal{G}$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \underbrace{\begin{bmatrix} A & 0 \\ 0 & B^{-1} \end{bmatrix}}_{\boldsymbol{M}} \begin{bmatrix} x \\ v^* \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & L^* \\ -L & 0 \end{bmatrix}}_{\boldsymbol{S}} \begin{bmatrix} x \\ v^* \end{bmatrix}$$

  - L. M. Briceño-Arias and PLC, A monotone+skew splitting model for composite monotone inclusions in duality, *SIAM J. Optim.*, vol. 21, 2011.

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- Even in minimization problems, such a framework cannot be reduced to a functional setting: monotone operator splitting is required!
- Possible limitation: linear inversions (in DR) or knowledge of $\|L\|$ (in FBF) are necessary

Revisiting the proximal point algorithm

- A maximally monotone, $(\gamma_n)_{n\in\mathbb{N}} \in \,]0,+\infty[^{\mathbb{N}}$, $\operatorname{zer} A \neq \varnothing$,

$$x_{n+1} = J_{\gamma_n A} x_n$$

  Classical results by Brézis&Lions (1978):
  - If $\sum_{n\in\mathbb{N}} \gamma_n^2 = +\infty$, then $x_n \rightharpoonup x \in \operatorname{zer} A$
  - If $A = \partial f$ $(f \in \Gamma_0(\mathcal{H}))$ and $\sum_{n\in\mathbb{N}} \gamma_n = +\infty$, then $x_n \rightharpoonup x \in \operatorname{zer} A = \operatorname{Argmin} f$
- Is the proximal point algorithm of any use?

Revisiting the proximal point algorithm

- $\boldsymbol{A}\colon \mathcal{H} \to 2^{\mathcal{H}}$ maximally monotone, $\boldsymbol{V}$ be a closed vector subspace of $\mathcal{H}$
- $\boldsymbol{A}_{\boldsymbol{V}}$: partial inverse of $\boldsymbol{A}$ w.r.t. $\boldsymbol{V}$ (Spingarn, 1983)

$$\operatorname{gra} \boldsymbol{A}_{\boldsymbol{V}} = \big\{(P_{\boldsymbol{V}}\boldsymbol{x} + P_{\boldsymbol{V}^{\perp}}\boldsymbol{u}, P_{\boldsymbol{V}}\boldsymbol{u} + P_{\boldsymbol{V}^{\perp}}\boldsymbol{x}) \mid (\boldsymbol{x}, \boldsymbol{u}) \in \operatorname{gra} \boldsymbol{A}\big\}$$

- Then (Spingarn, 1983):
  - $\boldsymbol{A}_{\boldsymbol{V}}$ is maximally monotone
  - $\boldsymbol{z} \in \operatorname{zer} \boldsymbol{A}_{\boldsymbol{V}} \Leftrightarrow (P_{\boldsymbol{V}}\boldsymbol{z}, P_{\boldsymbol{V}^{\perp}}\boldsymbol{z}) \in \operatorname{gra} \boldsymbol{A}$
  - $\boldsymbol{p} = J_{\boldsymbol{A}_{\boldsymbol{V}}}\boldsymbol{z} \Leftrightarrow P_{\boldsymbol{V}}\boldsymbol{p} + P_{\boldsymbol{V}^{\perp}}(\boldsymbol{z} - \boldsymbol{p}) = J_{\boldsymbol{A}}\boldsymbol{z}$

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- Aside – **Open problem 4:** Let $f \in \Gamma_0(\mathcal{H})$ and $V = \{0\}$. Then $(\partial f)_V = \partial f^*$, but how to define a partial conjugate of f? (in general $(\partial f)_V$ is not a subdifferential)

Revisiting the proximal point algorithm

- Back to the Kuhn-Tucker set

$$\boldsymbol{Z} = \big\{(x, v^*) \in \mathcal{H} \oplus \mathcal{G} \;\big|\; -L^*v^* \in Ax \;\text{ and }\; Lx \in B^{-1}v^*\big\}$$

- Define $\boldsymbol{\mathcal{H}} = \mathcal{H} \oplus \mathcal{G}$, $\boldsymbol{A} \colon (x, y) \mapsto Ax \times By$, and $\boldsymbol{V} = \big\{(x, y) \in \mathcal{H} \oplus \mathcal{G} \;\big|\; Lx = y\big\}$
- Apply the (monotone operator) proximal point algorithm to $\boldsymbol{A}_{\boldsymbol{V}}$

  - M. A. Alghamdi, A. Alotaibi, PLC, and N. Shahzad, A primal-dual method of partial inverses for composite inclusions, *Optim. Lett.*, March 2014.

Revisiting the proximal point algorithm

Theorem

Set $Q = (Id + L^*L)^{-1}$ *and assume that* $\text{zer}(A + L^*BL) \neq \varnothing$. *Let* $(\lambda_n)_{n\in\mathbb{N}} \in \,]0,2[^{\mathbb{N}}$, *such that* $\sum_{n\in\mathbb{N}} \lambda_n(2-\lambda_n) = +\infty$, *let* $x_0 \in \mathcal{H}$, $v_0^* \in \mathcal{G}$, *and set* $y_0 = Lx_0$, $u_0 = -L^*v_0^*$, *and*

$$(\forall n \in \mathbb{N})\quad \left\lfloor \begin{array}{l} p_n \approx J_A(x_n + u_n) \\ q_n \approx J_B(y_n + v_n^*) \\ r_n = x_n + u_n - p_n \\ s_n = y_n + v_n - q_n \\ t_n = Q(r_n + L^*s_n) \\ w_n = Q(p_n + L^*q_n) \\ x_{n+1} = x_n - \lambda_n t_n \\ y_{n+1} = y_n - \lambda_n L t_n \\ u_{n+1} = u_n + \lambda_n(w_n - p_n) \\ v_{n+1}^* = v_n^* + \lambda_n(Lw_n - q_n). \end{array} \right.$$

Then $x_n - w_n \to 0$, $y_n - Lw_n \to 0$, $u_n - r_n + t_n \to 0$, *and* $v_n^* - s_n + Lt_n \to 0$, $(x_n, v_n^*) \rightharpoonup (\overline{x}, \overline{v}^*) \in \boldsymbol{K}$.

A strongly convergent splitting method

- We have seen that the Kuhn-Tucker set

$$\boldsymbol{Z} = \big\{(x, v^*) \in \mathcal{H} \oplus \mathcal{G} \mid -L^* v^* \in Ax \;\text{ and }\; Lx \in B^{-1} v^*\big\}$$

  is a closed and convex
- The previous methods require efficient linear inversions schmes or knowledge of $\|L\|$, neither of which may be available in many situations (e.g., domain decomposition methods). In addition they require stringent additional hypotheses to guarantee strong convergence (to an unspecified Kuhn-Tucker point)
- We present a method which alleviates all these limitations.

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- Aside – **Open problem 5:** in infinite-dimensional problems is weak convergence relevant?

A strongly convergent splitting method

- An abstract Haugazeau scheme:
  Let $\boldsymbol{C}$ be a nonempty closed convex subset of $\mathcal{H}$ and let $\boldsymbol{x}_0 \in \mathcal{H}$. Iterate

  $$\begin{array}{l} \text{for } n = 0, 1, \ldots \\ \left\lfloor \begin{array}{l} \text{take } \boldsymbol{x}_{n+1/2} \in \mathcal{H} \text{ such that } \boldsymbol{C} \subset H(\boldsymbol{x}_n, \boldsymbol{x}_{n+1/2}) \\ \boldsymbol{x}_{n+1} = P_{H(\boldsymbol{x}_0, \boldsymbol{x}_n) \cap H(\boldsymbol{x}_n, \boldsymbol{x}_{n+1/2})} \boldsymbol{x}_0 \end{array} \right. \end{array}$$

  Suppose that, for every $\boldsymbol{x} \in \mathcal{H}$ and every strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$, $\boldsymbol{x}_{k_n} \rightharpoonup \boldsymbol{x} \Rightarrow \boldsymbol{x} \in \boldsymbol{C}$. Then $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is well defined and $\boldsymbol{x}_n \to P_{\boldsymbol{C}} \boldsymbol{x}_0$.
- Strategy: Apply this principle to $\boldsymbol{C} = \boldsymbol{Z}$, construct $\boldsymbol{x}_{n+1/2}$ by suitably choosing points in gra A and gra B
  - A. Alotaibi, PLC, and N. Shahzad, Best approximation from the Kuhn-Tucker set of composite monotone inclusions, *J. Nonlinear Convex Anal.*, to appear.

A strongly convergent splitting method

Theorem

Fix $(x_0, v_0^*) \in \mathcal{H} \times \mathcal{G}$ *and* $\varepsilon \in]0,1[$*, and iterate for every* $n \in \mathbb{N}$

$$
\left\lfloor
\begin{array}{l}
(\gamma_n, \mu_n) \in [\varepsilon, 1/\varepsilon]^2 \\
a_n = J_{\gamma_n A}(x_n - \gamma_n L^* v_n^*),\ l_n = Lx_n,\ b_n = J_{\mu_n B}(l_n + \mu_n v_n^*) \\
s_n^* = \gamma_n^{-1}(x_n - a_n) + \mu_n^{-1} L^*(l_n - b_n),\ t_n = b_n - La_n \\
\tau_n = \|s_n^*\|^2 + \|t_n\|^2,\ \lambda_n \in [\varepsilon, 1] \\
\theta_n = \lambda_n\big(\gamma_n^{-1}\|x_n - a_n\|^2 + \mu_n^{-1}\|l_n - b_n\|^2\big)/\tau_n \\
x_{n+1/2} = x_n - \theta_n s_n^*,\ v_{n+1/2}^* = v_n^* - \theta_n t_n \\
\chi_n = \big\langle x_0 - x_n \mid x_n - x_{n+1/2}\big\rangle + \big\langle v_0^* - v_n^* \mid v_n^* - v_{n+1/2}^*\big\rangle \\
\mu_n = \|x_0 - x_n\|^2 + \|v_0^* - v_n^*\|^2,\ \nu_n = \|x_n - x_{n+1/2}\|^2 + \|v_n^* - v_{n+1/2}^*\|^2 \\
\rho_n = \mu_n \nu_n - \chi_n^2 \\
\text{if } \rho_n > 0 \text{ and } \chi_n \nu_n \geqslant \rho_n \\
\left\lfloor
\begin{array}{l}
x_{n+1} = x_0 + (1 + \chi_n/\nu_n)(x_{n+1/2} - x_n) \\
v_{n+1}^* = v_0^* + (1 + \chi_n/\nu_n)(v_{n+1/2}^* - v_n^*)
\end{array}
\right. \\
\text{if(ETC... easy steps)}
\end{array}
\right.
$$

Then $(x_n, v_n^*) \to P_K(x_0, v_0^*)$