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Special Issue: Modern Convex Analysis

Guest Editors: Patrick L. Combettes · Jean-Baptiste Hiriart-Urruty · Michel Théra

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Preface

Patrick L. Combettes, Jean-Baptiste Hiriart-Urruty & Michel Théra

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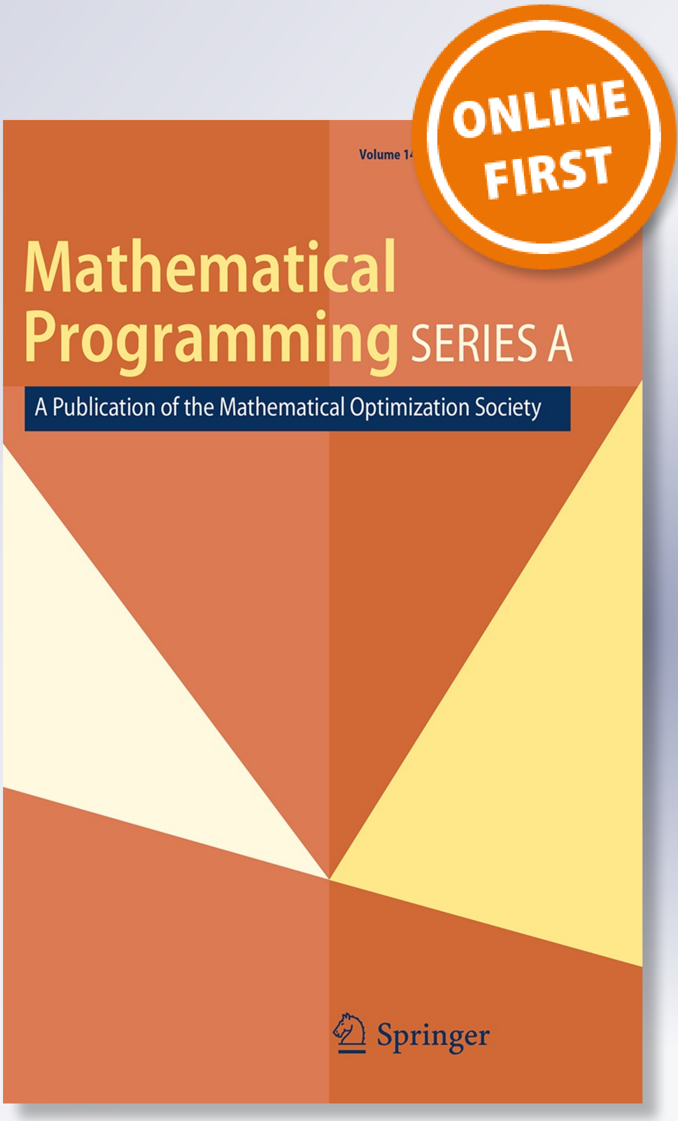

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PREFACE

Preface

Patrick L. Combettes · Jean-Baptiste Hiriart-Urruty · Michel Théra

Modern convex analysis and convex optimization were born about 50 years ago, in 1962–1963. This special issue of Mathematical Programming series B is intended to celebrate this 50th birthday, with a small sample of papers showing the vitality of this field and the renewed interest in the results and tools developed over the past decades. Sadly, as we were completing the edition of this special issue, Jean Jacques Moreau passed away (on January 9, 2014). Although this was not our original objective, this volume has also become a tribute to the memory of this outstanding mathematician motivated by mechanics, who contributed so much to the emergence of modern convex analysis.

The early days: 1962–1963

The development of convex analysis during the last fifty years owes much to W. Fenchel (1905–1988), J. J. Moreau (1923–2014), and R. T. Rockafellar (1935-), who contributed from different perspectives. Fenchel was very "geometrical" in his approach. On the other hand, Moreau's motivation was, in his own words, to do applied mechanics: he "applied mechanics to mathematics." In Rockafellar's work, which was motivated in part by problems in economics, the concept of a "dual problem" has been a

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constant leading thread [7]. The years 1962–1963 can be considered as birth date of *modern convex analysis* as the now familiar notions of *subdifferential, proximal mappings,* and *infimal convolution* all date back to this period. In two consecutive notes [1,2] published by the French Academy of Sciences in 1962, Moreau introduced the *proximal mapping* as a way of regularizing a convex function defined on a Hilbert space by performing an inf-convolution with the square of the norm. These preliminary works culminated with the 1965 paper [5], which remains for us the archetype of an elegant mathematical paper.

Various terms appeared in 1963 to name a vector s satisfying

$$f(x) + \langle y - x, s\rangle \leqslant f(y) \quad \text{for all } y.$$

In his 1963 Ph.D. thesis [6], R. T. Rockafellar called s "a differential of f at x". At the same time, in a note at the French Academy of Sciences [4], J. J. Moreau coined the term "sous-gradient," which became "subgradient" in English, and investigated the properties of the associated set-valued subdifferential operator ∂f. The term "la sous-différentielle" (a feminine word in French, closer to the classical "la différentielle" for differentiable functions) was initially used by Moreau, and it soon became "le sous-différentiel" (its masculine counterpart in French) and "the subdifferential" in English. As it often happens in research in mathematics, when times are ripe, concepts bloom in different places of the world at about the same time. Thus, in the USSR, researchers in Moscow and Kiev were interested in similar concepts. For instance, in 1962, N. Z. Shor published the first instance of the use of a subgradient method for minimizing a nonsmooth (more precisely, a piecewise linear) convex function (see also his 1964 thesis in Kiev).

The transformation $f \mapsto f^*$, which is central in duality theory, has its origins in a publication of A. Legendre (1752–1833), dated 1787. Since then, this transformation has received a number of names in the literature: conjugate, polar, maximum transformation, etc. However, it is now generally agreed that an appropriate terminology is *Legendre-Fenchel transform.*

The inf-convolution of two functions f and g is the function $f \,\Box\, g \colon x \mapsto \inf_{u+v=x}[f(u) + g(v)]$. This key operation in modern convex analysis has been used by Fenchel and, in a more general setting, by Moreau, who coined the term in [3]. The fundamental notions of proximal mapping, subdifferential, conjugation, and inf-convolution beautifully come together in Moreau's decomposition for a proper lower semicontinuous convex function f in a Hilbert space [5]:

$$\begin{cases} f \,\Box\, \frac{1}{2}\|\cdot\|^2 + f^* \,\Box\, \frac{1}{2}\|\cdot\|^2 = \frac{1}{2}\|\cdot\|^2 \\ \operatorname{prox}_f + \operatorname{prox}_{f^*} = \mathrm{Id} \\ \operatorname{prox}_{f^*} x \in \partial f(\operatorname{prox}_f x). \end{cases}$$

Developments and recent resurgences

Over the past decades, numerous papers and books have evidenced the continued interest and success of the field of convex analysis in various branches of applied mathematics; a good theory was put together and, as von Helmholtz used to say, "Nothing is more practical than a good theory." In addition, tools from convex and nonsmooth analysis are currently used to handle nonconvex variational problems, and they can be considered as "basics" when beginning to study variational analysis and optimization. In recent years, convex-analytical concepts have also become tools of choice tools in emerging areas such as signal and image processing, finance, risk analysis, and statistical learning. The 11 papers presented in this special issue reflect the vitality and the recent resurgence of the field of convex analysis. As we said before, J. J. Moreau was mainly a mathematician motivated by mechanics, he initiated the works on the so-called *sweeping processes* in mechanics; a paper in this issue is devoted to this topic.

The 11 papers presented in this special issue of Mathematical Programming (series B) reflect the vitality and recent resurgences of the field of convex analysis. We believe that the reader will find a delightful and valuable account on some new and fascinating applications of modern convex analysis:

- S. Adly, T. Haddad, and L. Thibault study both theoretically and numerically two new variants of the Moreau's sweeping process in the framework of measure differential inclusions and differential variational inequalities.
- J. Aragón Artacho, J.M. Borwein, V. Martín-Márquez, and L. Yao make a broad selection of applications of convex analysis within pure mathematics and give various applications of convex analysis in monotone operator theory.
- A. D'Aspremont, F. Bach, and L. El Ghaoui produce approximation bounds on a semidefinite programming relaxation for sparse principal component analysis.
- G. Bouchitté, I. Fragalà, and I. Lucardesi present a new approach to study the shape derivative of minimum problems involving integral functionals. Their study relies on the joint use of convex analysis, duality techniques and Gamma-convergence.
- B. Cox, A. Juditsky and A. Nemirovski propose an approach to solving nonsmooth optimization problems in which a Fenchel-type representation of the objective is available. This approach is based on applying first order algorithms to the dual problem and using associated accuracy certificates to recover the primal solution.
- D. Dentcheva and A. Ruszczyński consider the problem of quantification of risk preferences on the space of nondecreasing functions. Applying the conjugate duality theory, when the Banach space of bounded functions is paired with the space of finitely additive measures on a suitable algebra Σ, they develop a variational representation of quantile-based measures of risk. They introduce a notion of risk aversion in this context, and prove that the dual variables are in fact countably additive measures; they also derive an analogue of the Kusuoka representation of coherent law-invariant measures of risk.
- K. Emich, R. Henrion and W. Römisch study linear-quadratic two-stage stochastic optimization problems and their conditioning.
- A. Jofré, R.T. Rockafellar, and R. Wets demonstrate the role played by convex analysis in economic theory by applying it to a basic model of financial markets in which assets are traded and money can be lent or borrowed between the present and future.

- W. de Oliveira, C. Sagastizábal and C. Lemaréchal use convex analysis tools to analyze proximal bundle methods. They propose a convergence framework for a class of algorithms, handling inaccurate function and subgradient values in different manners.
- T. Pennanen studies the problem of optimal investment by embedding it in the general conjugate duality framework of convex analysis.
- R.T. Rockafellar and J.O. Royset, motivated by risk measures and their applications in mathematical finance and engineering, study distribution functions and related fundamental concepts of statistics from the point of view of convex analysis. They show that central notions in probability and statistics can be connected naturally (sometimes in a surprising fashion) to the theory of convex functions.

Final acceptance of all the papers in this volume were made through the normal refereeing procedure and standard practice of Mathematical Programming. We are grateful to the authors for their contributions to this special issue and to Danny Ralph for the opportunity to publish this volume. Finally we would like to thank all the dedicated referees who contributed to the quality of this volume through their constructive and insightful reviews.

P. L. Combettes, J.-B. Hiriart-Urruty, and M. Théra, July 2012–July 2014

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