NASH EQUILIBRIUM CONDITIONS — EXTENSIONS OF SOME CLASSICAL THEOREMS

Valeriu Ungureanu

1State University of Moldova
Republic of Moldova

Consider a non-cooperative game $\Gamma = \langle N, \{X_p\}_{p \in N}, \{f_p(x)\}_{p \in N}\rangle$, where $N = \{1, 2, ..., n\}$ is a set of players,

$X_p = \{x^p \in \mathbb{R}^{m_p} : g_i^p(x^p) \leq 0, i = 1, m_p, h_i^p(x^p) = 0, i = 1, l_p, x \in M_p\}$ is a set of strategies of player $p \in N, l_p, m_p, n_p < +\infty, p \in N$,

$g_i^p(x^p), h_i^p(x^p)$ are functions defined on $M_p, p \in N$,

$f_p(x)$ is a player’s $p \in N$ payoff function defined on $X = \times_{p \in N} X_p$.

Suppose that the players minimize the cost of their payoff functions.

Definition [1]. The outcome $\hat{x} \in X$ of the game is a Nash equilibrium (NE) if $f_p(x^p, \hat{x}^{\neq p}) \geq f_p(\hat{x}^p, \hat{x}^{\neq p}), \forall x^p \in X_p, \forall p \in N$. It is well known that the convex continuous compact games have NE [1].

Let $L_p(x, u^p, v^p) = f_p(x) + \sum_{i=1}^{m_p} u_i^p g_i^p(x^p) + \sum_{i=1}^{l_p} v_i^p h_i^p(x^p)$ be the Lagrange function for the player $p \in N$, where $u^p = (u_1^p, u_2^p, ..., u_{m_p}^p), v^p = (v_1^p, v_2^p, ..., v_{l_p}^p)$ are the Lagrange multipliers. Let $L = (L_1, ..., L_n)$ be the Lagrange vector-function.

Definition. $(\hat{x}, \hat{u}, \hat{v}) \in X \times R_{\geq}^{m_1} \times R^l_1 \times ... \times R_{\geq}^{m_n} \times R^l_n$ is a saddle point for $L$ if $L_p(\hat{x}, \hat{u}^p, \hat{v}^p) \leq L_p(\hat{x}, u^p, \hat{v}^p) \leq L_p(x^p, \hat{x}^{\neq p}, \hat{u}^p, \hat{v}^p) \forall (x^p, u^p, v^p) \in M_p \times R_{\geq}^{m_p} \times R^l_p, p \in N$.

Theorem 1. If $(\hat{x}, \hat{u}, \hat{v})$ is a saddle point for $L$, then $\hat{x}$ is a Nash equilibrium for $\Gamma$.

Consider $\Gamma$ with $X_p = \{x^p \in \mathbb{R}^{m_p} : g_i^p(x^p) \leq 0, i = 1, m_p\}$ where $g_i^p(x^p), i = 1, m_p$ are convex on $\mathbb{R}^{m_p}$. Let, also, $f_p(x^p, x^{\neq p}), p = 1, n$ be convex and continuous on $X_p$ for every fixed $x^{\neq p} \in X_{-p}$. $L_p(x, u^p) = f_p(x) + \sum_{i=1}^{m_p} u_i^p g_i^p(x^p)$.

Theorem 2. Let every $X_p$ be compact and satisfy the Slater regularity condition. The outcome $\hat{x} \in X$ is a NE for $\Gamma$ if and only if there exist $\hat{u}^p \geq 0, p = 1, n$ so that
(\hat{x}, \hat{u}) is a saddle point for L.

**Theorem 3.** Suppose that \( f_p(x), g^p_i(x^p), i = 1, m_p, \ p = 1, n \) are differentiable on \( \hat{x} \), every \( X_p \) is compact and satisfies the Slater regularity condition. The outcome \( \hat{x} \in X \) is a NE for \( \Gamma \) if and only if there exist \( \hat{u}^p \geq 0, p = 1, n \) so that

\[
\frac{\partial L_p(\hat{x}, \hat{u}^1, ..., \hat{u}^p)}{\partial x^p_j} = 0, \ \ j = 1, n_p, \quad \hat{u}^p_i g^p_i(\hat{x}^p) = 0, \ i = 1, m_p, \ p = 1, n.
\]

Theorems 1–3 are extensions on normal form strategic games of the well known theorems for matrix games. Theorem 3 is a Kuhn-Tucker type theorem for normal form strategic games. Other theorems of this kind are formulated and proved.

Games with vector payoffs are examined. The set of all Pareto-Nash equilibria may be defined as intersection of the graphs of efficient response mappings, analogically with definition of the Nash equilibria set for strategic form games \([2,3]\). Pareto-Nash equilibrium conditions are formulated and proved.

**Bibliographiche**

