ON THE REGULARITY OF THE VALUE FUNCTION FOR OPTIMAL CONTROL PROBLEMS

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We consider the control system

\[
\begin{cases}
    x'(s) = f(x(s), u(s)) \text{ for almost all } s \geq t \\
    x(t) = x \\
    u(\cdot) : [0, \infty) \rightarrow U \text{ is a measurable function}
\end{cases}
\]

(1)

Here and $f : \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ is continuous and there exists $L_0 > 0$ such that for all $x, y \in \mathbb{R}^N$ and for all $u \in U$

\[
|f(x, u) - f(y, u)| \leq L_0 |x - y|
\]

or

\[
F(x) \subseteq F(y) + L_0 |x - y| B
\]

\begin{itemize}
    \item \textbf{(H\textsubscript{1})} $F(x)$ is convex for all $x \in \mathbb{R}^N$
    \item \textbf{(H\textsubscript{2})} $f : \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ is continuous and there exists $L_0 > 0$ such that for all $x, y \in \mathbb{R}^N$ and for all $u \in U$
    \item \textbf{(H\textsubscript{3})} $f(\cdot, u)$ is differentiable for all $u \in U$ and there exists $L_1 > 0$ such that for all $x, y \in \mathbb{R}^N$ and for all $u \in U$
\end{itemize}

\[
|D_x f(x, u) - D_x f(y, u)| \leq L_1 |x - y|
\]

where $D_x f$ is the Jacobian matrix of $f$ with respect to $x$. 
Let $g : \mathbb{R}^N \to \mathbb{R}$ be a function. The value functions associated to the control system (1) and to $g$ are given by

$$V_1(t, x) = \sup_{u(\cdot) \in U(t)} g(x(T; t, x, u(\cdot))) \text{ for all } (t, x)$$

(2)

$$V_2(t, x) = \inf_{u(\cdot) \in U(t)} g(x(T; t, x, u(\cdot))) \text{ for all } (t, x)$$

(3)

where $x(\cdot; t, x, u(\cdot))$, denotes the solution of (1) starting from $(t, x)$.

We are interested to study the regularity of the value function. Under appropriate hypotheses on the dynamics, we prove that the value $V_2$ is primal-lower-nice (p.l.n.) (or $V_1$ is -p.l.n.) when the cost function $g$ is supposed to have the same regularity.

We recall the definition of p.l.n. regularity.

We recall that a lower semi-continuous (l.s.c.) function $h : \mathbb{R}^N \to \overline{\mathbb{R}}$ is primal-lower-nice (p.l.n.) at $\bar{x}$, a point where $h$ is finite, if there exist $R > 0$, $c > 0$, and $\varepsilon > 0$ with the property that

$$h(x') > h(\bar{x}) + \langle v, x' - \bar{x} \rangle - \frac{r}{2} |x' - x|^2$$

whenever $r > R$, $|v| < cr$, $v \in \partial h(\bar{x})$, $|x' - \bar{x}| < \varepsilon$ and $|x - \bar{x}| < \varepsilon$ with $x' \neq x$.

A function $h : \mathbb{R}^N \to \overline{\mathbb{R}}$ is - primal-lower-nice (- p.l.n.) at $\bar{x}$, if $-h$ is p.l.n. at $\bar{x}$.

Here the mapping $\partial h : \mathbb{R}^N \hookrightarrow \mathbb{R}^N$ and for all $x \in \mathbb{R}^N$, $\partial h(x)$ denotes the set of limiting proximal subgradients of $h$ at $x$.

Bibliographie
