NUMERICAL APPROXIMATION FOR SUPER-REPLICATION PROBLEMS UNDER GAMMA CONSTRAINTS

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In a financial market, consisting in a non-risky asset and some risky assets, people are interested to study the minimal initial capital needed in order to super-replicate a given contingent claim, under gamma constraints. In the literature, the super-replication price is characterized as the viscosity solution of an HJB-equation with terminal and boundary conditions. In a particular case, the dual formulation of the super-replication problem leads to a standard form of optimal stochastic control problem [4].

In this paper we study numerically an HJB-equation coming from the super-replication problem in dimension 2. We discretize the HJB equation using the Generalized Finite Differences scheme [2], [3], then we study existence and uniqueness of the discrete solution. Finally we prove the convergence of the numerical solution to the viscosity solution. In particular, we are interested on the HJB equation which comes from the two dimensional dual problem introduced in [4]:

\begin{equation}
\vartheta(t, x, y) = \sup_{(\rho, \xi) \in \mathcal{U}} \mathbb{E} \left[ g \left( X_{t,x,y}^{\rho,\xi}(T) \right) \right],
\end{equation}

where \((\rho, \xi)\) are valued in \([-1, 1] \times (0, \infty)\), the process \((X_{t,x,y}^{\rho,\xi}, Y_{t,y}^{\rho,\xi})\) is a 2-dimensional positive process, and \(g\) is a payoff function.

The main difficulty of the above problem is due to the non-boundness of the control set, this fact implies that the Hamiltonian associated to (1) is not bounded, and numerical approximation for such a problem becomes more complicate.

In the literature, problems with unbounded control have been studied by many authors (for example, [1], [5]). In all these cases, the authors decide to truncate the
set of controls to make it bounded. This truncation simplifies the numerical analysis of the problem. However, there is no theoretical result justifying this truncation. In this paper we do not truncate the set of controls, because we have a particular form of our HJB equation which leads us to avoid the difficulty of unbounded control. In fact, our HJB equation can be reformulated in the following way

$$\Lambda^{-}(J(t, x, y, D\theta(t, x, y), D^2\theta(t, x, y))) = 0,$$

where $J$ is a symmetric matrix differential operator associated to the Hamiltonian, and where $\Lambda^{-}(J)$ means the smallest eigenvalue of the matrix operator $J$. $J$ does not depend on the control, but when we look for the first time at this equation, it seems that it is very difficult to treat. From standard computations on algebra, we rewrite the smallest eigenvalue as follows:

$$\Lambda^{-}(J) = \min_{||\alpha||=1} \alpha^T J \alpha,$$

where $\alpha \in \mathbb{R}^2$. Then we have transformed our problem into a bounded control problem, and now the numerical analysis is possible.

We consider the discretization of the HJB equation, and recall the main properties of the Generalized Finite Differences Scheme and we prove the consistency of this scheme. Moreover, we prove existence and uniqueness of a bounded discrete solution, and finally we prove the convergence of the numerical approximation.

Bibliographie


