ON INFINITE-DIMENSIONAL GENERALIZATIONS OF
LYAPUNOV’S CONVEXITY THEOREM AND AUMANN’S
IDENTITY

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We present a new abstract bang-bang identity that generalizes the infinite-dimensional
extension of Lyapunov’s convexity theorem by Kingman and Robertson [1]. When
specialized to sub-Markovian kernels, it extends the Aumann identity and infinite-
dimensional Fatou lemma obtained by Rustichini and Yannelis [2].

The abstract bang-bang identity will now be described. Given a finite measure space
\((\Omega, \Sigma, \mu)\), let \(\Sigma^+\) be the collection of all non-null sets in \(\Sigma\). For \(E \in \Sigma^+\) let \(L^\infty(E)\)
be the space of all (equivalence classes of) essentially bounded scalar functions \(E\);
we identify it with the subspace of \(L^\infty(\Omega)\) that consists of all those \(\phi \in L^\infty(\Omega)\) for
which \(\phi(\omega) = 0\) a.e. on \(\Omega \setminus E\). Let \(V\) be a separable Banach space; the topological
dual of \(V\) is denoted by \(V'\). Let \(W \subset V\) be a closed convex cone with nonempty
interior and let \(e\) be a fixed element in the interior of \(W\). We define

\[
Q := \{v' \in V' : \langle e, v' \rangle \leq 1 \text{ and } \langle v, v' \rangle \geq 0 \text{ for all } v \in W\}. 
\]

The set of all extreme points in \(Q\) will be denoted by \(\text{ext } Q\). It is easy to see that

\[
\text{ext } Q = \{v' \in V' : \langle e, v' \rangle \in \{0, 1\} \text{ and } \langle v, v' \rangle \geq 0 \text{ for all } v \in W\}. 
\]

Let \(L^\infty_{V'}(\Omega)\) be the set of all \(V\)-scalarly measurable functions that are essentially
bounded for the dual norm on \(V'\). Then \(L^\infty_{V'}(\Omega)\) is the topological dual of the Bochner
space \(L^1_{V'}(\Omega)\) with respect to the usual \(L^1\)-norm. We define

\[
L^\infty_Q(\Omega) := \{b \in L^\infty_{V'}(\Omega) : b(\omega) \in Q \text{ a.e.}\}, 
L^\infty_{\text{ext } Q}(\Omega) := \{b \in L^\infty_{V'}(\Omega) : b(\omega) \in \text{ext } Q \text{ a.e.}\}. 
\]
The relative weak star topology will be used on these sets. We define a subset $F$ of $L_\infty^\infty(\Omega)$ to be rich if for every $b$ in the linear span of $F$ and for every $\phi \in L^\infty(\Omega)$ the function $\phi b$, defined by taking the pointwise product $\phi(\omega)b(\omega)$ a.e., belongs to the linear span $\text{lin } F$ of $F$. The following result generalizes the main result of [1], which has $V = \mathbb{R}$, $W = \mathbb{R}_+$ and $e = 1$.

**Theorem** Let $F$ be a rich and weak star closed face of $L_\infty^\infty(\Omega)$. Let $Y$ be a Hausdorff topological vector space and suppose that $\Lambda : \text{lin } F \to Y$ is a linear operator which is weak star continuous on $F$. Suppose also that for every $E \in \Sigma^+$ and $b \in \text{lin } F$, with $b \neq 0$ a.e. on $E$,

$$\phi \mapsto \Lambda(\phi b) : L^\infty(E) \to Y$$

is non-injective.

Then

$$\Lambda(\text{ext } F) = \Lambda(F),$$

and ext $F = F \cap L^\infty_{\text{ext } Q}(\Omega)$.

**Bibliographie**
