Discontinuous Feedback in Control Theory

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Outline of the talk

1. Introduction:
   dynamic programming,
   nonsmooth analysis
2. Stability and Lyapunov functions
3. Discontinuous feedbacks
4. Stabilizing feedback design

Note: detailed tutorial paper
Control theory of ordinary differential equations:

We consider throughout the system, for \( x \in \mathbb{R}^n \)

\[
\begin{align*}
(*) \quad \begin{cases} 
  x'(t) &= f(x(t), u(t)) \\
  u(t) &\in U
\end{cases}
\end{align*}
\]

Basic hypotheses:

\( U \) is compact, \( f \) is locally Lipschitz in \( x \)

\( U \neq \mathbb{R}^m \) nonlinear
Dynamic Programming

The minimal time function $T(\cdot)$, defined on $\mathbb{R}^n$ as follows: $T(\alpha)$ is the least time $T$ such that some trajectory $x(\cdot)$ satisfies $x(0) = \alpha$, $x(T) = 0$.

$T(\cdot)$ is an example of a value function.
The Principle of Optimality establishes some monotonicity properties for $T(\cdot)$:

A. If $x(\cdot)$ is any trajectory, and $0 < s < t$, then

$$T(x(s)) \leq (t - s) + T(x(t))$$

$s + T(x(s)) \leq t + T(x(t))$

B. If $x(\cdot)$ is an optimal trajectory joining $\alpha$ to 0, then an optimal trajectory from the point $x(t)$ is furnished by the truncation of $x(\cdot)$ to the interval $[t, T(\alpha)]$. Hence $T(x(t)) = T(\alpha) - t$; also $T(x(s)) = T(\alpha) - s$

$s + T(x(s)) = t + T(x(t))$
Conclusion: $t \mapsto t + T(x(t))$ is increasing for all $x(\cdot)$, constant for optimal $x(\cdot)$

$v 

\Rightarrow 1 + \langle \nabla T(x), f(x, u) \rangle \geq 0 \forall u \in U \quad (= \text{for } u \text{ optimal})$

$v 

\Rightarrow 1 + h(x, \nabla T(x)) = 0 \quad \text{Hamilton-Jacobi-Bellman equation}$

where $h$ is the lower Hamiltonian

\[ h(x, p) := \min_{u \in U} \langle p, f(x, u) \rangle \]

Suppose we solve the H-J-B equation (with $T(0) = 0$). How does knowing $T(\cdot)$ help?
For each $x$, let $k(x)$ be a point in $U$ such that

$$1 + \langle \nabla T(x), f(x, k(x)) \rangle = 0$$

(a steepest descent feedback). Then, for any $\alpha$, the solution to

$$x'(t) = f(x(t), k(x(t)), \ x(0) = \alpha$$

is optimal.

Here’s why:

$$\frac{d}{dt} T(x(t)) = \langle \nabla T(x(t)), x'(t) \rangle$$

$$= \langle \nabla T(x(t)), f(x(t), k(x(t))) \rangle = -1$$

$$\Rightarrow T(x(t)) - T(\alpha) = -t$$

Put $t = T(\alpha)$

$$\Rightarrow T(x(T(\alpha))) = 0$$

$$\Rightarrow x(T(\alpha)) = 0 \Rightarrow x(\cdot) \text{ optimal}$$

We obtain an optimal feedback synthesis
$x(0) = \alpha$
The murder of a beautiful theory by a gang of brutal facts

Serious difficulties in the dynamic programming approach:

- $T(\cdot)$ is nondifferentiable; replace $\nabla T$ in monotonicity?

- Need generalized solutions of H-J-B equation...

- Even if $T(\cdot)$ is smooth, there is no continuous $k(x)$ in general: what do we mean by a solution of $x' = f(x, k(x))$?
We find \( T(x, y) = \begin{cases} -y + \sqrt{2y^2 - 4x} & \text{left of } S \\ +y + \sqrt{2y^2 + 4x} & \text{right of } S \end{cases} \)

Example

\[ x''(t) = u(t) \in [-1, 1] \]

\[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix} \]

switching curve \( S \)

\[ 2y^2 + 4x = 0 \]

\[ u = +1 \]

\[ 2y^2 - 4x = 0 \]

\[ u = -1 \]

continuous but not Lipschitz
Resolving these issues has been an ongoing focus of activity for many years; suitable answers now exist, involving:

- **Nonsmooth analysis**
- **Generalized solutions of pde’s**
- **Sample-and-hold analysis of discontinuous feedbacks**

Next: a quick look at the first two of these
Generalized gradients and proximal normals: a glimpse (Clarke 1972)

Let $f$ be Lipschitz (locally): $|f(y) - f(x)| \leq K|y - x|$

Define

$$f^\circ(x; v) = \limsup_{t \downarrow 0, y \to x} \frac{f(y + tv) - f(y)}{t}$$

$$\partial_C f(x) = \left\{ \zeta \in X^*: f^\circ(x; v) \geq \langle \zeta, v \rangle \forall v \right\}$$

Example

$$f(x) = |x|$$

then

$$\partial_C f(0) = [-1, 1]$$
Calculus of generalized gradients

- $\partial_C f(x)$ is compact convex nonempty

- $\partial_C f(x) = \mathrm{co} \{ \lim_{x_i \to x} \nabla f(x_i), x_i \notin \Omega \}$

- $\partial_C (-f)(x) = -\partial_C f(x)$

- $\partial_C (f + g)(x) \subset \partial_C f(x) + \partial_C g(x)$

- $\partial_C \max_{1 \leq i \leq n} f_i(x) \subset \ldots$

- Mean value theorem, inverse functions...

- Tangent vectors and normals to closed sets
The proximal approach

\[ f'(\alpha) = \text{slope} \]
We can apply the 'local lower-approximation by parabolas' idea to nonsmooth (lsc) functions
The set of all `contact slopes' of lower locally supporting parabolas is the proximal subdifferential $\partial_P f(\alpha)$ has a very complete (but fuzzy!) calculus...

\[
\partial_P f(\alpha) = [-2, 1] \\
\partial_P f(\alpha) = \emptyset
\]

$\zeta \in \partial_P f(\alpha) \iff$

\[
f(x) \geq \langle \zeta, x - \alpha \rangle + f(\alpha) - \sigma |x - \alpha|^2 \text{ locally}
\]

$\partial_P f$ has a very complete (but fuzzy!) calculus...
Theorem

Let $\phi : \mathbb{R}^n \to \mathbb{R}_+$ be a continuous positive definite function such that

$$h(x, \zeta) + 1 = 0 \quad \forall \zeta \in \partial_P \phi(x), \forall x \neq 0.$$  

Then

$$\phi(\cdot) = T(\cdot)$$

Remark  Large literature on H-J-B equation:

- Clarke 1976 (Lipschitz, generalized gradients)
- Subbotin 1980 (invariance, Lipschitz, minimax)
- Crandall-Lions 1982 (comparison, continuous, viscosity)
- Clarke-Ledyaev 1994 (monotonicity, lsc, proximal)
- Fathi 1998 (KAM solutions)
- Dacorogna, DeVille... (almost everywhere)
We now study the stability of the system trajectories: \( x(t) \to 0 \), apparently very different

Recall For the ordinary differential equation
\[
x'(t) = g(x(t))
\]
we have:

**Theorem** Let \( g \) be continuous. The differential equation is stable if and only if there is a Lyapunov function for \( g \).

Lyapunov for the sufficiency

Massera, Barbashin and Krasovskii, and Kurzweil for the necessity:
converse Lyapunov theorems
RECALL:

A Lyapunov function $V$ for $g$ is $C^1$ and satisfies:

**Positive definiteness:**

$$V(0) = 0, \ V(x) > 0 \ \forall \ x \neq 0$$

**Properness:**

The level sets $\{x : V(x) \leq c\}$ are compact for every $c$. Equivalently, $V$ is radially unbounded:

$$V(x) \to +\infty \ \text{as} \ |x| \to +\infty$$

**Infinitesimal decrease:**

$$\langle \nabla V(x), g(x) \rangle < 0 \ \forall \ x \neq 0.$$
For the controlled differential equation

\[
(*) \begin{cases}
    x'(t) = f(x(t), u(t)) \\
    u(t) \in U
\end{cases}
\]

there are two principal scenarios:

• all trajectories go to 0, from any initial condition \text{ Strong stability}

• from any initial condition, some trajectory goes to 0 \text{ Weak stability?}
Strong stability:

\[
(*) \quad \begin{cases} 
  x'(t) = f(x(t), u(t)) \\
  u(t) \in U 
\end{cases}
\]

all trajectories go to 0, from any initial condition

**Theorem** [Clarke, Ledyaev, Stern 1998]

The system is strongly stable if and only if there exists a strong Lyapunov function for it: V smooth, positive definite, proper, such that

\[
\max_{u \in U} \langle \nabla V(x), f(x, u) \rangle < 0 \ \forall \ x \neq 0.
\]
Weak stability

\( \begin{align*}
\left\{ \begin{array}{l}
 x'(t) = f(x(t), u(t)) \\
u(t) \in U
\end{array} \right.
\end{align*} \)

from any initial condition, some trajectory goes to 0

The system is said to be open-loop Globally Asymptotically Controllable to the origin if:

For every \( \alpha \), there exists a control \( u_\alpha(t) \) and a state trajectory \( x(t) \) such that

\[ x'(t) = f(x(t), u_\alpha(t)), \quad x(0) = \alpha \text{ and } x(t) \to 0 \]

(plus a technical condition at 0)
In view of the above (and linear systems) we surmise:

**Theorem??**

The system is GAC if and only if there exists a weak Lyapunov function for it:

\[ V \text{ smooth, positive definite, proper, such that} \]

\[ h(x, \nabla V(x)) := \min_{u \in U} \langle \nabla V(x), f(x, u) \rangle < 0 \quad \forall x \neq 0. \]

**Definition:** Such a function is called a (smooth) CLF

**THEOREM** [Clarke, Ledyaev, Stern 1998]

Suppose that the system \((*)\) admits a smooth CLF. Then for every \(\delta > 0\), the following set is a neighborhood of 0:

\[ \{ v \in \co f(x, U) : x \in B(0, \delta) \}. \]

So any system which fails to satisfy this covering condition cannot admit a smooth CLF
Example: nonholonomic integrator (NHI)

\[
\begin{align*}
    x'_1(t) &= u_1(t) \\
    x'_2(t) &= u_2(t) \\
    x'_3(t) &= x_1(t)u_2(t) - x_2(t)u_1(t)
\end{align*}
\]

\[U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}\]

This is a nonlinear system (of real interest) that is close to being a classical linear system: it is linear in \( u \), linear in \( x \) (separately), with an ample control set.

It is easy to verify directly that the system is GAC.
RECALL: THEOREM

Suppose that the system (\(\ast\)) admits a smooth CLF. Then for every \(\delta > 0\), the following set is a neighborhood of 0:

\[
\{ v \in \co f(x, U) : x \in B(0, \delta) \}.
\]

For NHI:

\[
\begin{align*}
x'_1(t) &= u_1(t) \\
x'_2(t) &= u_2(t) \\
x'_3(t) &= x_1(t)u_2(t) - x_2(t)u_1(t)
\end{align*}
\]

\[
\begin{bmatrix}
0 \\
0 \\
\gamma
\end{bmatrix} \notin \co f(B(0, r), U) = f(B(0, r), U) \quad \text{if } \gamma \neq 0
\]

So: no smooth CLF.

The property GAC is not characterized by the existence of a smooth CLF.
Definition: \( V \) is a Dini CLF if it is continuous, proper, positive definite, and satisfies infinitesimal decrease in the Dini sense:

\[
\min_{u \in U} dV(x; f(x, u)) < -W(x) \quad \forall x \neq 0.
\]

**THEOREM [Sontag 1983]**

The system \((\ast)\) is GAC if and only if there exists a Dini CLF.

**THEOREM**

The system \((\ast)\) is GAC if and only if there exists a proximal CLF:

\[
\max_{\zeta \in \partial_P V(x)} \min_{u \in U} \langle \zeta, f(x, u) \rangle < -W(x) \quad \forall x \neq 0.
\]
The value function technique

How are CLF's found?

Let (*) be GAC. Fix $r>0$, and, for a given rate function $W$, define

$$\phi(\alpha) := \min \int_0^T W(x(t)) \, dt,$$

where the minimum is taken over all trajectories $x$ such that

$$x(0) = \alpha, \ x(T) \in B(0, r), \ T \text{ free}$$

The function $\phi$ is an example of a value function, in which $\alpha$ is the parameter. Such functions play a central role in pde's, optimization, and differential games.

Fact: $\phi$ satisfies infinitesimal decrease except on $B(0, r)$

$\phi$ is rather close to being a CLF for the system. But in which sense? Certainly not the smooth sense, for value functions are notoriously nonsmooth.
How are CLF’s found?

The “field of trajectories” approach

Exhibit a “reasonable, consistent” scheme for attaining a target $S$. Let $V(\alpha)$ be the time to the target, starting at $\alpha$, and according to the scheme.

Then $V$ is a Dini (and hence proximal) CLF (relative to the target $S$).
From $(x,y)$:

1. Go directly to the $x$-axis
   
   Time: $|y|$

2. Then go directly to the origin
   
   Time: $|x|$

\[ V(x,y) = |x| + |y| \]
More explicitly, find a feedback \( u = k(x) \) so that the differential equation is stable:

\[
\begin{align*}
x'(t) &= g(x(t)) := f(x(t), k(x(t))) \\
x(0) &= \alpha
\end{align*}
\]

For \( g(x) \) continuous, we require \( k(x) \) continuous.

Q: Does \( k \) exist? (The system is then said to be stabilizable.) How to construct such a feedback?
A necessary condition for this is GAC

Q: Is every (reasonable) GAC system stabilizable by feedback?

That is, can we synthesize the various open-loop controls $u_\alpha(t)$ into one coherent (continuous) feedback law $k(x)$?

This question has played a central role for decades.

The classic linear case:

\[ x'(t) = Ax(t) + Bu(t), \quad U = \mathbb{R}^m \]

Then: \( (*) \) is GAC $\iff$ \( (*) \) is stabilizable (by a linear feedback $k(x) = Kx$)
In applications, systems are rarely linear, yet linear systems theory is applied: linearization

\[ x'(t) = f(x(t), u(t)) \Rightarrow x'(t) = Ax(t) + Bu(t) \]

This requires:

- **smooth data**  false for some problems
- **nondegenerate linearization**  fails for NHI
- **small \((x,u)\)**  not the case in pursuit-evasion
- **all \((x,u)\) near \((0,0)\) available**  locally unrestricted state, control
A famous diagnostic tool for the feedback issue:

**Theorem** (Brockett 1983)

If (⋆) is stabilizable by a continuous feedback \( k \), then, for every \( r > 0 \), the set \( f(B(0, r), U) \) contains a neighborhood of 0.

For NHI (GAC and reasonable!):

\[
\begin{align*}
    x'_1(t) &= u_1(t) \\
    x'_2(t) &= u_2(t) \\
    x'_3(t) &= x_1(t)u_2(t) - x_2(t)u_1(t)
\end{align*}
\]

\[
U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}
\]

\[
\begin{bmatrix}
0 \\
0 \\
\gamma
\end{bmatrix} \notin f(B(0, r), U) \text{ if } \gamma \neq 0
\]

so: no continuous stabilizer

**Note:** The problem cannot be “approximated away”
So we reluctantly consider discontinuous feedbacks

\[ x'(t) = f(x(t), k(x(t))) \]

A natural solution concept: \textit{sample-and-hold}

This means that \( k(x) \) is applied on a piecewise-constant basis between sampling moments:

\[ x'(t) = f(x(t), k(x_i)), \quad \text{for } t_i \leq t \leq t_{i+1} \]

relative to a given partition \( \{t_i\} \) of the underlying interval
We say that $k$ stabilizes (in the s&h sense) if:
Given $B(0,R)$ and $B(0,r)$, then with sufficiently fine partitions, $k$ drives all points in $B(0,R)$ to $B(0,r)$

**Theorem** (Clarke, Ledyaev, Sontag, Subbotin 1997)

Any GAC system is stabilizable, with possibly discontinuous feedback, implemented in the sample-and-hold sense. (The converse is evident)

**Remarks:**
- The s&h stabilization is “meaningful”...
- There is robustness with respect to implementation, as well as small error; such analysis becomes possible
- Filippov solutions don’t work.
Theorem (Clarke, Ledyaev, Sontag, Subbotin 1997)

Any GAC system is stabilizable, with possibly discontinuous feedback, implemented in the sample-and-hold sense.  (The converse is evident)

The proof uses a nonsmooth Lyapunov function to construct the stabilizing feedback, which corresponds precisely to the third serious difficulty of the dynamic programming approach.

Recall:
Serious difficulties in the dynamic programming approach:

• $T(\cdot)$ is nondifferentiable; replace $\nabla T$ in monotonicity?

• Need generalized solutions of H-J-B equation...

• Even if $T(\cdot)$ is smooth, there is no continuous $k(x)$ in general: what do we mean by a solution of $x' = f(x, k(x))$?
From \((x,y)\):

1. Go directly to the \(x\)-axis
2. Then go directly to the origin

This defines a feedback, but a meaningless one.
From \((x,y)\):

1. Go directly to the x-axis
2. Then go directly to the origin

This defines a feedback, but a meaningless one
This is an example of the thin set fallacy:
a feedback whose effect depends on its values
on a set of measure zero

Lesson: In using discontinuous feedback,
take account from the beginning of the
implementation procedure.
Sample-and-hold forces one to do so.

This issue does not arise with continuous feedbacks.
So discontinuous feedbacks must be designed with
extra care. But they also have some advantages, such
as in blending and sliding.
If continuity is not an issue, then we can switch, in sample-and-hold, from $k_0$ to $k_1$. 

Goal: combine, or “blend”, the two feedbacks.

(We require overlap: $\{V_1(x) \leq \delta_1\} \subset \{V_0(x) \leq \Delta_0\}$)
Q: How to construct meaningful stabilizing feedbacks?

The case of a smooth CLF

\[
\inf_{u \in U} \langle \nabla V(x), f(x, u) \rangle < -W(x) \quad x \neq 0.
\]

Natural approach: choose \( k(x) \) in \( U \) so that

\[
\langle \nabla V(x), f(x, k(x)) \rangle < -W(x) \quad \forall x \neq 0.
\]

(A “steepest-descent” feedback induced by \( V \))

Theorem

\( k \) stabilizes the system in the s & h sense
THEOREM A steepest descent feedback $k$ stabilizes the system in the sample-and-hold sense.

PROOF. For ease of exposition, we shall suppose that $V$ on $\mathbb{R}^n$ and $\nabla V$ on $\mathbb{R}^n \setminus \{0\}$ are locally Lipschitz rather than merely continuous (otherwise, the argument is carried out with moduli of continuity). We also restrict attention to uniform partitions.

Let $B(0,R)$ and $B(0,r)$ be the initial values and target set under consideration. The properties of $V$ imply the existence of positive numbers $e < E$ such that

$$\{x : V(x) \leq e\} \subset B(0,r), \quad \{x : V(x) \leq E\} \supset B(0,R).$$

Fix $E' > E$. There exist positive constants $K,L,M$ such that, for all $x,y$ in the compact set $\{x : V(x) \leq E'\}$ and $u \in U$, we have

$$|V(x) - V(y)| \leq L|x-y|, \quad |f(x,u)| \leq M, \quad |f(x,u) - f(y,u)| \leq K|x-y|.$$

(1)

Now pick $e'$ and $e''$ so that $0 < e'' < e' < e$, and set

$$X := \{x : e' \leq V(x) \leq E'\}.$$

Then there exist constants $N$ and $\omega > 0$ such that

$$|\nabla V(x) - \nabla V(y)| \leq N|x-y|, \quad W(x) \geq \omega \forall x,y \in X. \quad (2)$$

Let $B(0,R)$ and $B(0,r)$ be the initial values and target set under consideration. The properties of $V$ imply the existence of positive numbers $e < E$ such that

$$\{x : V(x) \leq e\} \subset B(0,r), \quad \{x : V(x) \leq E\} \supset B(0,R).$$

Fix $E' > E$. There exist positive constants $K,L,M$ such that, for all $x,y$ in the compact set $\{x : V(x) \leq E'\}$ and $u \in U$, we have

$$|V(x) - V(y)| \leq L|x-y|, \quad |f(x,u)| \leq M, \quad |f(x,u) - f(y,u)| \leq K|x-y|.$$

(1)

Now pick $e'$ and $e''$ so that $0 < e'' < e' < e$, and set

$$X := \{x : e'' \leq V(x) \leq E'\}.$$

Then there exist constants $N$ and $\omega > 0$ such that

$$|\nabla V(x) - \nabla V(y)| \leq N|x-y|, \quad W(x) \geq \omega \forall x,y \in X. \quad (2)$$

Let $\pi$ be a uniform partition of diameter $\delta \in (0,1)$ such that

$$\delta LM < \min\{e-e',e-e'',E'-E\},$$

$$\delta (LK + MN)M < \omega/2. \quad (3)$$

Now let $x_0$ be any point in $B(0,R)$, and proceed to implement the feedback $k$ via the partition $\pi$. On the first time interval $[t_0,t_1]$ the trajectory $x$ corresponding to $k$ is generated by the differential equation

$$x'(t) = f(x(t),k(x_0)), \quad x(t_0) = x_0, \quad t_0 \leq t \leq t_1.$$

The solution to this differential equation exists on some interval of positive length, and is unique because $f$ is locally Lipschitz in the state variable. If the solution fails to exist on the entire interval, it is because blow-up has occurred. Then there exists a first $\tau \in (t_0,t_1]$ for which $V(x(\tau)) = E'$. On the interval $[t_0,\tau)$, the Lipschitz constant $L$ of (1) is valid, as well as the bound $M$, whence

$$V(x(t)) \leq V(x_0) + L|x(t) - x_0| \leq E + \delta LM \forall t \in [t_0,\tau).$$

But then $V(x(\tau)) \leq E + \delta LM < E'$ by (3), a contradiction. It follows that blow-up cannot occur, and that the solution of the differential equation exists on the entire interval $[t_0,t_1]$ and satisfies $V(x(t)) < E'$ there.

Case 1

$V(x_0) < e'$.

It follows then from $\delta LM < e - e'$ (see (3)) that we have

$$V(x(t)) < e' \forall t \in [t_0,t_1].$$

Case 2

$e' < V(x_0)$.

Now we have $x_0 \in X$ and

$$(\nabla V(x_0), f(x_0,k(x_0))) < -\omega.$$

from the way $k(x_0)$ is defined, and since $V(x_0) > \omega$.

Let $t \in (t_0,t_1]$; then, at least while $x(t)$ remains in the set $X$, we can argue as follows:

$$V(x(t)) - V(x(t_0)) = (\nabla V(x(t^*)), x(t^*)'\right) (t - t_0)$$

(by the Mean Value Theorem, for some $t^* \in (0,t)$)

$$\leq (\nabla V(x(t^*)), f(x(t^*),k(t_0))) (t - t_0) + \omega (t - t_0) + \omega (t - t_0)$$

(by definition of $k$)

$$= \{ -\omega + \delta (LK + MN)M \} (t - t_0) \leq -\omega/2 (t - t_0), \quad (4)$$

Thus the value of $V$ has decreased. It follows from this, together with the inequality $\delta LM < e - e''$ provided by (3), that $x(t)$ remains in $X$ throughout $[t_0,t_1]$, so that the estimates above apply.

To summarize, we have in Case 2 the following decrease property:

$$V(x(t)) - V(x(t_0)) \leq -\omega/2 (t - t_0) \forall t \in [t_0,t_1].$$

It follows that, in either case, we have $V(x(t)) \leq E$ for $t \in [t_0,t_1)$, and in particular $V(x_1) \leq E$, where $x_1 := x(t_1)$ is the next node in the implementation scheme.

We now repeat the procedure on the next interval $[t_1,t_2]$, but using the constant control value $k(x_1)$.

Precisely the same arguments as above apply to this and to all subsequent steps: either we are at a node $x_1$, for which $V(x_1) \leq e'$ (Case 1), or else $V(x(t))$ continues to decrease at a rate of $\omega/2$ (Case 2).

Since $V$ is nonnegative, the case of continued decrease cannot persist indefinitely. Let $x_J (J \geq 0)$ be the first node satisfying $V(x_J) \leq e'$. If $J > 0$, then

$$e' < V(x_{J-1}) \leq V(x_0) - (\omega/2)(t_{J-1} - t_0) = V(x_0) - (\omega/2)(J - 1)\delta,$$

whence

$$\omega/2(J-1)\delta < V(x_0) - e' \leq e' - e',$$

and so

$$J\delta < 2(\omega - e')/\omega + \delta \leq 2(\omega - e')/\omega + 1 = T,$$

which provides a uniform upper bound $T$ independent of $\delta$ for the time $J\delta$ required to attain the condition $V(x_J) \leq e'$. Once this condition is satisfied, the above analysis shows that for $t \geq t_J$, we have $V(x(t)) < e'$, which implies $x(t) \in B(0,0)$.

Thus for all $t \geq 0$ the trajectory $x$ satisfies $V(x(t)) \leq E$, and since $x : V \leq E \in B(0,R)$, there exists $C$ depending only on $R$ such that $V(x(t)) \leq C \forall t \geq 0$. This completes the proof that the required stabilization takes place.
Consider now a Dini CLF:
\[
\inf_{u \in U} dV(x; f(x, u)) < -W(x) \quad x \neq 0.
\]

Natural approach: choose \( k(x) \) in \( U \) so that
\[
dV(x; f(x, k(x))) < -W(x) \quad x \neq 0.
\]

But this can give a meaningless, non-stabilizing feedback
From \((x,y)\):

1. Go directly to the x-axis
2. Then go directly to the origin

By construction, the feedback is of steepest descent type for the CLF it induces:

\[ V(x,y) = |x| + |y| \]
SO: for a Dini CLF:
\[
\inf_{u \in U} dV(x; f(x, u)) < -W(x) \quad x \neq 0.
\]

the natural approach: choose \( k(x) \) in \( U \) so that
\[
dV(x; f(x, k(x))) < -W(x) \quad x \neq 0.
\]
can fail.

However, the natural steepest descent approach DOES work if \( V \) is semiconcave:

\( V \) is locally Lipschitz and (locally) \( V \) is locally Lipschitz and (locally)
\[
V(y) - V(z) - \langle \zeta, y - z \rangle \leq \sigma |y - z|^{1+\eta}
\]
\[
\forall \zeta \in \partial_C V(z)
\]

**Theorem**
\( k \) stabilizes the system in the s & h sense
4. If \( \phi \) is concave or \( C^{1,\eta} \) near \( x \), then \( \phi \) satisfies SC at \( x \).

5. The positive linear combination (and in particular, the sum) of a finite number of functions each of which satisfies SC at \( x \) also satisfies SC at \( x \).

6. If \( \phi = g \circ h \), where \( h : \mathbb{R}^n \to \mathbb{R}^m \) is \( C^{1,\eta} \) near \( x \), and where \( g : \mathbb{R}^m \to \mathbb{R} \) is concave, then \( \phi \) satisfies SC at \( x \).

7. If \( \phi = g \circ h \), where \( h : \mathbb{R}^n \to \mathbb{R} \) is concave, and where \( g : \mathbb{R} \to \mathbb{R} \) is \( C^{1,\eta} \) near \( h(x) \), then \( \phi \) satisfies SC at \( x \).

8. If \( \phi = gh \), where \( h \) is convex, and where \( g : \mathbb{R}^n \to (-\infty, 0] \) is \( C^{1,\eta} \) near \( x \), then \( \phi \) satisfies SC at \( x \).

9. If \( \phi = gh \), where \( g \) is \( C^{1,\eta} \) near \( x \), with \( g(x) > 0 \), and where \( h \) is concave, then \( \phi \) satisfies SC at \( x \).

10. If \( \phi = \min \phi_i \), where \( \{\phi_i\} \) is a finite family of functions each of which satisfies SC at \( x \), then \( \phi \) satisfies SC at \( x \).

11. If \( \phi \) satisfies SC at \( x \), then the directional derivative \( \phi'(x; v) \) exists for each \( v \), and one has

\[
\nabla \phi(x; v) = \phi'(x; v) = \min_{\zeta \in \partial C \phi(x)} \langle \zeta, v \rangle \quad \forall \, v \in \mathbb{R}^n.
\]
Two Dini CLF’s for NHI:

\[ V_1(x) := x_1^2 + x_2^2 + 2x_3^2 - 2|x_3|\sqrt{x_1^2 + x_2^2}. \]

\[ V_2(x) := \max \left\{ \sqrt{x_1^2 + x_2^2}, |x_3| - \sqrt{x_1^2 + x_2^2} \right\}. \]

Only one of these is semiconcave.

The collection of facts about operations that preserve that property (positive linear combinations, certain products and compositions, lower envelopes) allows us to see easily that \( V_1 \) is semiconcave.
The corresponding steepest-descent feedback induced by $V_1$ is given by

For $x \neq 0$:

When $\sigma \neq 0$ and $x_3 \neq 0$, set

$$k(x) = \begin{cases} 
(x_1, x_2)/\rho & \text{if } |x_3| - \rho \geq \rho|\rho \text{sgn}(x_3) - 2x_3| \\
-(x_1, x_2)/\rho & \text{if } \rho - |x_3| \geq \rho|\rho \text{sgn}(x_3) - 2x_3| \\
(x_2, -x_1)/\rho & \text{if } \rho(2x_3 - \rho \text{sgn}(x_3)) > |\rho - |x_3|| \\
-(x_2, -x_1)/\rho & \text{if } \rho(\rho \text{sgn}(x_3) - 2x_3) > |\rho - |x_3||
\end{cases}$$

where $\rho := \sqrt{x_1^2 + x_2^2}$.

When $\sigma = 0$ (then $x_3 \neq 0$), set $k(x) = (1, 1)/\sqrt{2}$.

When $x_3 = 0$ (then $\sigma \neq 0$), set $k(x) = -(x_1, x_2)/\sigma$

(Set $k(0)$ equal to any point in $U$)
Four types of regularity for Dini or proximal CLF's:

- Continuous
- Locally Lipschitz
- Semiconcave
- Smooth ($C^1$)

Theorem [Rifford 2000] The system is GAC if and only if it admits a semiconcave CLF.
What if $V$ is merely locally Lipschitz, not smooth or semiconcave?

Fact: given $r$ and $R$, then, for $\lambda$ sufficiently large, the steepest descent feedback generated by

$$V_\lambda(x) := \min_{z \in \mathbb{R}^n} \{ V(z) + (\lambda/2) \| x - z \|^2 \}.$$

stabilizes $B(0,R)$ to $B(0,r)$.

So we get feedbacks for “practical semiglobal stabilization”
Steepest descent for $V(x,y) = |x| + |y|$ (dither) \text{ not semiconcave!}

Steepest descent for $V_\lambda(x,y)$ (s & h stabilization)
Conclusions

Discontinuous feedbacks appear to be essential in nonlinear control settings.

They must be handled with more care than continuous ones, and require more effort, but they offer certain advantages.

There is a growing body of theory and techniques on the subject, based on sample-and-hold analysis.
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They must be handled with more care than continuous ones, and require more effort, but they offer certain advantages. There is a growing body of theory and techniques on the subject, based on sample-and-hold analysis.