Sensitivity of Darcy's Law to Discontinuities

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Abstract

We investigate the sensitivity of hydrostatic pressure of flows through porous media with respect to the position of the soil layers. Indeed, these induce discontinuities of the porosity which is a piecewise constant coefficient κ of the partial differential equation satisfied by the pressure u and it leads to the computation of the derivative of u with respect to changes in position of discontinuity surfaces of κ. The analysis relies on a mixed formulation of the problem. Preliminary numerical simulations are given to illustrate the theory. An application to a simple inverse problem is also given.

Sensibilité de la loi de Darcy aux discontinuités

Résumé

Nous étudions la dérivabilité de la solution de l'équation de Darcy par rapport à la porosité dans le cas où celle-ci est constante par morceaux. La variable est donc la position de la ligne de discontinuité de la porosité. Nous montrons que la dérivée existe et que pour la calculer le problème doit être posé sous forme variationnelle mixte. La dérivée est solution d'un problème identique avec une masse de Dirac au second membre. Des calculs numériques préliminaires sont donnés pour illustrer le résultat théorique et un problème d'identification de porosité discontinue est résolu par une méthode de gradient.

Keywords: Partial differential equations, flow through porous media, sensitivity, Darcy's equation, topological optimization.

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Introduction

The hydrostatic pressure u in a porous media of porosity κ can be found, in simple situations, by solving

$$-\nabla \cdot (\kappa \nabla u) = f$$

When the porous media has layers of different materials, κ is smooth except at the interface between layers where it is discontinuous. For ground identification from surface data it is important to compute the sensitivity of u with respect to the discontinuities of κ. This is similar to topological optimization (cf. [2]), but here we can make an explicit use of the fact that κ is piecewise regular and use the topology of the layers. Continuity in κ with respect to the L^∞ norm has been studied earlier (cf. Bruaset et al [4]), but such a formalism does not allow changes in the position of the discontinuities of κ.

Problem Statement More precisely, consider a bounded open set Ω of $\mathcal{R}^2$ and a family of closed curves $a \mapsto \Sigma(a)$ strictly inside Ω and function of a scalar parameter a separating Ω into two non overlapping sets $\Omega_i(a)$, $i = 1, 2$, with Lipschitz continuous boundaries:

$$\bar{\Omega} = \bar{\Omega}_1(a) \cup \bar{\Omega}_2(a) \quad \Omega_1(a) \cap \Omega_2(a) = 0 \qquad \bar{\Sigma}(a) = \bar{\Omega}_1(a) \cap \bar{\Omega}_2(a).$$

Note that we do not allow $\Sigma(a)$ to touch $\partial\Omega$. This is for mathematical convenience only and it is conjectured that the following results apply also to the general case both in $\mathcal{R}^2$ and $\mathcal{R}^3$.

Let $\kappa(a)$ be piecewise constant and equal to κ_i on $\Omega_i(a)$ and consider

$$-\nabla \cdot (\kappa(a) \nabla u(a)) = f \tag{1}$$

with Neumann or Dirichlet conditions on $\Gamma = \partial\Omega$. We wish to compute the derivative u' of $u(a)$ with respect to a at $a = 0$.
Again the following analysis is not hard to generalize to the case of a piecewise differentiable κ with similar discontinuity surfaces $\Sigma(a)$, and it is for sake of clarity that we present the piecewise constant case only.

The traditional approach fails Denoting by $I_{\Omega_2(a)}$ the characteristic function of $\Omega_2(a)$ and by $[\kappa]$ the jump $\kappa_2 - \kappa_1$, we have

$$\kappa(a) = \kappa_1 + [\kappa] I_{\Omega_2(a)}.$$

So it is tempting to write that

$$\int_\Omega \kappa(a) \nabla u' \nabla w = -\int_\Omega \kappa' \nabla u \nabla w = -\int_\Sigma [\kappa] \nabla u \nabla w$$

but the last integral makes no sense because ∇u is discontinuous across $\Sigma(a)$.

Similarly an "optimal shape design" approach seems to lead nowhere.
Indeed with the notation of [6], we assume that $\Sigma(a)$ depends on a via a given function α of $\mathcal{C}^1(\Sigma)$ and the following local variation in the "direction" α around $\Sigma(0)$ taken as a reference curve:

$$\Sigma(a) = \{x + a\alpha(x)n(x) \ : \ x \in \Sigma(0)\} \tag{2}$$

where n is the unit normal to $\Sigma(0)$ which points outside Ω_1.

For clarity we shall drop the parameter a when it is equal to 0 and write Σ instead of $\Sigma(0)$ whenever non ambiguous.

Recall that in smooth situations ($f \in \mathcal{C}^1(\Omega)$, see Pironneau [6], chap 3 for details) the following form of the mean value theorem is shown

$$\int_{\Omega_i(a)} f - \int_{\Omega_i(0)} f = -(-1)^i a \int_\Sigma f\alpha + o(|a|) \qquad i = 1, 2.$$

Therefore with the shorthand notation $\int_{\delta\Omega_i} f = \int_{\Omega_i(a)} f - \int_{\Omega_i(0)} f$, a formal differentiation of (1) would lead to

$$\begin{aligned} 0 &= \delta(\int_\Omega \kappa(a)\nabla u(a)\nabla w - \int_\Omega fw) = \int_{\delta\Omega_1} \kappa_1\nabla u\nabla w + \int_{\delta\Omega_2} \kappa_2\nabla u\nabla w \\ &+ \int_\Omega \kappa(a)\nabla\delta u\nabla w + \text{ higher order terms} \\ &= a\int_\Sigma (\kappa_1\frac{\partial u}{\partial n} - \kappa_2\frac{\partial u}{\partial n}) + \int_\Omega \kappa(a)\nabla\delta u\nabla w + \text{ higher order terms} \end{aligned} \tag{3}$$

However the jump of $\kappa\frac{\partial u}{\partial n}$ across Σ is zero so this calculation indicates that u' would be zero. The example below shows that it is not the case. Hence the "higher order terms" are wrongly called so.

An example Consider the same problem on (0,L) with $\Omega_1(a) = (0, a)$ and $\Omega_2(a) = (a, L),\ 0 < a < L$:

$$\nabla\cdot\kappa\nabla u = 0 \qquad u(0) = 0 \quad u(L) = b$$

Let $H(x)$ be the Heaviside function and

$$\kappa(x) = \kappa_1 + (\kappa_2 - \kappa_1)H(x - a) \qquad 0 < a < L$$

There is an analytical solution to the PDE

$$u = \left| \begin{array}{ll} a_1 x & x < a \\ a_2(x - L) + b & x > a \end{array} \right.$$

and the continuity of u and $\kappa\partial_n u$ requires

$$a_1 a = a_2(a - L) + b \quad \kappa_1 a_1 = \kappa_2 a_2 \Rightarrow a_2 = \frac{b\kappa_1}{L\kappa_1 + a[\kappa]} \quad a_1 = \frac{b\kappa_2}{L\kappa_1 + a[\kappa]}.$$

So the derivative u' of u with respect to a, at $a \neq 0$, is

$$u'(a) = \left| \begin{array}{cc} -xb[\kappa]\kappa_2(L\kappa_1 + a[\kappa])^{-2} & x < a \\ -(x-L)b[\kappa]\kappa_1(L\kappa_1 + a[\kappa])^{-2} & x > a \end{array} \right.$$

Notice that u' is discontinuous at a:

$$[u']_{x=a} = \frac{b[\kappa](a[\kappa] + L\kappa_1)}{(L\kappa_1 + a[\kappa])^2} = \frac{b[\kappa]}{L\kappa_1 + a[\kappa]} \neq 0 \text{ if } b \neq 0.$$

Outline The paper is organized as follows.

- First we recall the mixed formulation of the problem and we guess the result by giving a heuristic argument for the derivative.
- Then in Section 2 and 3 we prove existence, uniqueness and regularity for the solution of the mixed problem. The conjecture is proved in Section 4.
- A simple numerical illustration is given in Section 5; a Raviart-Thomas element is used, the derivative is compared with the finite difference approximation. Finally a simple inverse problem is analyzed numerically in Section 6; the curve of discontinuity of κ is recovered by least square (optimal control) from observations on a set distant from the curve.

1 The Conjecture

From now on we work with the curves $\Sigma(a)$ introduced in (2). For the sake of clarity and without loss of generality, let us consider a slightly different problem, for a given function f and a given vector valued function F:

$$\nabla \cdot (\kappa \nabla u) = f - \nabla \cdot (\kappa F) \text{ in } \Omega, \qquad \kappa \frac{\partial u}{\partial n}|_{\partial\Omega} = 0. \tag{4}$$

Let

$$\begin{array}{l} H(\text{div}, \Omega) = \{W \in L^2(\Omega)^d \ : \ \nabla \cdot W \in L^2(\Omega)\} \\ X = \{V \in H(\text{div}, \Omega) \ : \ V \cdot n = 0 \text{ on } \partial\Omega\} \\ L^2_0(\Omega) = \{w \in L^2(\Omega) \ : \ \int_\Omega w = 0\} \end{array}$$

Let F belong to $L^2(\Omega)^d$. Consider the mixed formulation with $U = \kappa(\nabla u + F)$.

$$\begin{array}{l} \text{Find } (U,u) \in X \times L^2_0(\Omega) \text{ with} \\ \left| \begin{array}{ll} \forall w \in L^2_0(\Omega) & \int_\Omega (\nabla \cdot U) w = \int_\Omega f w \\ \forall W \in X & \int_\Omega \frac{1}{\kappa} U \cdot W + \int_\Omega u \nabla \cdot W = \int_\Omega F \cdot W \end{array} \right. \end{array} \tag{5}$$

This is a slight departure from the standard mixed formulation (see Azaïez et al [3]) in that we have divided by κ.

Next we observe that with a piecewise constant κ

$$\frac{1}{\kappa(a)}(x) = \frac{1}{\kappa_1} + [\frac{1}{\kappa}]I_{\Omega_2} \qquad \text{therefore } \frac{d}{da}(\frac{1}{\kappa}) = \alpha\delta_\Sigma[\frac{1}{\kappa}]$$

with δ_Σ defined by

$$\forall f \in H^{1/2+\epsilon}(\Omega) \qquad \int_\Omega f\delta_\Sigma = \int_\Sigma f$$

So differentiating (5) yields

$$\begin{aligned} &\int_\Omega (\nabla \cdot U')w = 0 \\ &\int_\Omega (\frac{1}{\kappa}U' \cdot W + u'\nabla \cdot W) = -\int_\Sigma \alpha[\frac{1}{\kappa}U] \cdot W, \end{aligned} \tag{6}$$

assuming that the trace of $U \cdot W$ on Σ exist. Note, however, that if s, n denotes a tangent vector and a normal vector to Σ, $U \cdot n/\kappa$ jumps across Σ but $U \cdot s/\kappa = \partial u/\partial s$ does not because $[u]_\Sigma = 0$. Hence

$$[\frac{1}{\kappa}U] \cdot W = [\frac{1}{\kappa}U] \cdot nW \cdot n = [\frac{1}{\kappa}]U \cdot nW \cdot n.$$

and so it seems that $U \cdot n \in L^2(\Sigma)$ could be sufficient.

Conjecture 1 *The derivative $(V, v) \equiv (U', u')$ of (U, u) solution of (5) is given by*

Find $(V, v) \in X \times L^2_0(\Omega)$ with

$$\left| \begin{array}{ll} \forall w \in L^2_0(\Omega) & \int_\Omega (\nabla \cdot V)w = 0 \\ \forall W \in X & \int_\Omega (\frac{1}{\kappa}V \cdot W + v\nabla \cdot W) = \int_\Sigma gW \cdot n \end{array} \right. . \tag{7}$$

with

$$g = -\alpha[\frac{1}{\kappa}]U \cdot n$$

Remark 1 *In the distribution sense (7) is*

$$\nabla \cdot V = 0 \quad \frac{1}{\kappa}V - \nabla v = \delta_\Sigma g n \qquad V \cdot n|_\Gamma = 0$$

Note then that it is not possible to eliminate V and write a single equation for v because $\kappa\delta_\Sigma$ has no meaning on account of the discontinuity of κ on Σ.

Remark 2 *It is easy to see that the conjecture is true for the one-dimensional example described in the introduction.*

$$U' = \frac{d}{da}(\kappa\frac{du}{dx}) = -b\kappa_1\kappa_2[\kappa](L\kappa_1 + a[\kappa])^{-2} \;\Rightarrow\; \int_0^L U'\frac{dw}{dx} = U'\int_0^L \frac{dw}{dx} = 0.$$

$$\int_0^L (\frac{1}{\kappa}U'W + u'\nabla\cdot W) = -b\kappa_1\kappa_2[\kappa](L\kappa_1 + a[\kappa])^{-2}(\frac{1}{\kappa_1}\int_0^a W + \frac{1}{\kappa_2}\int_a^L W)$$
$$+ b\kappa_2[\kappa](L\kappa_1 + a[\kappa])^{-2}\int_0^a W \;+\; b\kappa_1[\kappa](L\kappa_1 + a[\kappa])^{-2}\int_a^L W$$
$$+u'W|_0^a + u'W|_a^L = W(a)\frac{b[\kappa]}{a[\kappa] + L\kappa_1} = -W(a)[\frac{1}{\kappa}]\kappa\frac{du}{dx}$$

2 Existence and Uniqueness

Assume $\Gamma = \partial\Omega$ of class $C^{1,1}$. Let Σ be a smooth closed curve inside Ω.
Let $\kappa > 0$ be a piecewise constant function discontinuous across Σ only. Then, ***provided that*** $W \mapsto \int_\Sigma gW.n$ ***is continuous on*** X, an existence and uniqueness result can be shown by adapting the proof of Theorem 2.1 in [3] to the case $\kappa \neq 1$. It is an application of the inf-sup lemma, namely, all bilinear forms being continuous,

$$\|V\|_{H(\mathrm{div},\Omega)} + \frac{\beta^*}{2}\|v\|_0 \le C\|g\|_{H^{1/2}(\Sigma)}$$

where β is the inf-sup constant of $(W, w) \mapsto (\nabla\cdot W, w)$:

$$\beta^* = \inf_{w\in L_0^2(\Omega)} \sup_{W\in X} \frac{(\nabla\cdot W, w)}{\|v\|_{H(\mathrm{div},\Omega)}\|w\|_0}$$

3 Regularity

For the sake of clarity, we assume for what follows that $F = 0$. Recall also that we have assumed that $\Gamma\cap\Sigma = 0$.

In (7) $U\cdot n|_\Sigma$ appears in an integral. We need to show that the integral exists. Functions of V have their normal component traces $V\cdot n$ on Σ in $H^{-1/2}(\Sigma)$. So we need to show that $U\cdot n \in H^{1/2}(\Sigma)$.

Proposition 1 *If Σ is regular and f is in $H^1(\Omega)$ then $U\cdot n$ belongs to $H^{1/2}(\Sigma)$.*

For clarity the proof is given in dimension 2. Assume that Σ is sufficiently regular so that in a neighborhood $\mathcal{O}$ of Σ we can define a coordinate system σ, ν in which the equation of Σ is $\nu = 0$, n the normal to Σ, is tangent to the curves $\sigma =$ constant and σ is its curvilinear abscissa.

Problem (4) in variational form writes

$$\forall w \in H^1(\Omega) \qquad \int_\Omega \kappa\nabla u\nabla w = \int_\Omega fw$$

By taking supp$w' \subset \mathcal{O}$, $w = \frac{\partial w'}{\partial\sigma}$ and by integrating by part in σ we find that

$$\forall w' \in H_0^1(\mathcal{O}) \qquad \int_{\mathcal{O}} \kappa\nabla\frac{\partial u}{\partial\sigma}\nabla w' = \int_{\mathcal{O}} \frac{\partial f}{\partial\sigma}w' \tag{8}$$

because κ is not a function of σ. This shows that if f is regular all partial derivatives in σ of u are in $H^1(\mathcal{O})$. Therefore $U \cdot \vec{s}$ has the same regularity.

Now $\frac{\partial U \cdot s}{\partial \sigma} = \kappa \frac{\partial^2 u}{\partial \sigma^2}$ belongs to $L^2(\mathcal{O})$. So by $\nabla \cdot U = 0$ we see that $\frac{\partial U \cdot n}{\partial \nu}|_{\mathcal{O}_i}$ is in $L^2(\mathcal{O}_i)$ for any open set $\mathcal{O}_i \subset \mathcal{O}$ not intersecting Σ.
Similarly, $U \cdot \vec{s} \in H^1(\mathcal{O})$ implies $\frac{\partial}{\partial \nu} U \cdot \vec{s} \in L^2(\mathcal{O})$, i.e. $\frac{\partial}{\partial \nu} \kappa \frac{\partial u}{\partial \sigma} \in L^2(\mathcal{O})$. Therefore $\frac{\partial U \cdot n}{\partial \sigma} = \kappa \frac{\partial^2 u}{\partial \nu \partial \sigma}$ is in $L^2(\mathcal{O} \backslash \Sigma)$. Hence U is in $H^1(\Omega \backslash \Sigma)$.

Corollary 1 *If Σ is regular and F belongs to $W^{2,\infty}(\Omega)$ then U is continuous in $\Omega \backslash \Sigma$ and U is in $L^\infty(\Omega)$.*

This is because (8) shows that $\frac{\partial u}{\partial \sigma}$ satisfies a partial differential equation of the same type as the one of u. So by the proposition above $\kappa \nabla \frac{\partial u}{\partial \sigma} \cdot n$ is in $H^{1/2}(\Sigma)$. Therefore $U.n$ is in $H^{3/2}(\Sigma)$, hence it is also continuous and bounded. By the maximum principle (in $\Omega \backslash \Sigma$, ΔU is bounded), U is bounded everywhere.

4 Differentiability

4.1 Continuity

In order to study the changes $\delta u, \delta U$ of u, U, when $a \to 0$, let

$$\eta = \kappa^{-1}, \quad \delta u = u(a) - u(0), \quad \delta U = U(a) - U(0), \quad \delta \eta = \eta(a) - \eta(0).$$

1. The first equation in (7) is easy to establish because

$$\int_\Omega (\nabla \cdot U) w = 0 \quad \int_\Omega (\nabla \cdot (U + \delta U)) w = 0 \Rightarrow \int_\Omega (\nabla \cdot \delta U) w = 0 \tag{9}$$

2. For the second equation we have

$$\int_\Omega ((\eta + \delta \eta)(U + \delta U) \cdot W - \eta U \cdot W + \delta u \nabla \cdot W) = 0$$

which is also

$$\int_\Omega ((\eta + \delta \eta) \delta U \cdot W + \delta u \nabla \cdot W) = -\int_\Omega \delta \eta U \cdot W. \tag{10}$$

Take $W = \delta U$ in (10), and use $w = \delta u$ in (9), then with $\kappa_{max} = \max(\kappa_1, \kappa_2)$,$\kappa_{min} = \max(\kappa_1, \kappa_2)$

$$\begin{aligned} &\int_\Omega (\eta + \delta \eta) |\delta U|^2 = -\int_\Omega \delta \eta U \cdot \delta U \leq \|\delta U\|_0 \|U\|_\infty \|\delta \eta\|_0 \\ \Rightarrow \quad &\frac{1}{\kappa_{max}} \|\delta U\|_0 \leq \|U\|_\infty \|\delta \eta\|_0 \end{aligned}$$

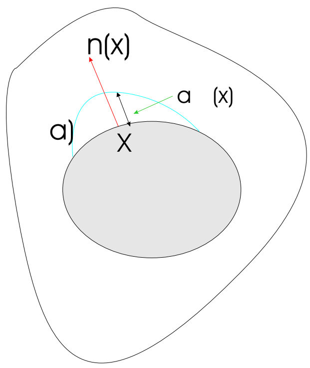

Figure 1: *The coefficient κ is constant on Ω_1 and constant on Ω_2 and discontinuous across Σ. When Σ becomes $\Sigma(a)$, the distance from $\Sigma(a)$ to Σ is $a\|\alpha\|_\infty$,the solution of the partial differential equation changes. We wish to find the derivative with respect to a for a given function α.*

The support $S(a)$ of $\delta\eta$ is a thin strip around Σ of width αa, and $\eta+\delta\eta$ is either κ_{max}^{-1} or κ_{min}^{-1}; therefore

$$\|\delta\eta\|_0 \le C_1|[\frac{1}{\kappa}]|\sqrt{a\|\alpha\|_\infty}$$

for some constant C_1. So we have the following result

Proposition 2 *If $U \in L^\infty(\Omega)$, the change $\{\delta u, \delta U\}$ in $\{u, U\}$ due to a is bounded in $L^2_0(\Omega)\times H(div,\Omega)$, verifies $\nabla\cdot\delta U = 0$ and*

$$\|\delta u\|_0 + \|\delta U\|_{H(div,\Omega)} \le C_2\sqrt{\|\alpha\|_\infty a}\|U\|_\infty \frac{|[\kappa]|}{\kappa_{min}}$$

4.2 Differentiability

Lemma 1 *Let $\mathcal{O}$ be a neighborhood of Σ containing $S(a)$, the support of $\eta(a)-\eta(0)$, but sufficiently thin near Σ so as to be able to extend n in $\mathcal{O}$. If U, W are continuous in $\mathcal{O}\backslash\Sigma$ and $U\cdot n, W\cdot n$ are continuous in $\mathcal{O}$ then, for a small enough,*

$$\lim_{a\to 0}|\int_{S(a)}\frac{\delta(\eta U)}{a}\cdot W - \int_\Sigma \alpha[\frac{1}{\kappa}]U\cdot nW\cdot n| = 0 \tag{11}$$

Proof We assume that W is smooth; the result can be extended later by density. Consider the case $a > 0$, $\alpha \ge 0$ everywhere. Then (see figure 1), assuming that n is oriented from $\Omega_1(a)$ to $\Omega_2(a)$, and denoting by $U_i(x,a)$ (resp $(U_i(\sigma, n, a))$ the value of U at $x \in \Omega_i$ (resp σ, n for the value a of the parameter,

$$\int_{S(a)}\frac{\delta(\eta U)}{a}\cdot W = \int_{S(a)}\frac{\eta_2 U_2(x,a) - \eta_1 U_1(x,0)}{a}\cdot W(x)dx$$

$$= \int_\Sigma d\sigma \int_0^{a\alpha(\sigma)} \frac{\eta_2 U_2(\sigma,n,0) - \eta_1 U_1(\sigma,n,0)}{a} \cdot W(\sigma,n)dn + O(\sqrt{a}) \quad (12)$$

where we have used the continuity with respect to a (Proposition 2), and where $S(a)$ is the support of $\delta\eta$, where $S(a)$ is given by

$$S(a) = \{(\sigma,n) : n \in [0, a\alpha(\sigma)\}.$$

By the mean value theorem, when g is continuous, there exists $n(\sigma)$ such that

$$\int_{S(a)} g = \int_\Sigma d\sigma \int_0^{a\alpha(\sigma)} g(\sigma,n)dn = \int_\Sigma a\alpha(\sigma) g(\sigma, n(\sigma))d\sigma.$$

Therefore (12) becomes

$$\int_{S(a)} \delta(\frac{\eta}{a}U) \cdot W = \int_\Sigma \alpha(\sigma)[\eta U(x,0)] \cdot W(x)d\sigma(x)$$

Naturally the same argument applies when $\alpha < 0$ and also to the general case by cutting Σ into pieces on which α does not change sign.
Finally (11) is found because

$$\eta_2 U_2 \cdot s - \eta_1 U_1 \cdot s = [\frac{\partial u}{\partial s}] = 0$$

◇

So with $\{v_\delta, V_\delta\} = \{\frac{\delta u}{a}, \frac{\delta U}{a}\}$, (9) gives

$$\forall w \in L^2_0(\Omega) \quad \int_\Omega (\nabla \cdot V_\delta) w = 0. \quad (13)$$

Let $\mathcal{O}$ be a thin strip around Σ containing $\Sigma(a)$ for all a under consideration below. After division by a equation (10) reads,

$$\forall W \in H(\text{div},\Omega) \quad \int_{\Omega \backslash \mathcal{O}} (\eta V_\delta \cdot W + u_\delta \nabla \cdot W) + \int_{\mathcal{O}} \frac{\delta(\eta U)}{a} \cdot W = 0.$$

Accordingly by Lemma 1 there exists a weakly converging subsequence of v_δ, V_δ and the limit v, V satisfies

$$\forall W \in H(\text{div},\Omega) \quad \int_\Omega (\eta V \cdot W + v \nabla \cdot W) + [\frac{1}{\kappa}] \int_\Sigma \alpha U \cdot n W \cdot n = 0$$

Remark 3 *With sufficient regularity we have in fact proved that*

$$|\int_{S(a)} \delta(\frac{\eta U}{a}) \cdot W - \int_\Sigma \alpha[\frac{1}{\kappa}] U \cdot n W \cdot n| \le aC \|\frac{\partial U}{\partial n} \cdot n\|_{1/2,\Sigma} \|\frac{\partial W}{\partial n} \cdot n\|_{1/2,\Sigma} \quad (14)$$

Let us summarize the result:

Theorem 1 *The solution of (5),* $\{u(a), U(a)\}$*, with*

$$\Sigma(a) = \{x + a\alpha(x)n(x) \ : \ x \in \Sigma\}$$

is differentiable in a *in the sense that*

$$v = \lim_{a\to 0} \frac{u(a) - u(0)}{a} \quad V = \lim_{a\to 0} \frac{U(a) - U(0)}{a}$$

is solution of (7) where the jump $[\frac{1}{\kappa}]$ *is*

$$x \in \Sigma \qquad [\frac{1}{\kappa}](x) = \lim_{a\to 0} \left(\frac{1}{\kappa(x + a\alpha(x)n(x))} - \frac{1}{\kappa(x - a\alpha(x)n(x))} \right)$$

5 Discretization and Numerical Test

5.1 Discretization

Consider a regular family of triangulations of Ω of maximum edge length h and two finite element spaces X_h and L_h to approximate $H(\mathrm{div}, \Omega)$ and $L^2_0(\Omega)$. Although precision is increased when the triangulations approximate Σ as an internal boundary, the theory works also without this hypothesis. The numerical discretization of problem (7) reads

$$\begin{array}{l} \text{Find } (V_h, v_h) \in X_h \times L_h \text{ with} \\ \left| \begin{array}{ll} \forall q \in L_h & \int_\Omega (\nabla \cdot V_h) q = 0 \\ \forall W \in X_h & \int_\Omega (\frac{1}{\kappa} V_h \cdot W + v_h \nabla \cdot W) = \int_\Sigma g W \cdot n \end{array} \right. \end{array} \tag{15}$$

Among the various admissible choices, we have selected the Raviart-Thomas element for X_h and the piecewise constant functions for L_h. This couple satifies the discrete inf-sup condition as shown in Achdou et al [1], Proposition 3.14:

$$\|V - V_h\|_0 + |v - v_h|_0 \le ch^s(\|V\|_{H^s} + \|v\|_{H^s})$$

when $(V, v) \in (H^s(\Omega))^2 \times H^s(\Omega), \quad 0 < s \le 1$.

5.2 Numerical Simulation

The numerical solution is calculated with `freefem++` [5]. To illustrate the theory we have solved the problem

$$-\nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega \qquad u|_\Gamma = xy \tag{16}$$

where Ω is the rectangle $(-5, 5) \times (-2.5, 2.5)$, κ is 6 inside an ellipse in the middle of the rectangle and 1 outside.

Then the ellipse is changed by ϵ according to

$$\{(x,y)\ :\ x=(2+\epsilon)(\sqrt{2}+\epsilon)\cos t, \quad y=(\sqrt{2}+\epsilon)\sin t \ \ t\in(0,2\pi)\}$$

yielding a new solution u_ϵ of (16). Then $u'_\epsilon=(u_\epsilon-u)/\epsilon$ is compared to the numerical solution of (15). The results are displayed on figure 2. Decreasing both ϵ

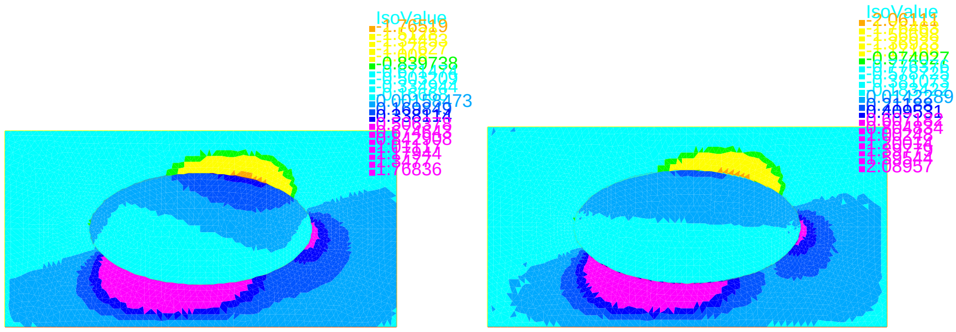

Figure 2: ***Left:*** u'. ***Right:*** u'_ϵ *when* $\epsilon=0.0125$. *Observe that both solutions are clearly discontinuous across* Σ

and the mesh size gives convergence:

ϵ	0.1	0.05	0.025	0.0125
L^2-error	0.96	0.71	0.56	0.46

6 Identification of a discontinuity

Consider an "observed" data u_d on an observation set S and the problem of finding the best κ (i.e. the best Σ) to fit this data. A least square approach leads to

$$\min_{\kappa\in K}\{J(\kappa)=\int_S \|u-u_d\|^2 : -\nabla\cdot(\kappa\nabla u)=0 \quad u|_\Gamma=x+y\}.$$

In this example the problem is driven by a non homogeneous boundary condition rather than by a right hand side F.

Obviously a normal change $a\alpha$ in the position of Σ induces a change in J and it is not hard to show that the derivative $J'=dJ/da$ is given by

$$J'(\kappa)=-\int_\Sigma \alpha[\frac{1}{\kappa}](\kappa\frac{\partial u}{\partial n})(\kappa\frac{\partial p}{\partial n})$$

with p solution of

$$-\nabla\cdot(\kappa\nabla p)=2(u-u_d)\delta_S \qquad p|_\Gamma=0$$

Assume that $\alpha = \alpha(r_1, r_2 ... r_m)$. A gradient method on the position of Σ with step size ρ, via the parameters $\{r_i\}_1^m$ would be to change r_i according to

$$r_i \quad \leftarrow \quad r_i + \rho \int_\Sigma \frac{\partial \alpha}{\partial r_i} [\frac{1}{\kappa}] (\kappa \frac{\partial u}{\partial n}) (\kappa \frac{\partial p}{\partial n})$$

a) We ran a preliminary test by taking

$$\Omega = (-5, 5) \times (-2.5, 2.5), \quad D = \{(x, y) \; : \; (x + 2)^2 + y^2 < 1\}$$
$$\Sigma(r_1, r_2) = \{(x, y) : \; x = (r_1 + r_2 \cos t) \cos t \;\; y = (r_1 + r_2 \cos t) \sin t\}$$

and the reference surface $\Sigma = \Sigma(\sqrt{2}, 0)$. As before $\kappa = 1$ outside Σ and 6 inside. We chose u_d to be the solution of the PDE for κ given by $\Sigma(\sqrt{2}, 0)$. Then we apply the steepest descent method with $\rho = 4$ starting from $\Sigma(0.3, 0.1)$. Figure 3 shows the convergence curves.

b) In the previous configuration D intersects Σ. Now we move D to the right and Σ to the left (figure 2) and ran the same test with $r_1 = 0.3$ and $r_2 = 0.5$ initially. The method converges but the final shape is close to but different from $\Sigma(\sqrt{2}, 0)$. This is because the numerical approximation does not "see" the right part of Σ which is too far.

c) In third test where $r_1 = 0$ and the descent is only on r_2, the exact solution was reached in 4 iterations. This indicates that the method is sound but a conjugate gradient is needed for test (b).

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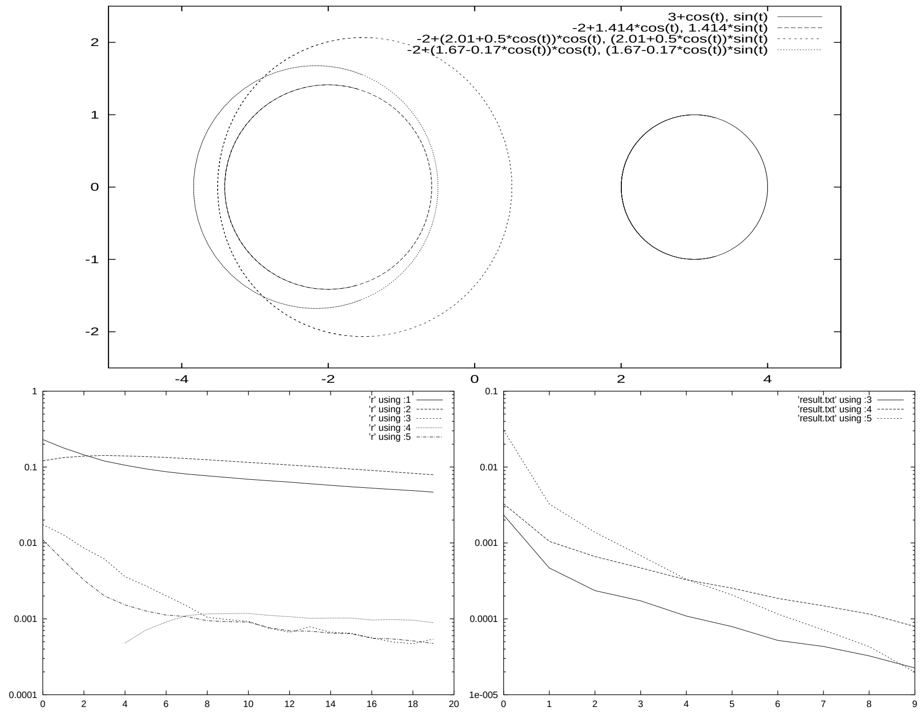

Figure 3: ***Center up:*** *Geometry of the second example showing the observation set (right circle), the exact and computed solution (inner circle and middle curve) and the initial shape (outer curve).* ***Bottom left:*** *Convergence curves for the identification problem: curve 1 and 2 are* $r_1 - \sqrt{2}, r_2$ *, curve 3 and 4 are* $\frac{\partial J}{\partial r_1}$, $\frac{\partial J}{\partial r_2}$ *and curve 5 is J; all tend to zero; the x-axis is the iteration count (from 1 to 20 here).* ***Bottom right:*** *Convergence curves showing the two gradients and the cost function for 10 iterations.*

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