Mini Course
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Cross-diffusion systems.

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0 Outline

These notes correspond to a lecture given at the University of Vienna in the spring 2019. After a short introduction in which both the modeling motivations and the ongoing open problems for cross-diffusion systems will be presented, the course comes in two parts. The first one is intended for students who *are not* specialized in PDEs ; several proofs that would be rather quickly sketched in a doctoral lecture will be written in details. In this first part we will focus on the *Kolmogorov equation*, a constitutive stone of cross-diffusion systems, and use it to present the general approximation-compactness procedure used in PDE to establish existence of solutions. Up to basic notions of functional analysis, the first part of the lecture is self-contained. The second part of the lecture is less scholar ; some results of the literature will be used as black boxes, other technical details will be left in exercise. The goal of this last part is to grasp the main tools necessary to establish the existence of global solutions to a wide class of cross-diffusion systems.

1 Introduction

1.1 Cross-diffusion systems

To describe the evolution of a given population along the time, Malthus proposed a model in the 18^{th} century, later refined by Verhulst :

$$u'(t) = au(t) - bu(t)^2,$$

where $a > 0$ and $b > 0$ are respectively the reproduction and competition rate for ressources. The Lotka-Volterra model generalizes this idea for two species in the context of a predator/prey :

$$\begin{aligned} u'(t) &= u(a_1 - c_1 v), \\ v'(t) &= -v(a_2 - c_2 u). \end{aligned}$$

These models are *homogeneous* in space. If one wants to take into account the repartition of the species in a given area, the simplest tool is the laplacian diffusion : individuals do not stand in one place, they tend to move in randomly and occupy the available space :

$$\begin{aligned} \partial_t u - d_1 \Delta u &= u(r_1 - s_{11} u - s_{12} v), \\ \partial_t v - d_2 \Delta v &= v(r_2 - s_{21} u - s_{22} v). \end{aligned}$$

When studied on a bounded domain Ω of the space variable, several boundary conditions are possible like homogeneous Neumann (confinement) or homogeneous Dirichlet (hostility at the boundary).

Drawback of the diffusive Lotka-Volterra models : steady states are all constant. This means that if one tries to follow the behavior of $u(t)$ and $v(t)$ when $t \to +\infty$, formally the only possible repartition are space-filling for both species. In particular, the phenomenon of *segregation* in which the two species share the available habitat, cannot be grasped by such models.

In 1979, Shigesada, Kawasaki and Teramoto proposed in [5] an alternative model to describe the interaction of two species, which writes (and that we call SKT)

$$\begin{aligned} \partial_t u - \Delta\Big[(d_1 + a_{11} u + a_{12} v) u\Big] &= u(r_1 - s_{11} u - s_{12} v), \\ \partial_t v - \Delta\Big[(d_2 + a_{21} u + a_{22} v) v\Big] &= v(r_2 - s_{21} u - s_{22} v), \end{aligned}$$

where a_{ij} is a coefficient of *self-diffusion* if $i = j$ and *cross-diffusion* if $i \neq j$. Contrary to the Lotka-Volterra models above, SKT systems allow the existence of stationnary solutions (u, v) for which $\{u \geq \varepsilon\} \cap \{v \geq \varepsilon\} = \varnothing$ for some $\varepsilon > 0$.

The structure of the diffusion operator used in the two equations of the SKT system is not Fickian. More precisely, the Fick law for the diffusion of a quantity z with diffusivity μ writes $\partial_t z - \operatorname{div}(\mu \nabla z) = 0$ whereas the equations that we have here are of the form

$$\partial_t z - \Delta(\mu z) = 0.$$

This is the Kolmogorov equation that we will explore in detail in Section 2. The tendency for each population of the SKT system to avoid each other (as examplified by the almost disjoint support of its steady states) has an link with the choice of diffusion operator. Formally, one can decompose the Kolmogorov diffusion operator in the following way

$$-\Delta(\mu z) = -\operatorname{div}(\mu \nabla z) - \operatorname{div}(z \nabla \mu),$$

and understand this operator as Fick diffusion to which one adds a transport in the opposite direction of the gradient of the diffusivity.

1.2 Mathematical framework and notations

In order to simplify the presentation we will work on $Q_T := [0,T) \times \mathbf{T}^d$ where $\mathbf{T}^d$ is the d-dimensionnal torus. All the result that we present in this lecture can be generalized to the case of a bounded open set of $\mathbf{R}^d$, with homogeneous Neumann boundary conditions. We denote by $\mathscr{D}(Q_T)$ the set of test functions (smooth with compact support in Q_T) and $\mathscr{D}'(Q_T)$ the corresponding set of distributions.

For any $1 \leq p, q \leq \infty$ we denote by $\|\cdot\|_{p,q}$ the norm of $\mathrm{L}^p(0,T;\mathrm{L}^q(\mathbf{T}^d))$ and simply by $\|\cdot\|_q$ the norm of $\mathrm{L}^q(\mathbf{T}^d)$.

Any element f of $\mathscr{C}^\infty(\mathbf{T}^d)$ admits a decomposition in terms of Fourier series

$$f = \sum_{k\in\mathbf{Z}^d} c_k(f) e_k, \tag{1}$$

where $e_k : x \mapsto e^{2i\pi k\cdot x}$ and $c_k(f) = \langle f, e_k\rangle_{\mathrm{L}^2(\mathbf{T}^d)}$; in the latter case, the convergence is uniform. This decomposition can be generalized up to $\mathscr{D}'(\mathbf{T}^d)$, but we will only use the hilbert framework : when $f \in \mathrm{L}^2(\mathbf{T}^d)$, (1) holds with a convergence in $\mathrm{L}^2(\mathbf{T}^d)$ of the right hand side.

The symbol $\lesssim$ will be frequently used, $A \lesssim B$ meaning that $A \leq CB$ for some universal constant C independent of A and B if the latter variables are quantified in the statement of the inequality ; $\dot\in$ means "bounded in" and $\ddot\in$ means "relatively compact in".

1.3 Mathematical analysis of cross-diffusion systems

For a PDE specialist, at first sight, the mathematical analysis of the SKT system does not seem *that* bad. It happens that this system provides a lot of (seemingly) simple questions, resisting to the mathematical community since forty years ago. Maybe the most striking one is the following, which concerns a particular instance of the system.

Question *Is there, for any smooth nonnegative initial condition, a global smooth nonnegative solution to the following system ?*

$$\partial_t u - \Delta\Big[(2+v)u\Big] = 0,$$
$$\partial_t v - \Delta\Big[(1+u)v\Big] = 0.$$

Local solutions (defined on some small intervall) are known to exist since the '90, thanks to a theory developped by Amann. The first global (weak) solution to the SKT system has been built by Chen and Jüngel [2] in 2006 (almost 30 years after the appearance of SKT), and up to now it is not known if this solution is smooth or not, for regular initial data.

In this lecture (mainly in Section 3) we will present the tools needed to establish the following generalization of [2], which is extracted from [3].

Theorem 1
For periodic boundary conditions and square-integrable nonnegative initial data, there exist a global (weak) solution (u,v) to the system

$$\partial_t u - \Delta\Big[(d_1 + a_{12}v^{\gamma_1})u\Big] = 0, \tag{2}$$
$$\partial_t v - \Delta\Big[(d_2 + a_{21}u^{\gamma_2})v\Big] = 0, \tag{3}$$

where all the coefficients are positive and arbitrary, the sole constraint being $\gamma_1\gamma_2 \leq 1$.

We will precise the meaning of solutions that we consider later on. When compared to the original SKT system, the previous system seems simpler because there is no self-diffusion and no reaction terms. We will explain why this assumption is harmless. However, the cross-diffusion terms are here more involved — SKT corresponds to $(\gamma_1,\gamma_2) = (1,1)$ — and we plan to justify the appearance of the enigmatic condition $\gamma_1\gamma_2 \leq 1$.

2 Kolmogorov equation

We focus here on the following equation

$$\partial_t z - \Delta(\mu z) = 0, \tag{4}$$
$$z(0,x) = z^0(x), \tag{5}$$

the unknown being z. Here μ is a given integrable function on Q_T for which we will always assume $\inf_{Q_T} \mu > 0$.

This first part of the lecture is somehow independent of the second, in terms of mathematical construction. The only result that we will use (in fact, a discretized version of it) in the sequel will be the duality lemma. However we use the specific example of the Kolmogorov equation to present a generic method of proof of existence for PDEs, based on the *a priori* estimate.

2.1 The "demanding" *a priori* estimate

Proposition 1
Assume that z and μ are smooth, with z solving (4). Then the following estimate holds

$$\int_{\mathbf{T}^d} z(t)^2 + \int_0^t \int_{\mathbf{T}^d} \mu |\nabla z|^2 \leq \exp\left(\int_0^t \sup_{\mathbf{T}^d} \Delta\mu(s)\,\mathrm{d}s\right) \int_{\mathbf{T}^d} z(0)^2. \tag{6}$$

Proof. Multiplying the Kolmogorov equation by its solution z and integrating by parts, we infer

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{T}^d} z(t)^2 + \int_{\mathbf{T}^d} \mu(t)|\nabla z(t)|^2 &= -\int_{\mathbf{T}^d} z(t)\nabla z(t)\cdot\nabla\mu(t)\\ &= \frac{1}{2}\int_{\mathbf{T}^d} z(t)^2\Delta\mu(t),\end{aligned}$$

so that the conclusion follows by Gronwall's inequality. □

2.2 The duality estimate

Another estimate exists, which demands less regularity on μ.

Proposition 2
Assume that z and μ are smooth, with z solving (4). Then the following estimate holds

$$\int_{Q_T} \mu z^2 \lesssim \left(1 + \int_{Q_T} \mu\right) \int_{\mathbf{T}^d} z(0)^2. \tag{7}$$

Remark 2.1 *When compared to estimate (6), we see that a lot less regularity is needed on μ (merely integrability on Q_T). The price is that no gradient of z is controlled : only its (μ-weighted) $\mathrm{L}^2(Q_T)$ norm.*

Before presenting the proof of this proposition, we start with a short lemma.

Lemma 1
For any smooth function Φ on Q_T vanishing at T, we have the following estimate

$$\sup_{[0,T]}\left\{\int_{\mathbf{T}^d}\Phi(t)^2 + \int_{\mathbf{T}^d}|\nabla\Phi(t)|^2\right\} + \int_{Q_T}\mu|\Delta\Phi|^2 \lesssim \left(1+\int_{Q_T}\mu\right)\int_{Q_T}\mu^{-1}(\partial_t\Phi + \mu\Delta\Phi)^2. \tag{8}$$

Proof. For simplicity, let's introduce $S := \partial_t\Phi + \mu\Delta\Phi$. When multiplying the previous equality by $\Delta\Phi$ we get after integration by parts on $\mathbf{T}^d$

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{T}^d}|\nabla\Phi(t)|^2 + \int_{\mathbf{T}^d}\mu(t)|\Delta\Phi(t)|^2 = \int_{\mathbf{T}^d} S(t)\Delta\Phi(t),$$

so that by Young's inequality, after integration on $[t,T]$ we infer

$$\int_{\mathbf{T}^d}|\nabla\Phi(t)|^2 + \int_t^T\int_{\mathbf{T}^d}\mu|\Delta\Phi|^2 \leq \int_{Q_T}\mu^{-1}S^2, \tag{9}$$

which contains a part of (8). By Parseval's identity, for any smooth function φ with zero mean on $\mathbf{T}^d$, we have

$$\int_{\mathbf{T}^d} \varphi^2 \leq 4\pi^2 \int_{\mathbf{T}^d} |\nabla\varphi|^2,$$

from which we infer for all smooth function the Poincaré-Wirtinger inequality

$$\int_{\mathbf{T}^d} \varphi^2 \lesssim \int_{\mathbf{T}^d} |\nabla\varphi|^2 + \left(\int_{\mathbf{T}^d} \varphi\right)^2,$$

so that it only remains to control the mean of Φ along the time. This can be done noticing, after direct integration of the equation on $[t, T] \times \mathbf{T}^d$,

$$\int_{\mathbf{T}^d} \Phi(t) = \int_t^T \int_{\mathbf{T}^d} (\mu\Delta\Phi - S),$$

from which we infer by Hölder's inequality

$$\left|\int_{\mathbf{T}^d} \Phi(t)\right|^2 \leq \left(\int_{Q_T} \mu\right)\left(\int_{Q_T} \mu|\Delta\Phi|^2 + \int_{Q_T} \mu^{-1}S^2\right),$$

and we obtain (8) after combining the previous inequality with the above Poincaré-Wirtinger inequality and (9). □

Proof of Proposition 2. What is the link with the original Kolmogorov equation ? The estimate given by Lemma 1 gives a control on any solution of the *dual equation*

$$\partial_t\Phi^S + \mu\Delta\Phi^S = S, \tag{10}$$
$$\Phi^S(T,\cdot) = 0, \tag{11}$$

where S is some smooth test function that we will choose later on. We admit for the moment (we'll have in Subsection 2.3 all the material to prove it, see Exercise 2) that a smooth solution Φ^S exists to the dual equation.

Now, by a direct integration by parts we have the following formula which links the Kolmogorov equation and its dual one :

$$\int_{Q_T} (\partial_t z - \Delta(\mu z))\Phi^S = -\int_{\mathbf{T}^d} z(0)\Phi^S(0) - \int_{Q_T} z(\partial_t\Phi^S + \mu\Delta\Phi^S),$$

so

$$\int_{Q_T} zS = -\int_{\mathbf{T}^d} z(0)\Phi^S(0),$$

from which we infer thanks to Lemma 1

$$\left|\int_{Q_T} zS\right| \lesssim \left(\int_{Q_T} z(0)^2\right)^{1/2}\left(1 + \int_{Q_T} \mu\right)^{1/2}\left(\int_{Q_T} \mu^{-1}S^2\right)^{1/2},$$

and therefore by duality (or taking $S := \mu^{1/2}z$), we recover (8). □

2.3 Existence

The Kolmogorov equation being linear, one can build (rather) weak solutions by compactness using the previous estimates. The question whether these solution are (enough) regular or not (to justify some formal manipulations) is intimately linked with the uniqueness issue, that we will tackle in the last subsection.

Definition 2.1
For any $\mu \in \mathrm{L}^1(Q_T)$ and $z_0 \in \mathrm{L}^1(\mathbf{T}^d)$ we say that a measurable function z is a distributionnal solution of (4) – (5) (or : solves the Kolmogorov equation with initial condition z_0) if $\mu z \in \mathrm{L}^1(Q_T)$ and if, for any $\varphi \in \mathscr{D}(Q_T)$, there holds

$$-\int_{Q_T} z(\partial_t \varphi + \mu \Delta \varphi) = \int_{\mathbf{T}^d} z_0\, \varphi(0).$$

Remark 2.2 *We skip it but one could define similar notion of solutions with terminal condition at time $t = T$ on $Q_0 := (0, T] \times \mathbf{T}^d$,* etc.

Solving a PDE in the sense of distribution is an extremely weak way to solve it. An intermediate way is to have Sobolev regularity in order to absorb "part" of the differential operators on the solution (and not apply it all on the test functions). Of course, when the derivatives exists in the classical sense, then we recover the standard formulation as is suggesting the following exercise.

Exercise 1 – *Prove that if z and μ are smooth, then the Kolmogorov equation holds in the classical sense, as well as the initial condition.*

Theorem 2
Consider $z^0 \in \mathrm{L}^2(\mathbf{T}^d)$ and $\mu \in \mathrm{L}^1(0,T;\mathrm{L}^2(\mathbf{T}^d)) + \mathrm{L}^2(0,T;\mathrm{L}^{p_d}(\mathbf{T}^d))$ where $p_d := (d+2)/2d$, such as $\inf_{Q_T} \mu > 0$ with $\Delta\mu \leq \rho$ for some $\rho \in \mathrm{L}^1(0,T)$. There exists at least one element z in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbf{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$ solving the Kolmogorov equation in $\mathscr{D}'(Q_T)$ with initial condition z^0 and satisfying

$$\sup_{[0,T]} \int_{\mathbf{T}^d} z(t)^2 + \int_{Q_T} \mu |\nabla z|^2 \leq \exp\left(\int_0^T \rho(s)\,\mathrm{d}s\right) \int_{\mathbf{T}^d} z_0^2.$$

Remark 2.3 *For $S \in \mathscr{D}'(Q_T)$, $S \geq 0$ means $\langle S, \varphi \rangle \geq 0$ for all $\varphi \in \mathscr{D}(Q_T)$ having nonnegative values.*

Remark 2.4 *The functional setting chosen for μ ensures precisely that μz is integrable, in order to match with Definition 2.1: by Sobolev embedding we have $\mathrm{H}^1(\mathbf{T}^d) \hookrightarrow \mathrm{L}^{p_d'}(\mathbf{T}^d)$.*

Proof. The assumptions on μ allow us to can consider a smooth approximation $(\mu_n)_n$ satisfying $\mu_n \in \mathscr{C}^\infty(Q_T)$, $\mu_n \geq \inf_{Q_T} \mu$, $\Delta\mu_n \leq \rho_n$, with $(\mu_n)_n \to \mu$ and $(\rho_n)_n \to \rho$ respectively in $\mathrm{L}^1(0,T;\mathrm{L}^2(\mathbf{T}^d)) + \mathrm{L}^2(0,T;\mathrm{L}^{p_d}(\mathbf{T}^d))$ and $\mathrm{L}^1(0,T)$.

We follow the Galerkine approach introducing, for $n \in \mathbf{N}$, the subspace $V_n := \mathrm{Span}(e_k : |k| \leq n)$ and searching $z^n \in \mathscr{C}^1([0,T];V_n)$ solving

$$\begin{aligned} \partial_t z_n - \Delta(\mu_n z_n) &\in \mathscr{C}^0([0,T];V_n^\perp), \\ z_n(0) &= \mathbb{P}_n z^0, \end{aligned}$$

where $\mathbb{P}_n$ denotes the $\mathrm{L}^2(\mathbf{T}^d)$ orthogonal projection on V_n. As n grows, one expects the previous orthogonality condition to merge with the Kolmogorov equation. Due to the finite dimension of V_n, the equation above reduces to

$$\forall |k| \leq n, \qquad \langle \partial_t z_n, e_k \rangle_{\mathrm{L}^2(\mathbf{T}^d)}(t) = \langle \Delta(\mu_n z_n), e_k \rangle_{\mathrm{L}^2(\mathbf{T}^d)}(t).$$

The belonging $z_n \in \mathscr{C}^1([0,T];V_n)$ means the existence of $\mathscr{C}^1([0,T];\mathbf{R})$ functions $(\alpha_k)_{|k|\leq n}$ such that $z = \sum_{|k|\leq n} \alpha_k e_k$, and the previous equation rewrites

$$Y'(t) = A(t)Y(t),$$

where Y is the vector $(\alpha_k)_{|k|\leq n}$, A is the matrix of coefficients $\langle \Delta(\mu_n e_\ell), e_k \rangle_{\mathrm{L}^2(\mathbf{T}^d)}$. Since μ_n is smooth, the standard theory of linear ODE applies to get the existence of the functions $(\alpha_k)_{|k|\leq n}$.

Since z_n takes its (time) values in V_n, we have for all time $t \in [0,T]$

$$\langle \partial_t z_n - \Delta(\mu_n z_n), z_n \rangle_{\mathrm{L}^2(\mathbf{T}^d)}(t) = 0,$$

and we have all the regularity needed to perform the computations that we have done when proving (6), si that we have for all $t \in [0,T]$

$$\int_{\mathbf{T}^d} z_n(t)^2 + \int_0^t \int_{\mathbf{T}^d} \mu_n |\nabla z_n|^2 \leq \exp\left(\int_0^t \rho_n(s)\,\mathrm{d}s\right) \int_{\mathbf{T}^d} z_n(0)^2.$$

Since $(\rho_n)_n \dot{\in} \mathrm{L}^1(0,T)$, $(z_n(0))_n \dot{\in} \mathrm{L}^2(\mathbf{T}^d)$ and $\mu_n \geq \inf_{Q_T} \mu > 0$, we infer from this estimate the bound $(z_n)_n \dot{\in} \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbf{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$. Extracting a weakly$(-\star)$ subsequence in these spaces, we can (up to a subsequence) consider that $(z_n)_n \rightharpoonup z$ in $\mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbf{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$. Fix $p \in \mathbf{N}$ and $w \in \mathscr{D}([0,T);V_p)$. For $n \geq p$ and $t \in [0,T]$ we have

$$\langle \partial_t z_n - \Delta(\mu_n z_n), w\rangle(t) = 0,$$

and after integration on $[0,T]$ and integration by parts becomes (since $z_n(0) = \mathbb{P}_n z^0$ with $V_p \subset V_n$)

$$-\int_{Q_T} z_n(\partial_t w + \mu_n \Delta w) = -\int_{\mathbf{T}^d} z^0 w,$$

which passes to the limit (since $(\mu_n)_n$ converges strongly "in duality" in front of $(z_n)_n$) to become

$$-\int_{Q_T} z(\partial_t w + \mu \Delta w) = -\int_{\mathbf{T}^d} z^0 w,$$

which holds now for any test function $w \in \mathscr{D}(Q_T)$ having a finite support for its modes. Fourier series of a smooth function converging uniformly towards it, the conclusion follows noting that z and $z\mu$ both belong (at least) to $\mathrm{L}^1(Q_T)$. □

Theorem 3
Consider $\mu \in \mathrm{L}^1(Q_T)$ such as $\inf_{Q_T} \mu > 0$ and a measurable function S such that $\mu^{-1/2} S \in \mathrm{L}^2(Q_T)$. There exists at least one element $\Phi^S \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbf{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\mathbf{T}^d))$ such that $\mu^{1/2}\Delta\Phi^S \in \mathrm{L}^2(Q_T)$, solving the dual equation $\partial_t \Phi^S + \mu\Delta\Phi^S = S$ in $\mathscr{D}'(Q_0)$ with the terminal condition 0 and satisfying the following estimate

$$\sup_{[0,T]} \left\{ \int_{\mathbf{T}^d} \Phi^S(t)^2 + \int_{\mathbf{T}^d} |\nabla \Phi^S(t)|^2 \right\} + \int_{Q_T} \mu |\Delta \Phi^S|^2 \lesssim \left(1 + \int_{Q_T} \mu\right) \int_{Q_T} \mu^{-1} S^2.$$

Proof. We consider smooth approximations $(\mu_n)_n$ and $(S_n)_n$ of μ and S, respectively in $\mathrm{L}^1(Q_T)$ and $\mathrm{L}^2(Q_T;\mathrm{d}\mu^{-1/2})$. The latter is possible thanks to standard measure theory. The Galerkine approximation consists here in the following system

$$\begin{aligned} \partial_t \Phi_n + \mu_n \Delta \Phi_n - S_n &\in \mathscr{C}^0([0,T];V_n^\perp), \\ \Phi_n(T,\cdot) &= 0. \end{aligned}$$

The *a priori* estimate (8) is here valid because $\Delta\Phi_n \in \mathscr{C}^1([0,T];V_n)$, and the rest of the proof is identical to what we have done for the proof of Theorem 2. □

Exercise 2 (hard) – *In the case when both μ and S are smooth, prove the existence of a smooth solution in $\mathscr{D}'(Q_0)$ for the dual equation, using the Sobolev (compact) embedding $\mathrm{H}^m(\mathbf{T}^d) \hookrightarrow \mathscr{C}^0(\mathbf{T}^d)$ when $m > d/2$; in a similar fashion of Exercise 1, the dual equation is then satisfied in the classical sense.*

Theorem 4
Consider $z^0 \in \mathrm{L}^2(\mathbf{T}^d)$ and $\mu \in \mathrm{L}^1(Q_T)$ such as $\inf_{Q_T} \mu > 0$. There exists at least one measurable function z such as $\sqrt{\mu} z \in \mathrm{L}^2(Q_T)$ solving the Kolmogorov equation in $\mathscr{D}'(Q_T)$ and satisfying

$$\int_{Q_T} \mu z^2 \lesssim \left(1 + \int_{Q_T} \mu\right) \int_{Q_T} z(0)^2.$$

Remark 2.5 *A careful inspection of the proof shows that $z^0 \in \mathrm{H}^{-1}(\mathbf{T}^d)$ is sufficient (changing the estimate adequately).*

Proof. Without loss of generality, we can assume that $\mu \in \mathscr{C}^\infty(\overline{Q_T})$. Indeed, one can introduce an approximation $(\mu_n)_n$ of μ in $\mathrm{L}^1(Q_T)$ such that $\mu_n \geq \inf_{Q_T} \mu$, and if z_n exists for all n with the estimate (with a constant behind $\lesssim$ independent of n)

$$\int_{Q_T} \mu_n z_n^2 \lesssim \left(1 + \int_{Q_T} \mu\right) \int_{Q_T} z(0)^2,$$

then $(z_n)_n \dot{\in} \mathrm{L}^2(Q_T)$ (because $\mu_n \geq \inf_{Q_T} \mu$) and therefore weakly converges (up to a subsequence) to some $z \in \mathrm{L}^2(Q_T)$. But $(\sqrt{\mu_n})_n \to \sqrt{\mu}$ in $\mathrm{L}^2(Q_T)$ (strong), so $(\sqrt{\mu_n} z_n)_n \rightharpoonup \sqrt{\mu} z$ in $\mathrm{L}^2(Q_T)$ (weak) because of the above estimate, and the conclusion follows.

If $\mu \in \mathscr{C}^\infty(\overline{Q_T})$, Theorem 2 gives us a solution z_n of the Kolmogorov equation in $\mathscr{D}'(Q_T)$ which is (at least) $\mathrm{L}^2(Q_T)$, and Theorem 3 produces $\Phi_n \in \mathrm{L}^\infty(0, T; \mathrm{H}^1(\mathbf{T}^d)) \cap \mathrm{L}^2(0, T; \mathrm{H}^2(\mathbf{T}^d))$ solving

$$\partial_t \Phi + \mu \Delta \Phi = \mu z,$$

with (in particular) the estimate

$$\sup_{[0,T]} \int_{\mathbf{T}^d} |\Phi(t)|^2 \lesssim \left(1 + \int_{Q_T} \mu\right) \int_{Q_T} \mu z^2.$$

The distributionnal formulation of the Kolmogorov equation is

$$\int_{Q_T} z(\partial_t \varphi + \mu \Delta \varphi) = -\int_{\mathbf{T}^d} z^0 \varphi(0),$$

and it holds for all $\varphi \in \mathscr{D}(Q_T)$ so we cannot choose $\varphi = \Phi$ directly. However, we infer from the previous

$$\forall \varphi \in \mathscr{D}(Q_T), \qquad \int_{Q_T} z(\partial_t \varphi + \mu \Delta \varphi) \leq \|z^0\|_2 \|\varphi\|_{\infty,2}.$$

Now since $z \in \mathrm{L}^2(Q_T)$ and $\mu \in \mathrm{L}^\infty(Q_T)$ (at least), we have $\partial_t \Phi \in \mathrm{L}^2(Q_T)$ and there exists $(\varphi_k)_k \in \mathscr{D}(Q_T)$ (standard properties of convolution) such that $(\partial_t \varphi_k)_k$ and $(\Delta \varphi_k)_k$ respectively approach $\partial_t \Phi$ and $\Delta \Phi$ in $\mathrm{L}^2(Q_T)$ with furthermore the bound $\|\varphi_k\|_{\infty,2} \leq \|\Phi\|_{\infty,2}$. In particular, $(\partial_t \varphi_k + \mu \Delta \varphi_k)_k \to \partial_t \Phi + \mu \Delta \Phi$ in $\mathrm{L}^2(Q_T)$, so that we get

$$\forall \varphi \in \mathscr{D}(Q_T), \qquad \int_{Q_T} z(\partial_t \varphi + \mu \Delta \varphi) \leq \|z^0\|_2 \|\Phi\|_{\infty,2},$$

and the conclusion follows using the above estimate on Φ. □

2.4 Uniqueness

The existence results of the previous section are weak in the sense that no informations are known on the solution : stability, uniqueness, maximum principle. Obviously, the *a priori* estimates seem to contain the uniqueness property : if z_1 and z_2 are solution of the same Kolmogorov equation, then $z_1 - z_2$ is a solution of the Kolmogorov equation with 0 initial data vanishes because of (6) or (7) ! The problem comes down to identify an amount of a regularity for the data (coefficients and initial value) and a class of solution for which these computations are justified, keeping in mind the limitation of the equation itself (one wants also existence !).

Expanding the Kolmogorov equation

$$\partial_t z - \operatorname{div}(\mu \nabla z) - \nabla z \cdot \nabla \mu - z \Delta \mu = 0,$$

one can look at it as a classical parabolic equation $\partial_t z - Lz = 0$ where L is some elliptic operator (here in divergence form) :

$$Lz = \sum_{i,j=1}^{d} \partial_j (a_{ij} \partial_i z) + b \cdot \nabla z + cz.$$

Under suitable assumptions on the coefficients, well-posedness is known and the following result can be seen as a particular instance of this general theory. However, for the Kolmogorov equation we will see that this result is far from optimality (at least concerning the uniqueness issue).

Theorem 5
If $\mu \in \mathrm{L}^\infty(0,T;\mathrm{W}^{2,\infty}(\mathbf{T}^d))$ and $z^0 \in \mathrm{L}^2(\mathbf{T}^d)$, there exists a unique element $z \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$ solution to the Kolmogorov equation with initial condition z_0. This solution is $\mathscr{C}^0([0,T];\mathrm{L}^2(\mathbf{T}^d))$.

Remark 2.6 *A more difficult — and complete — result can be written, using the theory developped by Le Bris and Lions in [1], which extends the DiPerna-Lions theory of transport equations to second order parabolic equations. If one only assumes that $\Delta\mu$ is bounded from above, with $\nabla\mu^{1/2} \in \mathrm{L}^2(Q_T)$ then uniqueness (and existence) holds among the class of bounded functions z such that $\mu^{1/2}\nabla z \in \mathrm{L}^2(Q_T)$.*

Before presenting the proof of Theorem 5 we need a technical lemma useful for the energy estimates.

Lemma 2
If $z \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$ and $\partial_t z \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{T}^d))$, then $z \in \mathscr{C}^0([0,T];\mathrm{L}^2(\mathbf{T}^d))$.

Proof. Consider a smooth approximation $(u_n)_n$ of u in $\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$ and such that $(\partial_t u_n)_n$ approximates $\partial_t u$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{T}^d))$. Without loss of generality, we can also assume the existence of $t_0 \in [0,T]$ such that $(u_n(t_0))_n \to u(t_0)$ in $\mathrm{L}^2(\mathbf{T}^d)$. We have by direct integration between an arbitrary t and t_0

$$\|u_n(t)-u_p(t)\|^2_{\mathrm{L}^2(\mathbf{T}^d)} \leq \|u_n(t_0)-u_p(t_0)\|^2_{\mathrm{L}^2(\mathbf{T}^d)} + 2\|\partial_t u_n - \partial_t u_p\|_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{T}^d))}\|u_n-u_p\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))},$$

which proves that $(u_n)_n$ is a Cauchy sequence in $\mathscr{C}^0([0,T];\mathrm{L}^2(\mathbf{T}^d))$. □

Proof of Theorem 5. The existence part comes from Theorem 2, and we want to establish the associated estimate. Using the previous lemma we can first check that $z \in \mathscr{C}^0([0,T];\mathrm{L}^2(\mathbf{T}^d))$. Indeed, $\mu z \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$ so that $\partial_t z \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{T}^d))$ comes from the equation itself.

Now, let's verify that $z(0)$ (which is well-defined element of $\mathrm{L}^2(\mathbf{T}^d)$) equals z_0 : if $(\rho_n)_n$ is a (time-) approximation of unity and $\psi \in \mathscr{D}(\mathbf{T}^d)$, we have by a change of variable

$$\int_{Q_T} z\rho_n \otimes \psi \to \int_{\mathbf{T}^d} z(0)\psi,$$

so that if $\eta_n(t) := \int_t^T \rho_n$, considering the test function $\varphi = \eta_n \otimes \psi$ in the distributionnal formulation, we have obtained for an arbitrary test function ψ

$$\int_{\mathbf{T}^d} z(0)\psi = \int_{\mathbf{T}^d} z_0\psi.$$

On the other hand, since we know that $\partial_t z \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{T}^d))$, we can strengthen a little bit the distributionnal formulation that we have at hand that we recall here :

$$\forall\varphi \in \mathscr{D}(Q_T), \qquad \int_{Q_T} z(\partial_t\varphi + \mu\Delta\varphi) = -\int_{\mathbf{T}^d} z_0\varphi(0).$$

First, replacing φ by $\varphi\eta$ where $\eta \in \mathscr{D}([0,T))$ we infer

$$\int_{Q_T} z\eta(\partial_t\varphi + \mu\Delta\varphi) = \int_{Q_T} z\varphi\eta' - \int_{\mathbf{T}^d} z_0\varphi(0).$$

Now if $(\eta_n) \rightharpoonup \mathbf{1}_{[0,t]}$ for some fixed $t \in (0,T)$ we have $(\eta_n') \rightharpoonup -\delta_t$ in $\mathscr{D}'(Q_T)$ and therefore for any $\psi \in \mathscr{D}(Q_T)$,

$$\int_{Q_T} \psi\varphi\eta' \underset{n\to+\infty}{\longrightarrow} \int_{\mathbf{T}^d} \psi(t)\varphi(t).$$

Using an approximation in $\mathscr{C}^0([0,T];\mathrm{L}^2(\mathbf{T}^d))$ one can replace ψ by z in the previous convergence and we therefore established the following extended formulation

$$\forall\varphi \in \mathscr{D}(Q_T), \qquad \int_0^t\int_{\mathbf{T}^d} z\,\partial_t\varphi - \int_0^t\int_{\mathbf{T}^d} \nabla(\mu z)\cdot\nabla\varphi = \int_{\mathbf{T}^d} z(t)\varphi(t) - \int_{\mathbf{T}^d} z_0\varphi(0),$$

where we used the belonging $\mu z \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$. This new formulation can be extended by density and continuity to all test function $\varphi \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d)) \cap \mathscr{C}^0([0,T];\mathrm{L}^2(\mathbf{T}^d))$ for which $\partial_t \varphi \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbf{T}^d))$; hence z is now an admissible test function and satisfies

$$\int_0^t \int_{\mathbf{T}^d} z\,\partial_t z - \int_0^t \int_{\mathbf{T}^d} \nabla(\mu z)\cdot\nabla z = \int_{\mathbf{T}^d} z(t)^2 - \int_{\mathbf{T}^d} z_0^2.$$

The identity

$$\int_0^t \int_{\mathbf{T}^d} \partial_t\varphi\,\varphi = \frac{1}{2}\int_{\mathbf{T}^d} \varphi(t)^2 - \frac{1}{2}\int_{\mathbf{T}^d}\varphi(0)^2,$$

that holds for any $\varphi \in \mathscr{C}^\infty(\overline{Q_T})$ extends by density and continuity of the above bilinear form to z, so that we have

$$\frac{1}{2}\int_{\mathbf{T}^d} z(t)^2 + \int_0^t \int_{\mathbf{T}^d} \nabla(\mu z)\cdot\nabla z = \frac{1}{2}\int_{\mathbf{T}^d} z(0)^2.$$

If all the functions were smooth we would write

$$-\int_0^t \int_{\mathbf{T}^d} \nabla(\mu z)\cdot\nabla z = -\int_0^t \int_{\mathbf{T}^d} \mu|\nabla z|^2 + \frac{1}{2}\int_0^t \int_{\mathbf{T}^d} z^2\Delta\mu \leq \frac{1}{2}\int_0^t z^2 \|\Delta\mu\|_{\mathrm{L}^\infty(Q_T)},$$

and as usual, we can justify this inequality between the two extreme sides of the equation by density. We have therefore established for any solution z of the Kolmogorov equation that belongs to $\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbf{T}^d))$, the inequality

$$\gamma(t) \leq \gamma(0) + \|\Delta\mu\|_{\mathrm{L}^\infty(Q_T)} \int_0^t \gamma(s)\,\mathrm{d}s,$$

where γ is the continuous function $t \mapsto \|z(t)\|^2_{\mathrm{L}^2(\mathbf{T}^d)}$, and uniqueness follows from the linearity of the equation. □

For the dual equation, the uniqueness issue is a bit surprising. On the one hand, the estimate of Theorem 3 seems to draw a functional framework in which uniqueness is credible : $\Phi \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbf{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\mathbf{T}^d)) \cap \mathscr{C}^0([0,T];\mathrm{L}^2(\mathbf{T}^d))$ with $\mu^{1/2}\Delta\Phi \in \mathrm{L}^2(Q_T)$ and $\mu \in \mathrm{L}^1(Q_T)$. On the other, when one tries to justify completely the estimate the following difficulty appears. Even though both sides of the inequality

$$\frac{1}{2}\sup_{t\in[0,T]} \int_{\mathbf{T}^d} |\nabla\Phi(t)|^2 \leq \int_0^T \int_{\mathbf{T}^d} \partial_t\Phi\,\Delta\Phi$$

make sense in the above functional spaces, this formula — which satisfied for regular functions — *cannot be justified by density.* The subtility is that smooth functions are not always dense in weighted Sobolev spaces, unless some specific conditions are required on the weight. A somehow straightforward case is when μ is bounded : in that case both $\partial_t\Phi$ and $\Delta\Phi$ belong to $\mathrm{L}^2(Q_T)$ and can be smoothly approximated in this space. It is this uniqueness property for bounded density that justifies the framework of the following uniqueness theorem. It is stronger than Theorem 5 in the sense that it demands very few regularity on z, however the diffusivity is assumed bounded.

Theorem 6
If $z_0 \in \mathrm{L}^2(\mathbf{T}^d)$ and $\mu \in \mathrm{L}^\infty(Q_T)$, there exists a unique solution $z \in \mathrm{L}^2(Q_T)$ of the Kolmogorov equation with initial data z_0.

Proof. First, let's notice that the dual equation $\partial_t\Phi^S + \mu\Delta\Phi^S = S$ (with 0 as terminal condition) is well-posed in $\mathrm{L}^\infty(0,T;\mathrm{H}^1(\mathbf{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(\mathbf{T}^d))$: since $\mu \in \mathrm{L}^\infty(Q_T)$, any solution Φ in this class satisfies $\partial_t\Phi \in \mathrm{L}^2(Q_T)$ which is sufficient to prove the *a priori* estimate of Theorem 3. Now, one can consider the following linear form on $\mathrm{L}^2(Q_T)$

$$\Psi : S \mapsto \int_{\mathbf{T}^d} -z_0\Phi^S(0).$$

Owing to the estimate of Theorem 3, Ψ is continuous and therefore represented by a unique $z \in \mathrm{L}^2(Q_T)$. □

Exercise 3 – *Prove that maximum principle : if z_0 is nonnegative, then so is the solution given by Theorem 6. Prove a stronger version of that theorem for the solution given by Theorem 5 : if the initial condition is bounded, then so is the solution (and give the bounds).*

→ *Click here for a (bit tedious) correction of this exercise.*

This framework of well-posedness allows to prove some strong sequential stability for the Kolmogorov equation, namely : if a sequence of bounded functions is such that $(\mu_n)_n \ddot{\in} \mathrm{L}^1(Q_T)$ with a bounded cluster point, then $(z_n)_n \ddot{\in} \mathrm{L}^2(Q_T)$. See [4] for more details.

3 Global weak solution of a cross-diffusion system

We turn now to the proof of Theorem 1. The notion of weake solution that we consider is the same as the one given in Definition 2.1 (for each equation). With regard to a single Kolmogorov equation, the main difficulty comes from the nonlinearity of the system (and that holds also in the case $(\gamma_1, \gamma_2) = (1,1)$ of SKT). It is not possible anymore to simply rely on the weak$(-\star)$ compactness of a sequence of approximating solutions. But that's not the only problem ! The approximation procedure itself is problematic. First of all, the two equations cannot be approximated separately ; secondly, it will be crucial to keep the nonnegativeness of the solutions during this approximation procedure.

3.1 The entropy of Chen and Jüngel

The first result of global existence for the system above when $(\gamma_1, \gamma_2) = (1,1)$ relies on a functionnal known as the *entropy of Chen and Jüngel* used in [2] and defined by

$$\mathrm{E}(u,v)(t) := \frac{1}{a_{12}}\int_{\mathbf{T}^d} \psi(u) + \frac{1}{a_{21}}\int_{\mathbf{T}^d} \psi(v),$$

where $\psi' = \ln$. One can indeed verify that any solution of the system aboves satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(u,v)(t) + \int_{\mathbf{T}^d}\left(\frac{d_1}{a_{12}}|\nabla\sqrt{u}|^2 + \frac{d_2}{a_{21}}|\nabla\sqrt{v}|^2 + 4|\nabla\sqrt{uv}|^2\right) = 0.$$

For the study of the SKT system, this is a valuable piece of information ; it opens the road to an approximation-compactness procedure, since the control of gradient could help to get rid of the oscillations.

3.2 A general method to produce Lyapunov functionals

Consider a system of I equations (possibly with $I > 2$)

$$\partial_t U - \Delta[A(U)] = 0,$$

where $U : Q_T \to \mathbf{R}^I$, $A : \mathbf{R}^I \to \mathbf{R}^I$. If one hopes to find a Lyapunov functional of the following form

$$\int_{\mathbf{T}^d} \Phi(U), \tag{12}$$

where $\Phi : \mathbf{R}^I \to \mathbf{R}_{\geq 0}$, it is natural to dot multiply the equation by $\nabla\Phi(U)$ and integrate on $\mathbf{T}^d$ so that the evolution term produces precisely the evolution of the above functional. More precisely we infer, using the repeated index convention,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{T}^d} \Phi(U) + \int_{\mathbf{T}^d} (\partial_k\Phi)(U)\partial_{jj}\big[A_k(U)\big] = 0,$$

where A_k stands for the k-th component of A. We have by integration by parts

$$\int_{\mathbf{T}^d} (\partial_k\Phi)(U)\partial_{jj}(A_k(U)) = \int_{\mathbf{T}^d} \partial_j\big[(\partial_k\Phi)(U)\big]\partial_j\big[A_k(U)\big],$$

applying the chain rule we infer (U_i is the i-th component of U)

$$\int_{\mathbf{T}^d} (\partial_k \Phi)(U)\partial_{jj}(A_k(U)) = \int_{\mathbf{T}^d} \partial_i\partial_k\Phi(U)\partial_j U_i \partial_\ell A_k \partial_j U_\ell$$
$$= \int_{\mathbf{T}^d} \langle \mathrm{D}^2(\Phi)(U)\partial_j U, \mathrm{D}(A)(U)\partial_j U\rangle.$$

Owing to the symmetry of the hessian matrix, and with a small abuse of notation we have therefore established

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{T}^d}\Phi(U) + \int_{\mathbf{T}^d}\langle \nabla U, \mathrm{D}^2(\Phi)(U)\mathrm{D}(A)(U)\nabla U\rangle = 0. \tag{13}$$

This identity tells us that, as soon as the symmetric part of $\mathrm{D}^2(\Phi)(U)\mathrm{D}(A)(U)$ is positive, we obtain a Lypanuov functional.

If we apply this analysis on the system we are interested in, we have

$$A(U) = A\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix}(d_1 + a_{12}v^{\gamma_1})u \\ (d_2 + a_{21}u^{\gamma_2})v\end{pmatrix},$$

so that

$$\mathrm{D}(A)(U) := \begin{pmatrix} d_1 + a_{12}v^{\gamma_1} & a_{12}\gamma_1 v^{\gamma_1 - 1}u \\ \gamma_2 a_{21}u^{\gamma_2-1}v & d_2 + a_{21}u^{\gamma_2}\end{pmatrix}. \tag{14}$$

For nonnegative u, v (which is expected !) we have $\mathrm{Tr}\,\mathrm{D}(A)(U) \geq 0$ and

$$\det \mathrm{D}(A)(U) = (d_1 + a_{12}v^{\gamma_1})(d_2 + a_{21}u^{\gamma_2}) - a_{12}a_{21}\gamma_1\gamma_2 v^{\gamma_1}u^{\gamma_2}, \tag{15}$$

which is nonnegative as soon as $\gamma_2\gamma_1 \leq 1$. This lack of symmetry of $\mathrm{D}(A_{\gamma_1,\gamma_2})(U)$ can be corrected using the left action of a diagonal matrix :

$$\begin{pmatrix} a_{21}\gamma_2 u^{\gamma_2 - 2} & 0 \\ 0 & a_{12}\gamma_1 v^{\gamma_1 - 2}\end{pmatrix}\mathrm{D}(A)(U) = a_{12}a_{21}\gamma_1\gamma_2 v^{\gamma_1-1}u^{\gamma_2-1}\begin{pmatrix}\star & 1 \\ 1 & \star\end{pmatrix}.$$

Now, since the above diagonal matrix has nonnegative coefficients, the trace of the resulting matrix remains nonnegative, and so is its determinant by straightforward multiplication ; the resulting matrix is therefore symmetric and nonnegative. The identity (13) informs us that, if Φ has an hessian matrix precisely given by this diagonal matrix the expression given in (12) is a Lyapunov functional.

In the particular case $\gamma_2 = \gamma_1 = 1$ (standard SKT), the integration of the hessian matrix allows to recover the entropy of Chen and Jüngel.

The identity (13) suggests $\Phi(U) \in \mathrm{L}^\infty(0, T; \mathrm{L}^1(\mathbf{T}^N))$ which is already a strong information, but we recover in fact a lot more : there is enough dissipation so as to control a part of the (spatial) gradients of the system ; this plays a crucial role in the construction of weak solutions. As a matter of fact we have the following estimate

Proposition 3
Assume that (u, v) are two smooth solutions of the system (2) – (3). For a convex nonnegative function Φ_{γ_1,γ_2} there holds

$$\int_{\mathbf{T}^d}\Phi_{\gamma_1,\gamma_2}(u,v)(t) + \int_0^t\left\{\int_{\mathbf{T}^d}|\nabla u^{\gamma_2/2}|^2 + \int_{\mathbf{T}^d}|\nabla v^{\gamma_1/2}|^2\right\} \lesssim \int_{\mathbf{T}^d}\Phi_{\gamma_1,\gamma_2}(u,v)(0),$$

Remark 3.1 *With (very) little effort it can be established that $\nabla(u^{\gamma_2/2}v^{\gamma_1/2})$ is also controlled in $\mathrm{L}^2(Q_T)$.*

Proof. We start from (13). The correspondance with the dynamic terms is straightforward, so we focus on the dissipation. We start by writing

$$\mathrm{D}(A)(U) = \begin{pmatrix} d_1 & 0 \\ 0 & d_2\end{pmatrix} + M(U),$$

and observe that

$$\int_{\mathbf{T}^d} \left\langle \nabla U, \mathrm{D}^2(\Phi)(U) \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \nabla U \right\rangle = \int_{\mathbf{T}^d} d_1 a_{21} \gamma_2 u^{\gamma_2 - 2} |\nabla u|^2 + \int_{\mathbf{T}^d} d_2 a_{12} \gamma_1 v^{\gamma_1 - 2} |\nabla v|^2,$$

where we recognize, up to irrelevant constants, $|\nabla u^{\gamma_2/2}|^2$ and $|\nabla v^{\gamma_1/2}|^2$. Next, we have

$$\begin{aligned} \mathrm{D}^2(\Phi)(U) M(U) &= a_{21} a_{12} \begin{pmatrix} \gamma_2 u^{\gamma_2 - 2} v^{\gamma_1} & \gamma_1 \gamma_2 v^{\gamma_1 - 1} u^{\gamma_2 - 1} \\ \gamma_1 \gamma_2 v^{\gamma_1 - 1} u^{\gamma_2 - 1} & \gamma_1 v^{\gamma_1 - 2} u^{\gamma_2} \end{pmatrix} \\ &= a_{21} a_{12} u^{\gamma_2} v^{\gamma_1} \begin{pmatrix} \gamma_2 u^{-2} & \gamma_1 \gamma_2 u^{-1} v^{-1} \\ \gamma_1 \gamma_2 u^{-1} v^{-1} & \gamma_1 v^{-2} \end{pmatrix} \\ &= a_{21} a_{12} u^{\gamma_2} v^{\gamma_1} \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix} \begin{pmatrix} \gamma_2 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & \gamma_1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix}. \end{aligned}$$

The assumption $\gamma_1 \gamma_2 \leq 1$ allows to check that the above quadratic form is nonnegative so that the estimate is established. □

3.3 Approximation scheme

One of the main difficulties of cross-diffusion systems is that the previous estimates need the nonnegativeness of the solution. If one wants to use this estimates to ensure the convergence (up to a subsequence) of the approximation procedure, the latter needs to produce only nonnegative functions. This forbids the use of a Galerkine method for instance. We propose here an approximation procedure which is different from the one followed by Chen and Jüngel. It applies to a wider set of systems. In order to do so we first need a simple lemma concerning the map A that we used before ; let's recall its definition: $A(U) = \begin{pmatrix} (d_1 + a_{12} v^{\gamma_1}) u \\ (d_2 + a_{21} u^{\gamma_2}) v \end{pmatrix}$, where $U = \begin{pmatrix} u \\ v \end{pmatrix}$.

Lemma 3
A is an homeomorphism from $\mathbf{R}^2_{>0}$ to itself.

Proof. Recall the Hadamard-Lévy Theorem : a $\mathscr{C}^1$ map $\Theta : \mathbf{R}^2 \to \mathbf{R}^2$ is a diffeomorphism if and only if it is proper and has no critical points. That's the case of $\Theta = \ln \circ A \circ \exp$, where the logarithm and exponential have to be understood component-wise. Indeed, A maps $\mathbf{R}^2_{>0}$ to itself (and is $\mathscr{C}^1$ on that set) so that $\Theta : \mathbf{R}^2 \to \mathbf{R}^2$ is well defined and of class $\mathscr{C}^1$. We have already established that $\det \mathrm{D}(A)$ is positive on $\mathbf{R}^2_{>0}$, and both ln and exp do not have critical points on their domain. As for the properness of Θ, if $(X_n)_n$ is unbounded then one component of $(\exp(X_n))_n$ goes to 0 or $+\infty$, and the same for $(A \circ \exp(X_n))_n$ which implies that $(\Theta(X_n))_n$ is unbounded. □

We will also need the following result of elliptic regularity:

Lemma 4
For $p \in (1, \infty)$ and $\varepsilon > 0$, for any $g \in \mathrm{L}^p(\mathbf{T}^d)$ there exists a unique $f \in \mathrm{W}^{2,p}(\mathbf{T}^d)$ solving $f - \varepsilon \Delta f = g$, and it satisfies furthermore $\|f\|_{\mathrm{W}^{2,p}(\mathbf{T}^d)} \lesssim \|g\|_p$.

The proof of Lemma 4 needs some tools of harmonic analysis and we will skip it ; on the torus, it is however less involved that the general case of a bounded open set. The idea is to translate the equation onto the Fourier coefficients so that it only remains to check that $k_i k_j / (1 + \varepsilon |k|^2)$ is an $\mathrm{L}^p(\mathbf{T}^d)$ multiplier (see [6] for instance). As usual when Fourier analysis is involed, the hilbertian case is a lot more practicable :

Exercise 4 – *Prove Lemma 4 in the case $p = 2$ and give two proofs of the following estimate $\|f\|_\infty \lesssim \|g\|_\infty$, when g is bounded.*

We now turn to the main result of this subsection.

Proposition 4
For any integer $N \geq 1$, and any $U^0 \in \mathscr{C}^0(\mathbf{T}^d; \mathbf{R}^2_{>0})$, there exists a family $(U^k)_{1 \leq k \leq N}$ of $\mathscr{C}^0(\mathbf{T}^d; \mathbf{R}^2_{>0})$

such that $A(U^k) \in \mathrm{W}^{2,q}(\mathbf{T}^d)$ for all $q \in [1,\infty)$ and all k and for which the following identity holds for $1 \le k \le N$

$$\frac{U^k - U^{k-1}}{\tau} - \Delta[A(U^k)] = 0,$$

where $\tau := T/N$.

Proof. By induction, the issue reduces to prove that for a given S in $\mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2_{>0})$ there exists U in the same set solving

$$U - \tau\Delta[A(U)] = S. \tag{16}$$

It is totally equivalent to find U such that, for $\widetilde{U} := A(U)$ we have

$$\widetilde{U} - \tau\Delta\widetilde{U} = A(U) - U + S.$$

This simple remark suggests a fixed-point procedure which, starting from U would produce $A(U)-U+S$, then $\widetilde{U}$ using the well-behaved operator $(\mathrm{Id} - \tau\Delta)^{-1}$ and finally $A^{-1}(\widetilde{U})$. However, to perform this last step one most ensure that $\widetilde{U}$ takes its values in $\mathbf{R}^2_{>0}$ (to invoke Lemma 3). The maximum principle precisely informs that $(\mathrm{Id} - \tau\Delta)^{-1}$ is a positive operator, but the source $A(U) - U + S$ is not necessarily positive. This can be solved noticing that for any $U \in \mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2)$, $\alpha A(|U|) - |U|$ takes its values in $\mathbf{R}^2_{\ge 0}$, where $\alpha^{-1} = \min(d_1, d_2)$ and the absolute value $|\cdot|$ is applied component-wise.

At the end of the day we get the following fixed-point procedure, defining (for a given $S \in \mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2_{>0})$)

$$\begin{aligned}
\Psi_1 : \mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2) &\longrightarrow \mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2_{>0})\\
U &\longmapsto \alpha A(|U|) - |U| + S\\
\Psi_2 : \mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2_{>0}) &\longrightarrow \mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2_{>0})\\
U &\longmapsto (\alpha\,\mathrm{Id} - \tau\Delta)^{-1}U.
\end{aligned}$$

For any $U \in \mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2_{>0})$, solving (16) is equivalent to fix the map $\Theta : A^{-1} \circ \Psi_2 \circ \Psi_1$. Θ sends the vector space $\mathscr{C}^0(\mathbf{T}^2;\mathbf{R}^2)$ on $\mathscr{C}^0(\mathbf{T}^d;\mathbf{R}^2_{>0})$, so we will invoke the Schaeffer fixed-point Theorem on the former space with the parametrized family of maps $\Lambda(\sigma,\cdot) := A^{-1} \circ \Psi_2 \circ \sigma\Psi_1$, where we extended the value of A^{-1} at $(0,0)$ by $(0,0)$ so that $\Lambda(0,\cdot) = 0$ is obvious. Note that A^{-1} is continuous at $(0,0)$: by properness of A if $(X_n)_n \to (0,0)$ then $(A^{-1}(X_n))_n$ is bounded, and it has $(0,0)$ as only possible limit point. The continuity and compactness of Λ come from Lemma 4 and the Sobolev (compact) embedding $W^{2,p}(\mathbf{T}^d) \hookrightarrow \mathscr{C}^0(\mathbf{T}^d)$ for p large enough. It therefore remains to ensure that all possible fixed points of $\Lambda(\sigma,\cdot)$ are contained in some fixed ball for the $\mathrm{L}^\infty(\mathbf{T}^d)$ norm, and we can assume $\sigma > 0$ for this task. If U is such a fixed point, this means that (note that U takes positive values)

$$\alpha A(U) - \tau\Delta A(U) = \alpha\sigma A(U) - \sigma U + \sigma S.$$

We introduce the linear form $\ell : \mathbf{R}^2 \to \mathbf{R}$ which maps (u,v) to $u+v$. We have

$$\alpha\ell(A(U)) - \tau\Delta\ell(A(U)) = \alpha\sigma\ell(A(U)) - \sigma\ell(U) + \sigma\ell(S).$$

This identity is satisfied a.e. and we can multiply it by the continuous function $\ell(A(U))$ to infer after appropriate (and justified) integration by parts

$$\int_{\mathbf{T}^d} \ell(A(U))\ell(U) \le \int_{\mathbf{T}^d} \ell(A(U))\ell(S). \tag{17}$$

Remark that all the integrands are nonnegative. We claim that for any U solving the previous equation,

$$\int_{\mathbf{T}^d} \ell(A(U)) \lesssim 1, \tag{18}$$

where the constant behind $\lesssim$ depends only on A and S. First, we have

$$\int_{\mathbf{T}^d} \ell(A(U))\ell(U) \ge \int_{\mathbf{T}^d} \ell(A(U))\ell(U)\mathbf{1}_{\ell(U)>R} \ge R\int_{\mathbf{T}^d} \ell(A(U))\mathbf{1}_{\ell(U)>R}.$$

We infer therefore from (17) that

$$\int_{\mathbf{T}^d}\ell(A(U)) = \int_{\mathbf{T}^d}\ell(A(U))\mathbf{1}_{\ell(U)\le R} + \int_{\mathbf{T}^d}\ell(A(U))\mathbf{1}_{\ell(U)>R}$$
$$\le \int_{\mathbf{T}^d}\ell(A(U))\mathbf{1}_{\ell(U)\le R} + \frac{2\|S\|_\infty}{R}\int_{\mathbf{T}^d}\ell(A(U)),$$

and for $R > 4\|S\|_\infty$ this implies

$$\int_{\mathbf{T}^d}\ell(A(U)) \le 2\int_{\mathbf{T}^d}\ell(A(U))\mathbf{1}_{\ell(U)\le R}.$$

But if $\ell(U)\le R$ this implies $\|U\|_\infty \le R$ and therefore (continuity of A) we have $\|A(U)\mathbf{1}_{\ell(U)\le R}\|_\infty \lesssim 1$, so that (18) is established. Now that we have a uniform $\mathrm{L}^1(\mathbf{T}^d)$ estimate on $A(U) := \widetilde{U}$ we go back to the elliptic equation satisfied by this $W^{2,p}(\mathbf{T}^d)$ function, which implies, using the positivity of the right hand side and U, in particular for $\varepsilon := \tau/\alpha$ and any $p\in(1,\infty)$

$$\|(\mathrm{Id}-\varepsilon\Delta)\widetilde{U}\|_p \le \|\sigma A(U) + \alpha^{-1}\sigma S\|_p.$$

Thanks to Lemma 4 and using the Sobolev embedding $\mathrm{W}^{2,p}(\mathbf{T}^d)\hookrightarrow \mathscr{C}^0(\mathbf{T}^d)$ for p large enough, we have therefore

$$\|\widetilde{U}\|_\infty \lesssim \sigma\|A(U)\|_\infty^{1/p'}\|A(U)\|_1^{1/p} + \alpha^{-1}\|S\|_\infty,$$

which, thanks to the Young's inequality, implies

$$\left(1-\frac{\sigma}{2}\right)\|\widetilde{U}\|_\infty \lesssim \|A(U)\|_1 + \alpha^{-1}\|S\|_\infty,$$

and we obtain therefore a uniform bound for $A(U) := \widetilde{U}$ (since $\sigma\in[0,1]$) which is directly transfered to U because of the nonnegativeness of this function and the positivity of d_1, d_2. □

Define, for $N\ge 1$ and the corresponding family $(U^k)_{1\le k\le N}$

$$\underline{U}(t,x) := \sum_{k=0}^{N-1} U^k(x)\mathbf{1}_{(k\tau,(k+1)\tau)}(t), \tag{19}$$

and by $\underline{u},\underline{v}$ the components of $\underline{U}$. Following carefully the computations done in the proof of Proposition 3, we can establish the following estimate for the same convex nonnegative function Φ_{γ_1,γ_2}

$$\max_{[0,T]}\int_{\mathbf{T}^2}\Phi_{\gamma_1,\gamma_2}(\underline{U}^N)(t) + \int_0^T\int_{\mathbf{T}^d}|\nabla \underline{u}^{\gamma_2/2}|^2\int_{\mathbf{T}^d}|\nabla\underline{v}^{\gamma_1/2}|^2 \lesssim \int_{\mathbf{T}^2}\Phi_{\gamma_1,\gamma_2}(U^0). \tag{20}$$

3.4 Oscillations

The Aubin-Lions Lemma is a tool frequently used in the study of nonlinear evolution PDE to recover compactness from a sequence of approximating solutions. It generally states (the operator ∇ acts only the space variable) something like

$$\text{“}\quad (\partial_t u_n)_n \mathrel{\dot\in} \mathrm{L}^p(0,T;\mathrm{H}^{-m}(\mathbf{T}^d)) + (\nabla u_n)_n \mathrel{\dot\in} \mathrm{L}^q(0,T;\mathrm{L}^\alpha(\mathbf{T}^d)) \Rightarrow (u_n)_n \mathrel{\ddot\in} \mathrm{L}^q(0,T;\mathrm{L}^\beta(\mathbf{T}^d))\text{”},$$

for appropriate exponents α,β and arbitrary integer $m\in\mathbf{N}$. Unfortunately the estimate that we have at hand on the sequence U_n does not control directly the gradient of U_n but only a function (power law) of it.

We propose here a modified Aubin-Lions lemma which applies when only $\nabla\Theta(U_n)$ is controlled, under appropriate conditions for the nonlinear function Θ. Note that it's hopeless to get a compactness result without specifying something for Θ: in the extreme case where this function is constant, $(\nabla\Theta(u_n))_n$ is always a bounded sequence.

Lemma 5
Consider a Lipschitz function Θ *such that* $\{\Theta' = 0\}$ *has* 0 *measure and a sequence* $(u_n)_n \dot{\in} \mathrm{L}^2(Q_T)$ *having Sobolev regularity in the space variable and satisfying* $(\nabla\Theta(u_n))_n \dot{\in} \mathrm{L}^2(Q_T)$ *and furthermore* $(\partial_t u_n)_n \dot{\in} \mathscr{M}^1(0,T;\mathrm{H}^{-m}(\mathbf{T}^d))$ *for some arbitrary integer* $m \in \mathbf{N}$*. Then one has* $(u_n)_n \ddot{\in} \mathrm{L}^2(Q_T)$.

Remark 3.2 *It is possible to lower the regularity assumption of* Θ *and* u_n*, but we focused on a simple version. However, we pay a little effort to include boundedness in the measure sense, so that we can apply this result to our semi-discrete scheme.*

We first have to establish the following general lemma that allows to pass into the limit for a product of two weakly converging sequences.

Lemma 6
Consider two sequences $(a_n)_n$ *and* $(b_n)_n$ *of* $\mathrm{L}^2(Q_T)$ *respectively weakly converging to* a *and* b *in that space. and such that, furthermore,* $(\nabla a_n)_n \dot{\in} \mathrm{L}^2(Q_T)$ *and* $(\partial_t b_n)_n \dot{\in} \mathscr{M}^1(0,T;\mathrm{H}^{-m}(\mathbf{T}^d))$ *for some arbitrary integer* $m \in \mathbf{N}$*. Then*

$$\int_{Q_T} a_n b_n \underset{n\to+\infty}{\longrightarrow} \int_{Q_T} ab.$$

Proof. Without loss of generality, we can assume that $a = b = 0$. Consider a sequence $(\varphi_k)_k$ of even mollifiers on the torus $\mathbf{T}^d$ and denote by $\star$ the convolution acting only on the space variable. Owing to the classical estimate $\|\tau_h f - f\|_2 \leq |h|\,\|\nabla f\|_2$, we infer the convergence $\sup_{n\in\mathbf{N}} \|a_n - a_n \star \varphi_k\|_2 \to_k 0$. On the other hand, since φ_k is even, we have

$$\int_{Q_T} (a_n \star \varphi_k) b_n = \int_{Q_T} a_n (b_n \star \varphi_k).$$

We have therefore, by Cauchy-Schwarz's inequality,

$$\left|\int_{Q_T} a_n b_n\right| \leq \left|\int_{Q_T} a_n (b_n \star \varphi_k)\right| + \sup_{n\in\mathbf{N}} \|b_n\|_2 \|a_n - a_n \star \varphi_k\|_2.$$

By weak convergence we have that $(b_n)_n$ is bounded in $\mathrm{L}^2(Q_T)$, so for k large enough the second term is as small as we want. When k is fixed, we have $(b_n \star \varphi_k)_n \dot{\in} \mathrm{BV}(Q_T)$, so we have a.e. convergence (up to a subsequence) and even (strong) convergence in $\mathrm{L}^2(Q_T)$ because the sequence is uniformly bounded. The conclusion follows. □

Proof of Lemma 5. Let's define for $\varepsilon > 0$

$$\beta_\varepsilon(z) = \int_0^z \min\left\{1, \frac{|\Theta'(r)|}{\varepsilon}\right\} \mathrm{d}r.$$

We have

$$\beta_\varepsilon(z) - z = \int_0^z \left(\frac{|\Theta'(r)|}{\varepsilon} - 1\right) \mathbf{1}_{|\Theta'(r)|<\varepsilon} \,\mathrm{d}r,$$

so that $\|\beta_\varepsilon(z) - z\|_\infty \leq |\{\Theta' < \varepsilon\}|$, which converges to the measure of $\{\Theta' = 0\}$ as $\varepsilon \to 0$. Since u_n has Sobolev (in space) regularity and Θ is Lipschitz, we have

$$\nabla\beta_\varepsilon(u_n) = \frac{\beta_\varepsilon'(u_n)}{\Theta'(u_n)} \nabla\Theta(u_n),$$

with β_ε precisely erasing the critical points of Θ in the sense that $\beta_\varepsilon'/\Theta'$ is uniformly bounded $1/\varepsilon$. From $(u_n)_n \dot{\in} \mathrm{L}^2(Q_T)$ we infer, for any fixed ε that $\beta_\varepsilon(u_n) \dot{\in} \mathrm{L}^2(Q_T)$. Up to subsequences we have therefore $(u_n)_n \rightharpoonup u$ and $(\beta_\varepsilon(u_n))_n \rightharpoonup u_\varepsilon$ in $\mathrm{L}^2(Q_T)$. Thanks to Lemma 6 we have

$$\int_{Q_T} u_n \beta_\varepsilon(u_n) \underset{n\to+\infty}{\longrightarrow} \int_{Q_T} u\, u_\varepsilon.$$

On the other hand, we have

$$\sup_{k\in\mathbf{N}} \|u_k - \beta_\varepsilon(u_k)\|_{\mathrm{L}^2(Q_T)} \lesssim R_\varepsilon := \|\mathrm{Id} - \beta_\varepsilon\|_\infty \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

So by weak lower semi-continuity

$$\|u - u_\varepsilon\|_{\mathrm{L}^2(Q_T)} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Now we write

$$\|u_n\|^2_{\mathrm{L}^2(Q_T)} - \|u\|^2_{\mathrm{L}^2(Q_T)} = \int_{Q_T} u(u-u_\varepsilon) + \left\{\int_{Q_T} u\, u_\varepsilon - \int_{Q_T} u_n \beta_\varepsilon(u_n)\right\} + \int_{Q_T} u_n(u_n - \beta_\varepsilon(u_n)),$$

from which we infer by Cauchy-Scwharz's inequality

$$\left|\|u_n\|^2_{\mathrm{L}^2(Q_T)} - \|u\|^2_{\mathrm{L}^2(Q_T)}\right| \leq \|u\|_{\mathrm{L}^2(Q_T)}\|u-u_\varepsilon\|_{\mathrm{L}^2(Q_T)} + \left|\int_{Q_T} u\, u_\varepsilon - \int_{Q_T} u_n\beta_\varepsilon(u_n)\right| + R_\varepsilon \sup_{k\in\mathbf{N}} \|u_k\|_{\mathrm{L}^2(Q_T)},$$

so that picking first ε small enough and letting after $n \to +\infty$ we recover the strong convergence of $(u_n)_n$ in $\mathrm{L}^2(Q_T)$. □.

3.5 Concentration

With Lemma 5 at hand and estimate (20), we are able to recover strong convergence for our sequence of approximated solutions. The last problem that could appear is the concentration issue. This is especially true for the term uv^{γ_1} of the system if for instance $\gamma_1 \gg 1$ (which is possible if $\gamma_2 \ll 1$). For this severe nonlinearity, we have a.e. convergence but nothing protects us from concentration to a Dirac mass. A way out is given by the duality lemma, that we presented in Subsection 2.2. In the context of a system of equations that writes

$$\partial_t U - \Delta[A(U)] = 0$$

where $A_i(U) = \mu_i(U)u_i$ (which is the case for the SKT system and its variant), summing up all the lines we obtain a single Kolmogorov equation

$$\partial_t z - \Delta(\mu z) = 0,$$

where

$$z = u_1 + \cdots + u_I,$$
$$\mu = \frac{\sum_{i=1}^I \mu_i(U)u_i}{\sum_{i=1}^I u_i}.$$

If for all i we have $\mu_i \geq \alpha$, then we also have $\mu \geq \alpha$, as long as the entries u_i are all nonnegative. With this simple remark and the previous duality estimate, we obtain the following *a priori* one

$$\int_{Q_T} U \otimes A(U) \lesssim \left(1 + \int_{Q_T} \mu\right) \|U_0\|^2_{\mathrm{L}^2(\mathbf{T}^d)}.$$

This is enough to handle all the possible concentrations of the system. There is a remaining technicality : the proof of the duality lemma has to be discretized, which is the object of the last exercise of these lecture notes.

Exercise 5 – *Consider a family* $(\mu_k)_{0\leq k\leq N}$ *of* $\mathrm{L}^1(\mathbf{T}^d)$ *such that* $\inf_k \inf_{\mathbf{T}^d} \mu_k > 0$. *Consider also a family* $(z_k)_{0\leq k\leq N-1}$ *of* $\mathrm{L}^\infty(\mathbf{T}^d)$ *of nonnegative functions such that* $\mu_k z_k \in \mathrm{H}^2(\mathbf{T}^d)$ *for all* k, *with the following equations satisfied a.e. for some* $\tau > 0$

$$\frac{z_k - z_{k-1}}{\tau} - \Delta(\mu_k z_k) = 0.$$

Using the same notation as in (19), *prove that*

$$\int_{Q_T} \underline{\mu}\,\underline{z}^2 \lesssim \left(1 + \int_{Q_T} \underline{\mu}\right) \int_{\mathbf{T}^d} \underline{z_0}^2.$$

Correction of Exercise 3 – For the first maximum principle, we start by proving that for $S \geq 0$ the unique solution Φ^S of the dual equation is nonpositive. The function $\Psi(t,x) := -\Phi^S(T-t,x)$, satisfies $\partial_t \Psi - \mu \Delta \Psi \geq 0$ with 0 as initial condition and our goal is now to prove $\Psi \geq 0$. Letting $\Psi^\varepsilon := \Psi + \varepsilon t$, we have $\partial_t \Psi^\varepsilon - \mu \Delta \Psi^\varepsilon \geq \varepsilon$ with same initial data. Note that due to the functional setting that we have, the previous inequality makes sense in a.e. since $\partial_t \Psi^\varepsilon$ and $\Delta \Psi^\varepsilon$ both belong to $\mathrm{L}^2(Q_T)$.

If Ψ^ε was smooth, then we could argue like this. If $t^\star := \sup\{t \in [0,T] : \min_{\mathbf{T}^d} \Psi^\varepsilon(t) \geq 0\} < T$, since $\Psi^\varepsilon(0,\cdot) = 0$ we would have $\min_{\mathbf{T}^d} \Psi^\varepsilon(t^\star) = 0$. If $x^\star \in \mathbf{T}^d$ is a corresponding argminimum, then $\Delta \Psi^\varepsilon(t^\star, x^\star) \geq 0$ so that $\partial_t \Psi^\varepsilon(t^\star, x^\star) > \varepsilon$ and we have therefore by continuity, for some $\delta^\star > 0$, $\Psi^\varepsilon > 0$ on $[t^\star, t^\star + \delta) \times B(x_\star, \delta)$. By continuity, such a $\delta > 0$ exists all the more for an $x \in \mathbf{T}^d$ such that $\Psi^\varepsilon(t^\star, x) > 0$ so taking a finite covering of $\mathbf{T}^d$ we obtain $\min_{\mathbf{T}^d} \Psi^\varepsilon(t) > 0$ for some $t > t^\star$, which is a contradiction. We recover afterwards $\Psi \geq 0$ by letting $\varepsilon \to 0$.

To treat the general case, we can use Exercise 2 : regularizing μ and S we have a sequence of smooth solutions Φ_n which, by uniqueness, converges (at least weakly) to Φ.

Returning back to the Kolmogorov equation, we use Stampacchia's argument : if $z^0 \geq 0$ and $z^\pm := \max(\pm z, 0)$, we have $\langle z, z^- \rangle_{\mathrm{L}^2(Q_T)} = -\langle z^0, \Phi^{z^-}(0) \rangle_{\mathrm{L}^2(\mathbf{T}^d)} \geq 0$ thanks to the above analysis, and $z = z^+ - z^-$ implies $\langle z, z^- \rangle_{\mathrm{L}^2(Q_T)} = -\|z^-\|_2^2$, so $z^- = 0$. Important remark : the proof of Theorem 6 can be easy generalized to include a source term G in the Kolmogorov equation if G is enough integrable (for instance $G \in \mathrm{L}^2(Q_T)$), and the previous proof applies to check that if $G \geq 0$ and $z^0 \geq 0$ then so is z.

In the stronger functional setting $\mu \in \mathrm{L}^\infty(0,T; \mathrm{W}^{2,\infty}(\mathbf{T}^d))$, if for some $\alpha > 0$, $z_0 \leq \alpha$, then letting $\varphi := \alpha \exp(t \|(\Delta\mu)^+\|_\infty)$, we have $\partial_t \varphi - \Delta(\mu\varphi) = \varphi(\|(\Delta\mu)^+\|_\infty - \Delta\mu) \geq 0$, so that $\widetilde{z} := \varphi - z$ satisfies the Kolmogorov equation with nonnegative initial condition and source term : $\widetilde{z} \geq 0$. If $z_0 \geq -\alpha$, then working with $-z$ we get $z \geq -\varphi$ so that we established $z^0 \in \mathrm{L}^\infty(\mathbf{T}^d) \Rightarrow \|z\|_\infty \leq \|z^0\|_\infty \exp(T \|(\Delta\mu)^+\|_\infty)$.

Back to Exercise 3.

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