Brief summary in english of the results of *Étude minutieuse de processus moins indécis que les autres*

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The paper is concerned with the long-time behaviour of the semi-group $P_t = e^{t\mathcal{L}_a}$ on $\mathbb{T} \times \{-1, 1\}$ with generator

$$\mathcal{L}_a f(x, y) = y\partial_x f(x, y) + a\big(f(x, -y) - f(x, y)\big).$$

Its unique equilibrium μ is the uniform law on $\mathbb{T} \times \{-1, 1\}$. The operator norm on $L^2(\mu)$

$$\begin{aligned}\|P_t - \mu\| &= \sup_{f \in L^2(\mu)\setminus\{0\}} \frac{\|P_t f - \int f \mathrm{d}\mu\|}{\|f\|} \\ &= \sup\left\{\frac{\|f_t\|}{\|f_0\|},\ \partial_s f_s = \mathcal{L}_a f_s \text{ for } s \in [0, t],\ f_0 \neq 0,\ \int f_0 \mathrm{d}\mu = 0\right\}\end{aligned}$$

is explicitly computed.

Theorem 1. *For $a \geq 1$, the spectral gap is $\lambda = a - \sqrt{a^2 - 1}$ and for $a \leq 1$, $\lambda = a$. More precisely, for all $t > 0$,*

- *If $a > 1$ then, denoting $\omega = \sqrt{a^2 - 1}$ and $\gamma = e^{-2\omega t}$,*

$$\begin{aligned}\|P_t - \mu\| &= e^{(-a+\sqrt{a^2-1})t}\sqrt{1 + \frac{2}{\omega^2\left(\frac{1+\gamma}{1-\gamma}\right) + a\sqrt{1 + \omega^2\left(\frac{1+\gamma}{1-\gamma}\right)^2} - 1}} \\ &= 1 - \frac{t^3}{3} + \underset{t\to 0}{o}(t^3) \\ &\underset{t\to+\infty}{\sim} \frac{a^2}{a^2 - 1} e^{\lambda t}\end{aligned}$$

- *If $a = 1$ then*

$$\begin{aligned}\|P_t - \mu\| &= e^{-t}\sqrt{1 + \frac{2}{\sqrt{1 + \frac{1}{t^2}} - 1}} \\ &= 1 - \frac{t^3}{3} + \underset{t\to 0}{o}(t^3) \\ &\underset{t\to+\infty}{\sim} 2te^{-t}\end{aligned}$$

- *If $a < 1$ then*

$$\begin{aligned}\|P_t - \mu\| &= e^{-at}\sqrt{g(t)} \\ &= 1 - \frac{at^3}{3} + \underset{t\to 0}{o}(t^3)\end{aligned}$$

with g such that

$$\sup_{t>0} g(t) = \limsup_{t>0} g(t) = \frac{1+a}{1-a}$$

$$\liminf_{t>0} g(t) = 1$$

and, denoting $\nu = 2\sqrt{1-a^2}$, if $t \in \left[0, \frac{\pi}{\nu}\right]$ then

$$g(t) = \left(1 + \frac{2}{\sqrt{\frac{\nu^2}{a^2}\frac{1}{2(1-\cos(\nu t))}+1}-1}\right).$$

Sketch of proof. The 2-dimensional planes $V_n = \{f : (x,y) \mapsto e^{inx}g(y),\ g \in \mathbb{C}^{\{-1,1\}}\}$ for $n \in \mathbb{N}$ are fixed by $\mathcal{L}_a$, hence by P_t for all $t \geqslant 0$. As a consequence, the problem is reduced to the computation of the L^2 operator norm of 2×2 matrices, which is explicit (and involves the scalar product between the two (possibly generalized) eigenvectors). As the second step, the norm of these restrictions are compared, and it turns out the norm of the restriction on V_1 is, most of the time, the largest one (and thus is the norm on the whole space). The only case where it may not be the case is for $a < 1$ and $t > \pi/\nu$, in which case the prefactor g is the supremum of a sequence of periodic functions g_n with amplitude and period going both to 0 as n goes to infinity (see fig. 1).

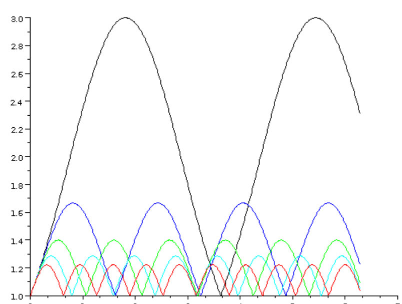

Figure 1: Even when $a < 1$, most of the time, the maximum of the g_n is g_1, so that the operator norm of $P_t - \mu$ is the norm of its restriction to V_1.

□

Note that the distinction $a \lesseqgtr 1$ corresponds, at a spectral level, to the following: for $a > 1$, the spectral gap is attained at a unique, real eigenvalue (i.e. the restriction of P_t on V_1 is diagonalizable with a real spectrum), for $a < 1$, the spectral gap is attained at two conjugated complex eigenvalues, and at $a = 1$, the spectral gap is attained at a unique, defective eigenvalue (with a 2×2 Jordan block).

The rest of the paper is concerned with the same kind of results but for the persistent walk on $\mathbb{Z}/N\mathbb{Z}$, which is a discrete Markov chain which converges as $N \to \infty$ to the Markov process associated to the generator $\mathcal{L}_a$, called the telegraph process. The convergence of the (first coordinate of the) telegraph process toward the Brownian motion on $\mathbb{T}$ is also established.