Periodic asymptotic dynamics of the measure solutions to an equal mitosis equation

Pierre Gabriel [*]   Hugo Martin [†]

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Abstract

We prove, in the framework of measure solutions, that the equal mitosis equation present persistent asymptotic oscillations. To do so we adopt a duality approach, which is also well suited for proving the well-posedness when the division rate is unbounded. The main difficulty for characterizing the asymptotic behavior is to define the projection onto the subspace of periodic (rescaled) solutions. We achieve this by using the generalized relative entropy structure of the dual problem.

Keywords: growth-fragmentation equation, self-similar fragmentation, measure solutions, long-time behavior, general relative entropy, periodic semigroups

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1   Introdution

We are interested in the following nonlocal transport equation

$$\frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}\big(x\,u(t,x)\big) + B(x)u(t,x) = 4B(2x)u(t,2x), \qquad x > 0. \quad (1)$$

It appears as an idealized size-structured model for the bacterial cell division cycle [7, 55], and it is an interesting and challenging critical case of the general linear growth-fragmentation equation, as we will explain below. The unknown $u(t,x)$ represents the population density of cells of size x at time t, which evolves according to two phenomena: the individual exponential growth which results in the transport term $\partial_x(xu(t,x))$, and the equal mitosis corresponding to the nonlocal infinitesimal term $2B(2x)u(t,2x)2\mathrm{d}x - B(x)u(t,x)\mathrm{d}x$.

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[*]Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 45 Avenue des États-Unis, 78035 Versailles cedex, France. Email: pierre.gabriel@uvsq.fr

[†]Laboratoire Jacques-Louis Lions, CNRS UMR 7598, Sorbonne université, 4 place Jussieu, 75005 Paris, France. Corresponding author, Email: martin@ljll.math.upmc.fr

Equation (1) is also the Kolmogorov (forward) equation of the underlying piecewise deterministic branching process [22, 24, 28, 45]. Let us explain this briefly and informally. Consider the measure-valued branching process $(Z_t)_{t \geqslant 0}$ defined as the empirical measure

$$Z_t = \sum_{i \in V_t} \delta_{X_t^i}$$

where V_t is the set of individuals alive at time t and $\{X_t^i : i \in V_t\}$ the set of their sizes. For each individual $i \in V_t$ the size X_t^i grows exponentially fast following the deterministic flow $\frac{\mathrm{d}}{\mathrm{d}t} X_t^i = X_t^i$ until a division time T_i which occurs stochastically in a Poisson-like fashion with rate $B(X_t^i)$. Then the individual i dies and gives birth to two daughter cells i_1 and i_2 with size $X_{T_i}^{i_1} = X_{T_i}^{i_2} = \frac{1}{2} X_{T_i}^i$. Taking the expectancy of the random measures Z_t, we get a family of measures indexed by t

$$u(t, \cdot) = \mathbb{E}(Z_t)$$

which is a weak solution to the Kolmogorov Equation (1).

Another Kolmogorov equation is classically associated to $(Z_t)_{t \geqslant 0}$, which is the dual equation of (1)

$$\frac{\partial}{\partial t} \varphi(t, x) = x \frac{\partial}{\partial x} \varphi(t, x) + B(x) \big[2\varphi(t, x/2) - \varphi(t, x) \big], \qquad x > 0. \qquad (2)$$

This second equation is sometimes written in its backward version where $\frac{\partial}{\partial t} \varphi(t, x)$ is replaced by $-\frac{\partial}{\partial t} \varphi(t, x)$, and is then usually called the backward Kolmogorov equation. Nevertheless since the division rate $B(x)$ does not depend on time, we prefer here writing this backward equation in a forward form. Indeed in this case we have that for an observation function f,

$$\varphi(t, x) := \mathbb{E}[f(Z_t) \,|\, Z_0 = \delta_x] := \mathbb{E}\left[\sum_{i \in V_t} f(X_t^i) \,\Big|\, Z_0 = \delta_x \right]$$

is the solution to (2) with initial data $\varphi(0, x) = f(x)$.

Equation (1) is then naturally defined on a space of measure, while Equation (2) is defined on a space of functions.

As we already mentioned, Equation (1) is a particular case of the growth-fragmentation equation which reads in its general form

$$\frac{\partial}{\partial t} u(t, x) + \frac{\partial}{\partial x} \big(g(x) u(t, x) \big) + B(x) u(t, x) = \int_x^\infty k(y, x) B(y) u(t, y) \mathrm{d}y.$$

In this model the deterministic growth flow is given by $\frac{\mathrm{d}x}{\mathrm{d}t} = g(x)$ and the mitosis $x \to \frac{x}{2}$ is replaced by the more general division $x \to y < x$ with a kernel $k(x, \mathrm{d}y)$. Equation (1) then corresponds to the case $g(x) = x$ and $k(x, \mathrm{d}y) = 2\delta_{y = \frac{x}{2}}$. The long time behavior of the growth-fragmentation equation is strongly related to

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the existence of steady size distributions, namely solutions of the form $\mathcal{U}(x)\mathrm{e}^{\lambda t}$ with $\mathcal{U}$ nonnegative and integrable. It is actually equivalent to say that $\mathcal{U}$ is a Perron eigenfunction associated to the eigenvalue λ. Such an eigenpair $(\lambda, \mathcal{U})$ typically exists when, roughly speaking, the fragmentation rate B dominates the growth speed g at infinity and on the contrary g dominates B around the origin (see [26, 27, 29] for more details). In most cases where this existence holds, the solutions behave asymptotically like the steady size distribution $\mathcal{U}(x)\mathrm{e}^{\lambda t}$. This property, known as asynchronous exponential growth [57], has been proved by many authors using various methods since the pioneering work of Diekmann, Heijmans, and Thieme [25]. Most of these results focus on one of the two special cases $g(x) = 1$ (linear individual growth) or $g(x) = x$ (exponential individual growth). When $g(x) = 1$ it has been proved for the equal mitosis or more general kernels $k(x, y)$ by means of spectral analysis of semigroups [9, 25, 38, 41, 49], general relative entropy method [23, 48] and/or functional inequalities [2, 18, 43, 50, 51, 52], Doeblin's type condition [3, 14, 16, 54] combined with the use of Lyapunov functions [5, 13], coupling arguments [6, 21, 44], many-to-one formula [22], or explicit expression of the solutions [58]. For the case $g(x) = x$ asynchronous exponential growth is proved under the assumption that $k(x, \mathrm{d}y)$ has an absolutely continuous part with respect to the Lebesgue measure: by means of spectral analysis of semigroups [9, 41, 49], general relative entropy method [31, 48] and/or functional inequalities [2, 17, 18, 37], Foster-Lyapunov criteria [13], Feynman-Kac [10, 11, 12, 20] or many-to-one formulas [46].

The assumption that the fragmentation kernel has a density part when $g(x) = x$ is a crucial point, not only a technical restriction. In the mitotic case of Equation (1) for instance, asynchronous exponential growth does not hold. It can be easily understood through the branching process $(Z_t)_{t \geqslant 0}$. If at time $t = 0$ the population is composed of only one individual with deterministic size $x > 0$, then for any positive time t and any $i \in V_t$ we have that $X_t^i \in \{xe^t 2^{-k} : k \in \mathbb{N}\}$. This observation was made already by Bell and Anderson in [7] and it has two important consequences.

First the solution $\mathbb{E}(Z_t)$ cannot relax to a steady size distribution and it prevents Equation (1) from having the asynchronous exponential growth property. The dynamics does not mix enough the trajectories to generate ergodicity, and the asymptotic behavior keeps a strong memory of the initial data. This situation has been much less studied than the classical ergodic case. In [25, 42] Diekmann, Heijmans, and Thieme made the link with the existence of a nontrivial boundary spectrum: all the complex numbers $1 + \frac{2ik\pi}{\log 2}$, with k lying in $\mathbb{Z}$, are eigenvalues. As a consequence the Perron eigenvalue $\lambda = 1$ is not strictly dominant and it results in persistent oscillations, generated by the boundary eigenfunctions. The convergence to this striking behavior was first proved by [38] in the space $L^1([\alpha, \beta])$ with $[\alpha, \beta] \subset (0, \infty)$. More recently it has been obtained in $L^1(0, \infty)$ for monomial division rates and smooth initial data [56], and in $L^2((0, \infty), x/\mathcal{U}(x)\,\mathrm{d}x)$ [8].

Second, it highlights the lack of regularizing effect of the equation. If the initial distribution is a Dirac mass, then the solution is a Dirac comb for any

time. It contrasts with the cases of density fragmentation kernels for which the singular part of the measure solutions vanishes asymptotically when times goes to infinity [23], and gives an additional motivation for studying Equation (1) in a space of measures.

The aim of the present paper is to prove the convergence to asymptotic oscillations for the measure solutions of Equation (1).

Measure solutions to structured populations dynamics PDEs have attracted increasing attention in the last few years, and there exist several general well-posedness results [15, 19, 32, 33, 39]. However they do not apply here due to the unboundedness of the function B, which is required for the Perron eigenfunction $\mathcal{U}$ to exist, see Section 2.2. We overcome this difficulty by adopting a duality approach in the spirit of [4, 5, 30, 35], which is also convenient for investigating the long time behavior.

In [38] Greiner and Nagel deduce the convergence from a general result of spectral theory of positive semigroups, valid in L^p spaces with $1 \leqslant p < \infty$ [1, C-IV, Th. 2.14]. To be able to apply this abstract result they need to consider on a compact size interval $[\alpha, \beta]$. In [56] van Brunt $et~al.$ take advantage of the Mellin transform to solve Equation (1) explicitly and deduce the convergence in $L^1(0, \infty)$. But this method requires the division rate to be monomial, namely $B(x) = x^r$ with $r > 0$, and $u(0, \cdot)$ to be a $\mathcal{C}^2$ function with polynomial decay at 0 and ∞. In [8] the authors combine General Relative Entropy inequalities and the Hilbert structure of the space $L^2((0, \infty), x/\mathcal{U}(x)\,\mathrm{d}x)$ to prove that the solutions converge to their orthogonal projection onto the closure of the subspace spanned by the boundary eigenfunctions. The general relative entropy method has been recently extended to the measure solutions of the growth-fragmentation equation with smooth fragmentation kernel [23], but this cannot be applied to the singular case of the mitosis kernel. Our approach rather relies on the general relative entropy of the dual equation (2). It allows us to both define a projector on the boundary eigenspace despite the absence of Hilbert structure and prove the convergence to this projection.

The paper is organized as follows. In the next section, we state our main result. In Section 3, we prove the well-posedness of Equation (1) in the framework of measure solutions. Section 4 is devoted to the analysis of the long time asymptotic behavior. Finally, in a last section, we draw some future directions that can extend the present work.

2   Preliminaries and the main result

Before stating our main result, we introduce the space of weighted signed measures in which we will work and we recall existing spectral results about Equation (1).

2.1   Weighted signed measures and measure solutions

A particular feature of Equation (1) is the exponential growth of the total mass. Indeed, a formal integration against the measure $x\,\mathrm{d}x$ over $(0,\infty)$ leads to the balance law

$$\int_0^\infty xu(t,x)\mathrm{d}x = \mathrm{e}^t \int_0^\infty xu(0,x)\mathrm{d}x.$$

Due to this property, the weighted Lebesgue space $L^1((0,\infty), x\,\mathrm{d}x)$ provides a natural framework for studying Equation (1). In the measure solutions framework, the space $\mathcal{M}$ of finite signed Borel measures on $(0,\infty)$ extends the space $L^1((0,\infty),\mathrm{d}x)$. Thus a possible choice for the setting of our work could be the subspace

$$\left\{ \mu \in \mathcal{M}, \int_0^\infty x\,|\mu|(\mathrm{d}x) < \infty \right\}$$

where the positive measure $|\mu|$ is the total variation of μ, see [53] for instance. However this subspace of $\mathcal{M}$ does not contain $L^1((0,\infty), x\,\mathrm{d}x)$, so we prefer to define a more relevant *ad hoc* space.

Denote by $\dot{\mathcal{M}}_+$ the cone of positive measures μ on $(0,\infty)$ such that

$$\int_0^\infty x\,\mu(\mathrm{d}x) < \infty.$$

We define the space of *weighted signed measures* $\dot{\mathcal{M}}$ as the quotient space

$$\dot{\mathcal{M}} := \left. \dot{\mathcal{M}}_+ \times \dot{\mathcal{M}}_+ \middle/ {}_\sim \right.$$

where $(\mu_1, \mu_2) \sim (\tilde{\mu}_1, \tilde{\mu}_2)$ if $\mu_1 + \tilde{\mu}_2 = \tilde{\mu}_1 + \mu_2$. Clearly $\dot{\mathcal{M}}$ is isomorphic to $\mathcal{M}$ through the canonical mapping

$$\begin{cases} \dot{\mathcal{M}} & \to & \mathcal{M} \\ \mu & \mapsto & \left\{ A \mapsto \int_A x\,\mu_1(\mathrm{d}x) - \int_A x\,\mu_2(\mathrm{d}x) \right\} \end{cases} \tag{3}$$

where (μ_1, μ_2) is any representative of the equivalence class μ, and this motivates the notation $\mu = \mu_1 - \mu_2$. Through this isomorphism the Hahn-Jordan decomposition of signed measures ensures that for any $\mu \in \dot{\mathcal{M}}$ there exists a unique couple $(\mu_+, \mu_-) \in \dot{\mathcal{M}}_+ \times \dot{\mathcal{M}}_+$ of mutually singular measures such that $\mu = \mu_+ - \mu_-$, and we can define its total variation $|\mu| := \mu_+ + \mu_-$. We endow $\dot{\mathcal{M}}$ with the weighted total variation norm

$$\|\mu\|_{\dot{\mathcal{M}}} := \int_0^\infty x\,|\mu|(\mathrm{d}x) = \int_0^\infty x\,\mu_+(\mathrm{d}x) + \int_0^\infty x\,\mu_-(\mathrm{d}x)$$

which makes it a Banach space, the isomorphism (3) being actually an isometry if $\mathcal{M}$ is endowed with the standard total variation norm.

Notice that in general an element $\mu = \mu_1 - \mu_2$ of $\dot{\mathcal{M}}$ is not strictly speaking a measure since if $A \subset (0, \infty)$ is a Borel set which touches the origin then one can have $\mu_1(A) = \mu_2(A) = +\infty$, so that $\mu(A)$ does not make sense. Nevertheless, the isomorphism (3) ensures that it becomes a measure once multiplied by the weight function $x \mapsto x$, and this motivates calling it a weighted signed measure. Another motivation is the analogy with weighted L^1 spaces: we can naturally associate to a function $f \in L^1((0, \infty), x\,\mathrm{d}x)$ the weighted measure $\mu(\mathrm{d}x) = f_+(x)\mathrm{d}x - f_-(x)\mathrm{d}x$, thus defining a canonical injection of $L^1((0, \infty), x\,\mathrm{d}x)$ into $\dot{\mathcal{M}}$.

Now that the convenient space $\dot{\mathcal{M}}$ is defined, we give some useful properties of its natural action on measurable functions. Denote by $\dot{\mathcal{B}}$ the space of Borel functions $f : (0, \infty) \to \mathbb{R}$ such that the quantiy

$$\|f\| := \sup_{x>0} \frac{|f(x)|}{x} \tag{4}$$

is finite. An element μ of $\dot{\mathcal{M}}$ defines a linear form on $\dot{\mathcal{B}}$ through

$$\mu(f) := \int_0^\infty f \,\mathrm{d}\mu_+ - \int_0^\infty f \,\mathrm{d}\mu_-.$$

We also define the subset $\dot{\mathcal{C}} \subset \dot{\mathcal{B}}$ of continuous functions, and the subset $\dot{\mathcal{C}}_0 \subset \dot{\mathcal{C}}$ of the functions such that the ratio $f(x)/x$ vanishes at zero and infinity. The isomorphism (3) combined with the Riesz representation theorem, which states that $\mathcal{M} \simeq \mathcal{C}_0(0, \infty)'$, ensures that $\dot{\mathcal{M}} \simeq \dot{\mathcal{C}}_0'$ with the identity

$$\|\mu\|_{\dot{\mathcal{M}}} = \sup_{\|f\| \leqslant 1} \mu(f)$$

where the supremum is taken over $\dot{\mathcal{C}}_0$. Actually the supremum can also be taken over $\dot{\mathcal{B}}$ and in this case it is even a maximum (take $f(x) = x$ on the support of μ_+ and $f(x) = -x$ on the support of μ_-).

Since the boundary eigenvalues are complex, it is also useful to consider the space $\dot{\mathcal{C}}^{\mathbb{C}}$ of continuous functions $f : (0, \infty) \to \mathbb{C}$ such that $\|f\| < \infty$, where the norm $\| \cdot \|$ is still defined by (4) but with $| \cdot |$ denoting the modulus instead of the absolute value, as well as the space of weighted complex measures $\dot{\mathcal{M}}^{\mathbb{C}} := \dot{\mathcal{M}} + i\dot{\mathcal{M}}$. The action of $\dot{\mathcal{M}}^{\mathbb{C}}$ of $\dot{\mathcal{C}}^{\mathbb{C}}$ is naturally defined by

$$\mu(f) := (\mathrm{Re}\mu)(\mathrm{Re}f) - (\mathrm{Im}\mu)(\mathrm{Im}f) + i\big[(\mathrm{Re}\mu)(\mathrm{Im}f) + (\mathrm{Im}\mu)(\mathrm{Re}f)\big].$$

It remains to define a notion of solutions in the space $\dot{\mathcal{M}}$ for Equation (1). Let us first define the operator $\mathcal{A}$ acting on the space $\mathcal{C}^1(0, \infty)$ of continuously differentiable functions via

$$\mathcal{A}f(x) := xf'(x) + B(x)[2f(x/2) - f(x)].$$

With this notation the dual equation (2) simply reads

$$\partial_t \varphi = \mathcal{A}\varphi.$$

The definition we choose for the measure solutions to Equation (1) is of the "mild" type in the sense that it relies on an integration in time, and of the "weak" type in the sense that it involves test functions in space.

Definition 1. *A family $(\mu_t)_{t \geqslant 0} \subset \dot{\mathcal{M}}$ is called a measure solution to Equation (1) if for all $f \in \dot{\mathcal{C}}$ the mapping $t \mapsto \mu_t f$ is continuous, and for all $t \geqslant 0$ and all $f \in \mathcal{C}_c^1(0, \infty)$*

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(\mathcal{A}f)\, ds. \tag{5}$$

2.2 Dominant eigenvalues and periodic solutions

As we already mentioned in the introduction, the long term behavior of Equation (1) – as well as that of Equation (2) – is strongly related to the associated Perron eigenvalue problem, which consists in finding a constant λ together with nonnegative and nonzero $\mathcal{U}$ and ϕ such that

$$\big(x\,\mathcal{U}(x)\big)' + (B(x) + \lambda)\,\mathcal{U}(x) = 4B(2x)\mathcal{U}(2x), \tag{6}$$

$$- x\phi'(x) + (B(x) + \lambda)\,\phi(x) = 2B(x)\phi\Big(\frac{x}{2}\Big). \tag{7}$$

This problem has been solved under various assumptions on the division rate B in [27, 31, 40, 47]. The most general result is the one obtained as a particular case of [27, Theorem 1], which supposes that B satisfies the following conditions

$$\begin{cases} B : (0, \infty) \to [0, \infty) \text{ is locally integrable,} \\ \operatorname{supp} B = [b, +\infty) \text{ for some } b \geqslant 0, \\ \exists z_0, \gamma_0, K_0 > 0, \ \forall x < z_0, \quad B(x) \leqslant K_0 x^{\gamma_0} \\ \exists z_1, \gamma_1, \gamma_2, K_1, K_2 > 0 \ \forall x > z_1 \quad K_1 x^{\gamma_1} \leqslant B(x) \leqslant K_2 x^{\gamma_2}. \end{cases} \tag{A}$$

Theorem 1 ([27]). *Under assumption (A), there exists a unique nonnegative eigenfunction $\mathcal{U} \in L^1(0, \infty)$ solution to (6) and normalized by $\int_0^\infty x\,\mathcal{U}(x)dx = 1$. It is associated to the eigenvalue $\lambda = 1$ and to the adjoint eigenfunction $\phi(x) = x$ solution to (7). Moreover,*

$$\forall \alpha \in \mathbb{R}, \qquad x^\alpha \mathcal{U} \in L^1(0, \infty) \cap L^\infty(0, \infty).$$

As already noticed in [25] (see also example 2.15, p.354 in [1]), the Perron eigenvalue $\lambda = 1$ is not strictly dominant in the present case. There is an infinite

number of (complex) eigenvalues with real part equal to 1. More precisely for all $k \in \mathbb{Z}$ the triplet $(\lambda_k, \mathcal{U}_k, \phi_k)$ defined from $(\lambda, \mathcal{U}, \phi)$ by

$$\lambda_k = 1 + \frac{2ik\pi}{\log 2}, \qquad \mathcal{U}_k(x) = x^{-\frac{2ik\pi}{\log 2}} \mathcal{U}(x), \qquad \phi_k(x) = x^{1 + \frac{2ik\pi}{\log 2}},$$

verifies (6)-(7). In such a situation the asynchronous exponential growth property cannot hold, since for any $k \in \mathbb{Z} \setminus \{0\}$ the functions

$$\mathrm{Re}\big(\mathcal{U}_k(x)\mathrm{e}^{\lambda_k t}\big) = \\ \left[\cos\Big(\frac{2k\pi}{\log 2}\log x\Big) \cos\Big(\frac{2k\pi}{\log 2}t\Big) - \sin\Big(\frac{2k\pi}{\log 2}\log x\Big) \sin\Big(\frac{2k\pi}{\log 2}t\Big) \right] \mathcal{U}(x)\mathrm{e}^t$$

and

$$\mathrm{Im}\big(\mathcal{U}_k(x)\mathrm{e}^{\lambda_k t}\big) = \\ \left[\cos\Big(\frac{2k\pi}{\log 2}\log x\Big) \sin\Big(\frac{2k\pi}{\log 2}t\Big) - \sin\Big(\frac{2k\pi}{\log 2}\log x\Big) \cos\Big(\frac{2k\pi}{\log 2}t\Big) \right] \mathcal{U}(x)\mathrm{e}^t$$

are solutions to Equation (1) that oscillate around $\mathcal{U}(x)\mathrm{e}^t$.

2.3 Statement of the main result

In [8] it is proved that the family $(\mathcal{U}_k(x)\mathrm{e}^{\lambda_k t})_{k \in \mathbb{Z}}$ of solutions is enough to get the long time behavior of all the others in the space $L^2((0, \infty), x/\mathcal{U}(x)\,\mathrm{d}x)$. More precisely it is proved that

$$\left\| u(t, \cdot)\mathrm{e}^{-t} - \sum_{k \in \mathbb{Z}} (u(0, \cdot), \mathcal{U}_k)\,\mathcal{U}_k\,\mathrm{e}^{(\lambda_k - 1)t} \right\|_{L^2((0,\infty),\,x/\mathcal{U}(x)\,\mathrm{d}x)} \xrightarrow[t \to +\infty]{} 0 \qquad (8)$$

where $(\cdot, \cdot)$ stands for the canonical inner product of the complex Hilbert space $L^2((0, \infty), x/\mathcal{U}(x)\,\mathrm{d}x)$. Such an oscillating behavior also occurs in a L^1 setting, as shown by [25, 38, 56]. But the techniques used in these papers (abstract theory of semigroups or Mellin transform) do not allow to make appear explicitly the eigenelements $(\lambda_k, \mathcal{U}_k, \phi_k)$ in the limiting dynamic equilibrium. The objective of the present paper is threefold: prove the well-posedness of Equation (1) in $\dot{\mathcal{M}}$, extend the previous results about the long time behavior to this large space, and characterize the oscillating limit in terms of $(\lambda_k, \mathcal{U}_k, \phi_k)$.

To prove the existence and uniqueness of solutions to Equation (1) in the sense of Definition 1, we make the following assumption on the division rate:

$$B : (0, \infty) \to [0, \infty) \text{ is continuous and bounded around } 0. \qquad (A2)$$

Since we work in $\dot{\mathcal{M}}$, it is convenient to define the family of complex measures $\nu_k \in \dot{\mathcal{M}}^{\mathbb{C}}$ with Lebesgue density $\mathcal{U}_k$, i.e.

$$\nu_k(\mathrm{d}x) = \mathcal{U}_k(x)\,\mathrm{d}x.$$

The following theorem summarizes the main results of the paper.

Theorem 2. *Let $\mu_0 \in \dot{\mathcal{M}}$. If Assumption (A2) is verified, then there exists a unique measure solution $(\mu_t)_{t\geqslant 0}$ to Equation (1) in the sense of Definition 1. If B satisfies additionally (A), then there exists a unique $\log 2$-periodic family $(\rho_t)_{t\geqslant 0} \subset \dot{\mathcal{M}}$ such that for all $f \in \dot{\mathcal{C}}_0$*

$$\mu_t(f)\,\mathrm{e}^{-t} - \rho_t(f) \xrightarrow[t\to+\infty]{} 0.$$

Moreover, for any $t \geqslant 0$, the weighted measure ρ_t is characterized through a Fejér type sum: for all $f \in \mathcal{C}^1_c(0,\infty)$

$$\rho_t(f) = \lim_{N\to\infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right)\mu_0(\phi_k)\nu_k(f)\mathrm{e}^{\frac{2ik\pi}{\log 2}t}.$$

Let us make some comments about these results:

(i) It is worth noticing that the well-posedness of Equation (1) do not require any upper bound for the division rate. It contrasts with existing results in Lebesgue spaces where at most polynomial growth is usually assumed.

(ii) In [8, 38, 56] the convergence to the oscillating behavior is proved to occur in norm. Here we extend the convergence to a much larger set of initial data but only for the weak-* topology.

(iii) In (8) the dynamic equilibrium is characterized as a Fourier type series. In our result it is replaced by a Fejér sum, namely the Cesàro means of the Fourier series.

(iv) Even though all the ν_k have a density with respect to the Lebesgue measure, the limit ρ_t does not in general. Indeed, as noticed in the introduction, if for instance $\mu_0 = \delta_x$ then $\mathrm{supp}\,\mu_t \subset \{x\mathrm{e}^t 2^{-k} : k \in \mathbb{N}\}$, and consequently $\mathrm{supp}\,\rho_t \subset \{x\mathrm{e}^t 2^{-k} : k \in \mathbb{Z}\}$ and it is thus a Dirac comb.

(v) We easily notice in the explicit formula of ρ_t that if μ_0 is such that $\mu_0(\phi_k) = 0$ for all $k \neq 0$, then there is no oscillations and the solution behaves asymptotically like $\mathcal{U}(x)\mathrm{e}^t$, similarly to the asynchronous exponential growth case. Such initial distributions actually do exist, as for instance the one proposed in [56] which reads in our setting

$$\mu_0(\mathrm{d}x) = \frac{1}{x^2}\mathbb{1}_{[1,2]}(x)\,\mathrm{d}x$$

where $\mathbb{1}_{[1,2]}$ denotes the indicator function of the interval $[1,2]$.

3 Well-posedness in the measure setting

Our method consists in two main steps. In the first place, we prove the well-posedness of Equation (2) by means of fixed point techniques. Afterwards, we combien this result and a duality property to define the unique solutions to Equation (1).

3.1 The dual equation

We actually prove slightly more than the well-posedness of Equation (2) in $\dot{\mathcal{B}}$. Let us first introduce some useful functional spaces. For a subset $\Omega \subset \mathbb{R}^d$, we denote by $\mathcal{B}_{loc}(\Omega)$ the space of functions $f : \Omega \to \mathbb{R}$ that are bounded on $\Omega \cap B(0, r)$ for any $r > 0$, and by $\mathcal{B}(\Omega)$ the (Banach) subspace of bounded functions endowed with the supremum norm $\|f\|_\infty = \sup_{x \in \Omega} |f(x)|$. Using these spaces allows us to prove the well-posedness without needing any upper bound at infinity on the division rate B.

In the following proposition, we prove that for any $f \in \mathcal{B}_{loc}(0, \infty)$ there exists a unique solution $\varphi \in \mathcal{B}_{loc}([0, \infty) \times (0, \infty))$ to Equation (2) in a mild sense (Duhamel formula) with initial condition $\varphi(0, \cdot) = f$. Moreover, we show that if $f \in \mathcal{C}^1(0, \infty)$ then φ is also continuously differentiable and verifies Equation (2) in the classical sense.

Proposition 3. *Assume that B satisfies* (A2). *Then for any $f \in \mathcal{B}_{loc}(0, \infty)$ there exists a unique $\varphi \in \mathcal{B}_{loc}([0, \infty) \times (0, \infty))$ such that for all $t \geqslant 0$ and $x > 0$*

$$\varphi(t, x) = f(xe^t)e^{-\int_0^t B(xe^s)ds} + 2\int_0^t B(xe^\tau)e^{-\int_0^\tau B(xe^s)ds}\varphi\Big(t - \tau, \frac{xe^\tau}{2}\Big)d\tau.$$

Moreover if f is nonnegative/continuous/continuously differentiable, then so is φ. In the latter case φ verifies for all $t, x > 0$

$$\frac{\partial}{\partial t}\varphi(t, x) = \mathcal{A}\varphi(t, \cdot)(x) = x\frac{\partial}{\partial x}\varphi(t, x) + B(x)\big[2\varphi(t, x/2) - \varphi(t, x)\big].$$

Proof. Let $f \in \mathcal{B}_{loc}(0, \infty)$ and define on $\mathcal{B}_{loc}([0, \infty) \times (0, \infty))$ the mapping Γ by

$$\Gamma g(t, x) = f(xe^t)e^{-\int_0^t B(xe^s)\,ds} + 2\int_0^t B(xe^\tau)e^{-\int_0^\tau B(xe^s)\,ds}g\Big(t - \tau, \frac{xe^\tau}{2}\Big)\,d\tau.$$

For $T, K > 0$ define the set $\Omega_{T,K} = \{(t, x) \in [0, T] \times (0, \infty), \; xe^t < K\}$. Clearly Γ induces a mapping $\mathcal{B}(\Omega_{T,K}) \to \mathcal{B}(\Omega_{T,K})$, still denoted by Γ. To build a fixed point of Γ in $\mathcal{B}_{loc}([0, \infty) \times (0, \infty))$ we prove that it admits a unique fixed point in any $\mathcal{B}(\Omega_{T,K})$, denoted $\varphi_{T,K}$, that we will build piecewisely on subsets of $\Omega_{T,K}$.

Let $K > 0$ and $t_0 < 1/(2\sup_{(0,K)} B)$. For any $g_1, g_2 \in \mathcal{B}(\Omega_{t_0,K})$ we have

$$\|\Gamma g_1 - \Gamma g_2\|_\infty \leqslant 2t_0 \sup_{(0,K)} B \, \|g_1 - g_2\|_\infty$$

and Γ is a contraction. The Banach fixed point theorem then guarantees the existence of a unique fixed point $\varphi_{t_0,K}$ of Γ in $\mathcal{B}(\Omega_{t_0,K})$. The first step to construct the unique fixed point of Γ on $\mathcal{B}(\Omega_{T,K})$ is to set $\varphi_{T,K|\Omega_{t_0,K}} := \varphi_{t_0,K}$. We can repeat this argument on $\mathcal{B}(\Omega_{t_0,Ke^{-t}})$ with f being replaced by $\varphi_{t_0,K}(t_0, \cdot)$ to obtain a fixed point $\varphi_{t_0,Ke^{-t}}$. Then we set $\varphi_{T,K|\Omega_{t_0,Ke^{-t}}}(\cdot + t_0, \cdot) := \varphi_{t_0,Ke^{-t}}$, thus defining $\varphi_{T,K}$ on $\Omega_{2t_0,K}$. Iterating the procedure we finally get a unique fixed point $\varphi_{T,K}$ of Γ in $\mathcal{B}(\Omega_{T,K})$.

For $T' > T > 0$ and $K' > K > 0$ we have $\varphi_{T',K'|\Omega_{T,K}} = \varphi_{T,K}$ by uniqueness of the fixed point in $\mathcal{B}(\Omega_{T,K})$, and we can define φ by setting $\varphi_{|\Omega_{T,K}} = \varphi_{T,K}$ for any $T, K > 0$. Clearly the function φ thus defined is the unique fixed point of Γ in $\mathcal{B}_{loc}([0,\infty) \times (0,\infty))$.

Since Γ preserves the closed cone of nonnegative functions if f is nonnegative, the fixed point $\varphi_{t_0,K}$ is necessarily nonnegative when f is so. Then by iteration $\varphi_{T,K} \geqslant 0$ for any $T, K > 0$, and ultimately $\varphi \geqslant 0$. Similarly, the closed subspace of continuous functions being invariant under Γ when f is continuous, the fixed point φ inherits the continuity of f.

Consider now that f is continuously differentiable on $(0,\infty)$. Unlike the sets of nonnegative or continuous functions, the subspace $\mathcal{C}^1(\Omega_{t_0,K})$ is not closed in $\mathcal{B}(\Omega_{t_0,K})$ for the norm $\|\cdot\|_\infty$. For proving the continuous differentiability of φ we repeat the fixed point argument in the Banach spaces

$$\{g \in \mathcal{C}^1(\Omega_{T,K}), \ g(0,\cdot) = f\}$$

endowed with the norm

$$\|g\|_{\mathcal{C}^1} = \|g\|_\infty + \|\partial_t g\|_\infty + \|x \partial_x g\|_\infty.$$

Differentiating Γg with respect to t we get

$$\partial_t(\Gamma g)(t,x) = \left[x e^t f'(x e^t) - B(x e^t) f(x e^t) \right] e^{-\int_0^t B(x e^s) ds}$$
$$+ 2B(x e^t) e^{-\int_0^t B(x e^s) ds} g\left(0, \frac{x e^t}{2}\right)$$
$$+ 2 \int_0^t B(x e^\tau) e^{-\int_0^\tau B(x e^s) ds} \partial_t g\left(t - \tau, \frac{x e^\tau}{2}\right) d\tau$$
$$= \mathcal{A}f(x e^t) e^{-\int_0^t B(x e^s) ds} + 2 \int_0^t B(x e^\tau) e^{-\int_0^\tau B(x e^s) ds} \partial_t g\left(t - \tau, \frac{x e^\tau}{2}\right) d\tau \tag{9}$$

and differentiating the alternative formulation

$$\Gamma g(t,x) = f(x e^t) e^{-\int_x^{x e^t} B(z) \frac{dz}{z}} + 2 \int_x^{x e^t} B(y) e^{-\int_x^y B(z) \frac{dz}{z}} g\left(t - \log\left(\frac{y}{x}\right), \frac{y}{2}\right) \frac{dy}{y}$$

with respect to x we obtain

$$x \partial_x(\Gamma g)(t,x) = \left[\mathcal{A}f(x e^t) + B(x) f(x e^t) \right] e^{-\int_x^{x e^t} B(z) \frac{dz}{z}} - 2B(x) g\left(t, \frac{x}{2}\right)$$
$$+ 2B(x) \int_x^{x e^t} B(y) e^{-\int_x^y B(z) \frac{dz}{z}} g\left(t - \log\left(\frac{y}{x}\right), \frac{y}{2}\right) \frac{dy}{y}$$
$$+ 2 \int_x^{x e^t} B(y) e^{-\int_x^y B(z) \frac{dz}{z}} \partial_t g\left(t - \log\left(\frac{y}{x}\right), \frac{y}{2}\right) \frac{dy}{y}$$
$$= \mathcal{A}f(x e^t) e^{-\int_0^t B(x e^s) ds} + \Gamma g(t,x) - 2B(x) g\left(t, \frac{x}{2}\right)$$
$$+ 2 \int_0^t B(x e^\tau) e^{-\int_0^\tau B(x e^s) ds} \partial_t g\left(t - \tau, \frac{x e^\tau}{2}\right) d\tau.$$

We deduce that for $g_1, g_2 \in \mathcal{C}^1(\Omega_{t_0, K})$ such that $g_1(0, \cdot) = g_2(0, \cdot) = f$

$$\|\Gamma g_1 - \Gamma g_2\|_{\mathcal{C}^1} \leqslant 4t_0 \sup_{(0,K)} B \, \|g_1 - g_2\|_\infty + 6t_0 \sup_{(0,K)} B \, \|\partial_t g_1 - \partial_t g_2\|_\infty$$

$$\leqslant 6t_0 \sup_{(0,K)} B \, \|g_1 - g_2\|_{\mathcal{C}^1}.$$

Thus Γ is a contraction for $t_0 < 1/(6 \sup_{(0,K)} B)$ and, with the same iterative procedure as above, this guarantees that the fixed point φ necessarily belongs to $\mathcal{C}^1([0, \infty) \times (0, \infty))$. We also get that

$$\partial_t(\Gamma g)(t, x) - x \partial_x(\Gamma g)(t, x) = B(x)\left[2g\left(t, \frac{x}{2}\right) - \Gamma g(t, x)\right]$$

and accordingly the fixed point satisfies $\partial_t \varphi = \mathcal{A}\varphi$. $\qquad\qquad\square$

From now on, we assume that the division rate B satisfies (A2). From Proposition 3 we deduce that Equation (2) generates a positive semigroup on $\dot{\mathcal{B}}$ by setting for any $t \geqslant 0$ and $f \in \mathcal{B}_{loc}(0, \infty)$

$$M_t f := \varphi(t, \cdot).$$

Corollary 4. *The family $(M_t)_{t \geqslant 0}$ is a semigroup of positive operators on $\mathcal{B}_{loc}(0, \infty)$. If $f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0, \infty)$ then the function $(t, x) \mapsto M_t f(x)$ is continuously differentiable on $(0, \infty) \times (0, \infty)$ and satisfies*

$$\partial_t M_t f(x) = \mathcal{A} M_t f(x) = M_t \mathcal{A} f(x).$$

Moreover the subspaces $\dot{\mathcal{B}}$ and $\dot{\mathcal{C}}$ are invariant under M_t, and for any $f \in \dot{\mathcal{B}}$ and any $t \geqslant 0$

$$\|M_t f\| \leqslant \|f\| \, \mathrm{e}^t.$$

Proof. The semigroup property $M_{t+s} = M_t M_s$ follows from the uniqueness of the fixed point in the proof of Proposition 3, $(t, x) \mapsto M_{t+s} f(x)$ and $(t, x) \mapsto M_t(M_s f)(x)$ being both solutions with initial distribution $M_s f \in \mathcal{B}_{loc}(0, \infty)$.

The positivity of M_t is given by Proposition 3.

Proposition 3 also provides the regularity of $(t, x) \mapsto M_t f(x)$ when $f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0, \infty)$, as well as the identity $\partial_t M_t f = \mathcal{A} M_t f$. Besides, if $f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0, \infty)$ then $\mathcal{A}f \in \mathcal{B}_{loc} \cap \mathcal{C}^1(0, \infty)$ and (9) with $g(t, x) = M_t f(x)$ ensures, still by uniqueness of the fixed point, that $\partial_t M_t f = M_t \mathcal{A} f$.

It is easy computations to check that if $f(x) = x$ then $M_t f(x) = x \mathrm{e}^t$. Together with the positivity of M_t it guarantees that $\|M_t f\| \leqslant \|f\| \mathrm{e}^t$ for any f in $\dot{\mathcal{B}}$. In particular $\dot{\mathcal{B}}$ is invariant under M_t, and $\dot{\mathcal{C}}$ also by virtue of Proposition 3. $\qquad\qquad\square$

We give now another useful property of the positive operators M_t, namely that they preserve increasing pointwise limits.

Lemma 5. *Let $f \in \mathcal{B}_{loc}(0, \infty)$ and let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{B}_{loc}(0, \infty)$ be an increasing sequence that converges pointwise to f, i.e for all $x > 0$*

$$f(x) = \lim_{n \to \infty} \uparrow f_n(x).$$

Then for all $t \geqslant 0$ and all $x > 0$

$$M_t f(x) = \lim_{n \to \infty} M_t f_n(x).$$

Proof. Let f and $(f_n)_{n \in \mathbb{N}}$ satisfy the assumptions of the lemma. For all $t \geqslant 0$, the positivity of M_t ensures that the sequence $(M_t f_n)_{n \in \mathbb{N}}$ is increasing and bounded by $M_t f$. Denote by $g(t, x)$ the limit of $M_t f_n(x)$. Using the monotone convergence theorem, we get by passing to the limit in

$$M_t f_n(x) = f_n(x e^t) e^{-\int_0^t B(x e^s) ds} + 2 \int_0^t B(x e^\tau) e^{-\int_0^\tau B(x e^s) ds} M_{t-\tau} f_n \left(\frac{x e^\tau}{2} \right) d\tau$$

that

$$g(t, x) = f(x e^t) e^{-\int_0^t B(x e^s) ds} + 2 \int_0^t B(x e^\tau) e^{-\int_0^\tau B(x e^s) ds} g \left(t - \tau, \frac{x e^\tau}{2} \right) d\tau.$$

By uniqueness property we deduce that $g(t, x) = M_t f(x)$. $\qquad\square$

3.2 Construction of a measure solution

Using the results in Section 3.1, we define a left action of the semigroup $(M_t)_{t \geqslant 0}$ on $\dot{\mathcal{M}}$. To do so we first set for $t \geqslant 0$, $\mu \in \dot{\mathcal{M}}_+$, and $A \subset (0, \infty)$ Borel set

$$(\mu M_t)(A) := \int_0^\infty M_t \mathbb{1}_A \, d\mu$$

and verify that μM_t such defined is a positive measure on $(0, \infty)$.

Lemma 6. *For all $\mu \in \dot{\mathcal{M}}_+$ and all $t \geqslant 0$, μM_t defines a positive measure. Additionally $\mu M_t \in \dot{\mathcal{M}}_+$ and for any $f \in \dot{\mathcal{B}}$*

$$(\mu M_t)(f) = \mu(M_t f).$$

Proof. Let $\mu \in \dot{\mathcal{M}}_+$ and $t \geqslant 0$. We first check that μM_t is a positive measure.

Clearly $\mu M_t(A) \geqslant 0$ for any Borel set A, and $\mu M_t(\varnothing) = \int_0^\infty M_t \mathbf{0} \, d\mu = 0$.

Let $(A_n)_{n \in \mathbb{N}}$ be a countable sequence of disjoint Borel sets of $(0, \infty)$ and define $f_n = \sum_{k=0}^n \mathbb{1}_{A_k} = \mathbb{1}_{\bigsqcup_{k=0}^n A_k}$. For every integer n, one has

$$\mu M_t \left(\bigsqcup_{k=0}^n A_k \right) = \int_0^\infty M_t f_n \, d\mu = \sum_{k=0}^n \int_0^\infty M_t(\mathbb{1}_{A_k}) d\mu = \sum_{k=0}^n \mu M_t(A_k).$$

The sequence $(f_n)_{n\in\mathbb{N}}$ is increasing and its pointwise limit is $f = \mathbb{1}_{\bigsqcup_{k=0}^{\infty} A_k}$, which belongs to $\mathcal{B}_{loc}(0,\infty)$. We deduce from Lemma 5 and the monotone convergence theorem that

$$\lim_{n\to\infty} \mu M_t \left(\bigsqcup_{k=0}^{n} A_k \right) = \lim_{n\to\infty} \int_0^{\infty} M_t f_n \, \mathrm{d}\mu = \int_0^{\infty} M_t f \, \mathrm{d}\mu = \mu M_t \left(\bigsqcup_{k=0}^{\infty} A_k \right)$$

where the limit lies in $[0,+\infty]$. This ensures that

$$\mu M_t \left(\bigsqcup_{k=0}^{\infty} A_k \right) = \sum_{k=0}^{\infty} \mu M_t(A_k)$$

and μM_t thus satisfies the definition of a positive measure.

By definition of μM_t, the identity $(\mu M_t)(f) = \mu(M_t f)$ is clearly true for any simple function f. Since any nonnegative measurable function is the increasing pointwise limit of simple functions, Lemma 5 ensures that it is also valid in $[0,+\infty]$ for any $f \in \mathcal{B}_{loc}(0,\infty)$ nonnegative. Considering $f(x) = x$ we get $(\mu M_t)(f) = \mu(f)\mathrm{e}^t < +\infty$, so that $\mu M_t \in \dot{\mathcal{M}}_+$. Finally, decomposing $f \in \dot{\mathcal{B}}$ as $f = f_+ - f_-$ we readily obtain that $(\mu M_t)(f) = \mu(M_t f)$. $\qquad \square$

Now for $\mu \in \dot{\mathcal{M}}$ and $t \geqslant 0$, we naturally define $\mu M_t \in \dot{\mathcal{M}}$ by

$$\mu M_t = \mu_+ M_t - \mu_- M_t.$$

It is then clear that the identity $(\mu M_t)(f) = \mu(M_t f)$ is still valid for $\mu \in \dot{\mathcal{M}}$ and $f \in \dot{\mathcal{B}}$.

Proposition 7. *The left action of $(M_t)_{t\geqslant 0}$ defines a positive semigroup in $\dot{\mathcal{M}}$, which satisfies for all $t \geqslant 0$ and all $\mu \in \dot{\mathcal{M}}$*

$$\|\mu M_t\|_{\dot{\mathcal{M}}} \leqslant \mathrm{e}^t \|\mu\|_{\dot{\mathcal{M}}}.$$

Proof. Using the duality relation $(\mu M_t)(f) = \mu(M_t f)$, it is a direct consequence of Corollary 4. $\qquad \square$

Finally we prove that the (left) semigroup $(M_t)_{t\geqslant 0}$ yields the unique measure solutions to Equation (1).

Theorem 8. *For any $\mu \in \dot{\mathcal{M}}$, the family $(\mu M_t)_{t\geqslant 0}$ is the unique solution to Equation (1), in the sense of Definition 1, with initial distribution μ.*

Proof. Let $\mu \in \dot{\mathcal{M}}$. We first check that $t \mapsto (\mu M_t)(f)$ is continuous for any $f \in \dot{\mathcal{C}}$ by writing

$$|(\mu M_t)(f) - \mu(f)| \leqslant \left| \int_0^\infty f(xe^t)e^{-\int_0^t B(xe^s)ds} - f(x)\,\mu(dx) \right|$$

$$+ \left| 2 \int_0^\infty \int_0^t B(xe^\tau)e^{-\int_0^\tau B(xe^s)ds} M_{t-\tau}f\left(\frac{xe^\tau}{2}\right)d\tau\,\mu(dx) \right|$$

$$\leqslant \int_0^\infty \left| f(xe^t)e^{-\int_0^t B(xe^s)ds} - f(x) \right| |\mu|(dx)$$

$$+ \|f\|e^t \int_0^\infty (1 - e^{-\int_0^t B(xe^s)ds})x\,|\mu|(dx).$$

The two terms in the right hand side vanish as t tends to 0 by dominated convergence theorem and the continuity of $t \mapsto (\mu M_t)(f)$ follows from the semigroup property.

Now consider $f \in \mathcal{C}_c^1(0, \infty)$. Integrating $\partial_t M_t f = M_t \mathcal{A}f$ between 0 and t we obtain for all $x > 0$

$$M_t f(x) = f(x) + \int_0^t M_s(\mathcal{A}f)(x)\,ds.$$

Since f is continuously differentiable and compactly supported, the function $\mathcal{A}f$ is so and thus belongs to $\dot{\mathcal{B}}$. We deduce that $|M_s(\mathcal{A}f)(x)| \leqslant \|\mathcal{A}f\|e^s x$ and we can use Fubini's theorem to get by integration against μ

$$\mu(M_t f) = \mu(f) + \mu\left(\int_0^t M_s(\mathcal{A}f)ds\right) = \mu(f) + \int_0^t \mu(M_s(\mathcal{A}f))ds.$$

The duality relation $(\mu M_t)(f) = \mu(M_t f)$ then guarantees that (μM_t) satisfies (5).

It remains to check the uniqueness. Let $(\mu_t)_{t \geqslant 0}$ be a solution to Equation (1) with $\mu_0 = \mu$. Recall that it implies in particular that $t \mapsto \mu_t(f)$ is continuous for any $f \in \dot{\mathcal{C}}$, and consequently $t \to \mu_t$ is locally bounded for the norm $\|\cdot\|_{\dot{\mathcal{M}}}$ due to the uniform boundedness principle. We want to verify that $\mu_t = \mu M_t$ for all $t \geqslant 0$. Fix $t > 0$ and $f \in \mathcal{C}_c^1(0, \infty)$, and let us compute the derivative of the mapping

$$s \mapsto \int_0^s \mu_\tau(M_{t-s}f)\,d\tau$$

defined on $[0, t]$. For $0 < s < s + h < t$ we have

$$\frac{1}{h}\left[\int_0^{s+h} \mu_\tau(M_{t-s-h}f)\,d\tau - \int_0^s \mu_\tau(M_{t-s}f)\,d\tau \right] = \frac{1}{h}\int_s^{s+h} \mu_\tau(M_{t-s}f)\,d\tau$$

$$+ \int_s^{s+h} \mu_\tau\left(\frac{M_{t-s-h}f - M_{t-s}f}{h}\right)d\tau + \int_0^s \mu_\tau\left(\frac{M_{t-s-h}f - M_{t-s}f}{h}\right)d\tau.$$

The convergence of the first term is a consequence of the continuity of $\tau \mapsto \mu_\tau(M_{t-s}f)$

$$\frac{1}{h} \int_s^{s+h} \mu_\tau M_{t-s}f \, \mathrm{d}\tau \xrightarrow[h \to 0]{} \mu_s M_{t-s}f.$$

For the second term we use that

$$M_{t-s}f - M_{t-s-h}f = M_{t-s-h} \int_0^h \partial_\tau M_\tau f \, \mathrm{d}\tau = M_{t-s-h} \int_0^h M_\tau \mathcal{A}f \, \mathrm{d}\tau$$

to get, since $\tau \mapsto \|\mu_\tau\|_{\mathcal{M}}$ is locally bounded,

$$\left| \int_s^{s+h} \mu_\tau \left(\frac{M_{t-s-h}f - M_{t-s}f}{h} \right) \mathrm{d}\tau \right| \leqslant h \sup_{\tau \in [0,t]} \|\mu_\tau\|_{\mathcal{M}} \|\mathcal{A}f\| \mathrm{e}^{t-s} \xrightarrow[h \to 0]{} 0.$$

For the last term we have, by dominated convergence and using the identity $\partial_t M_t f = \mathcal{A} M_t f$,

$$\int_0^s \mu_\tau \frac{M_{t-s-h}f - M_{t-s}f}{h} \, \mathrm{d}\tau \xrightarrow[h \to 0]{} - \int_0^s \mu_\tau \mathcal{A} M_{t-s}f \, \mathrm{d}\tau.$$

Finally we get

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^s \mu_\tau(M_{t-s}f) \, \mathrm{d}\tau = \mu_s(M_{t-s}f) - \int_0^s \mu_\tau(\mathcal{A} M_{t-s}f) \, \mathrm{d}\tau = \mu_0(M_{t-s}f).$$

Integrating between $s = 0$ and $s = t$ we obtain, since $\mu_0 = \mu$,

$$\int_0^t \mu_\tau(f) \, \mathrm{d}\tau = \int_0^t \mu(M_{t-s}f) \, \mathrm{d}s = \int_0^t (\mu M_\tau)(f) \, \mathrm{d}\tau$$

then by differentiation with respect to t

$$\mu_t(f) = (\mu M_t)(f).$$

By density of $\mathcal{C}_c^1(0, \infty)$ in $\dot{\mathcal{C}}_0$, it ensures that $\mu_t = \mu M_t$. $\qquad\square$

4 Long time asymptotics

To study the long time behavior of the measure solutions to Equation (1) we proceed by duality by first analyzing Equation (2). The method relies on the general relative entropy structure of this dual problem. From now on, we assume that the division rate B satisfies (A), so that the Perron eigenelements exist.

Lemma 9 (General Relative Entropy). *Let $H : \mathbb{R} \to \mathbb{R}$ be a differentiable convex function. Then for all $f \in \dot{\mathcal{B}} \cap \mathcal{C}^1(0, \infty)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty x \mathcal{U}(x) H \left(\frac{M_t f(x)}{x \, \mathrm{e}^t} \right) \mathrm{d}x = -D^H[\mathrm{e}^{-t} M_t f] \leqslant 0$$

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with D^H defined on $\dot{\mathcal{B}}$ by

$$D^H[f] = \int_0^\infty x B(x) \mathcal{U}(x)$$
$$\left[H'\left(\frac{f(x)}{x}\right)\left(\frac{f(x)}{x} - \frac{f(x/2)}{x/2}\right) + H\left(\frac{f(x/2)}{x/2}\right) - H\left(\frac{f(x)}{x}\right) \right] \mathrm{d}x.$$

Proof. For $f \in \dot{\mathcal{B}} \cap \mathcal{C}^1(0,\infty)$ the function $(t,x) \mapsto M_t f(x)$ is continuously differentiable and verifies $\partial_t M_t f(x) = \mathcal{A} M_t f(x)$, see Corollary 4. Simple computations then yield, using that $\mathcal{U}$ satisfies (6),

$$\left(\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}\right)\left(x\mathcal{U}(x) H\left(\frac{M_t f(x)}{x\, \mathrm{e}^t}\right)\right)$$
$$= x\mathcal{U}(x) B(x) H'\left(\frac{M_t f(x)}{x\, \mathrm{e}^t}\right)\left(\frac{M_t f(x/2)}{x/2} - \frac{M_t f(x)}{x}\right)$$
$$- H\left(\frac{M_t f(x)}{x\, \mathrm{e}^t}\right) x\left(4B(2x)\mathcal{U}(2x) - B(x)\mathcal{U}(x) - \mathcal{U}(x)\right)$$

and the conclusion follows by integration. $\qquad\square$

This result reveals the lack of coercivity of the equation in the sense that the dissipation $D^H[f]$ does not vanish only for $f(x) = \phi(x) = x$ but for any function f such that $f(2x) = 2f(x)$ for all $x > 0$. In particular all the eigenfunctions ϕ_k satisfy this relation, so $D^H[\mathrm{Re}(\phi_k)] = D^H[\mathrm{Im}(\phi_k)] = 0$. More precisely we have the following result about the space

$$X := \left\{ f \in \dot{\mathcal{C}}^{\mathbb{C}} \mid \forall x > 0,\ f(2x) = 2f(x) \right\}.$$

Lemma 10. *We have the identity*

$$X = \overline{\mathrm{span}}(\phi_k)_{k \in \mathbb{Z}}$$

and more specifically any $f \in X$ is the limit in $(\dot{\mathcal{C}}^{\mathbb{C}}, \|\cdot\|)$ of a Fejér type sum

$$f = \lim_{N \to \infty} \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N}\right)\nu_k(f)\phi_k.$$

Proof. The vector subspace X contains all the ϕ_k and is closed in $(\dot{\mathcal{C}}^{\mathbb{C}}, \|\cdot\|)$, so it contains $\overline{\mathrm{span}}(\phi_k)_{k \in \mathbb{Z}}$.

To obtain the converse inclusion, we consider $f \in X$ and we write it as

$$f(x) = x\,\theta(\log x)$$

with $\theta : \mathbb{R} \to \mathbb{C}$ a continuous $\log 2$-periodic function. The Fejér theorem ensures that the Fejér sum, namely the Cesàro means of the Fourier series

$$\sigma_N(\theta)(y) := \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\hat{\theta}(k)\mathrm{e}^{\frac{2ik\pi}{\log 2}y} = \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N}\right)\hat{\theta}(k)\mathrm{e}^{\frac{2ik\pi}{\log 2}y}$$

where

$$\hat{\theta}(k) = \frac{1}{\log 2} \int_0^{\log 2} \theta(y) \mathrm{e}^{-\frac{2ik\pi}{\log 2}y} \mathrm{d}y$$

converges uniformly on $\mathbb{R}$ to θ. We deduce that the sequence $(F_N(f))_{N \geqslant 1} \subset$ span$(\phi_k)_{k \in \mathbb{Z}}$ defined by

$$F_N(f)(x) := x \, \sigma_N(\theta)(\log x) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \hat{\theta}(k) \phi_k(x)$$

converges to f in norm $\|\cdot\|$.

To conclude it remains to verify that $\hat{\theta}(k) = \nu_k(f)$. Since $\int_0^\infty x\,\mathcal{U}(x)\mathrm{d}x = 1$ by definition and $\lambda_k \neq \lambda_l$ when $k \neq l$, we have that $\nu_k(\phi_l) = \delta_{kl}$, the Kronecker delta function. We deduce that for any positive integer N

$$\nu_k(F_N(f)) = \begin{cases} 0 \text{ if } N < |k|, \\ \left(1 - \frac{|k|}{N}\right)\hat{\theta}(k) \text{ otherwise.} \end{cases}$$

As a consequence for all $N \geqslant |k|$ we have

$$|\nu_k(f) - \hat{\theta}(k)| \leqslant \|f - F_N(f)\| + \frac{|k|}{N}\|f\|$$

and this gives the desired identity by letting N tend to infinity.

$\square$

We have shown in the proof of Lemma 10 that the Fejér sums F_N can be extended to $\dot{\mathcal{C}}^{\mathbb{C}}$ by setting

$$F_N(f) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right)\nu_k(f)\phi_k.$$

The limit when $N \to \infty$, provided it exists, is a good candidate for defining a relevant projection on X. Using Lemma 9 we prove in the following theorem that the sequence $(F_N(f))_{n \geqslant 1}$ converges in X for any $f \in \mathcal{C}_c^1(0, \infty)$, and that the limit extends into a linear operator $\dot{\mathcal{C}}_0 \to X$ which provides the asymptotic behavior of $(M_t)_{t \geqslant 0}$ on $\mathcal{C}_0$.

Theorem 11. *For any $f \in \mathcal{C}_c^1(0, \infty)$ and any $t \geqslant 0$ the sequence*

$$F_N(\mathrm{e}^{-t}M_t f) = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right)\nu_k(f)\mathrm{e}^{\frac{2i\pi k}{\log 2}t}\phi_k$$

converges in $\dot{\mathcal{C}}$ and the limit $R_t f$ defines a $\log 2$-periodic family of bounded linear operators $R_t : \dot{\mathcal{C}}_0 \to X \cap \dot{\mathcal{C}}$. Moreover for all $f \in \dot{\mathcal{C}}_0$

$$\mathrm{e}^{-t}M_t f - R_t f \xrightarrow[t \to \infty]{} 0$$

locally uniformly on $(0, \infty)$.

Notice that R_0 is actually a projector from $\dot{\mathcal{C}}_0 \oplus X$ onto X.

Proof. We know from Corollary 4 that $\mathrm{e}^{-t}M_t$ is a contraction for $\|\cdot\|$. Let $f \in \mathcal{C}_c^1(0,\infty)$. We have $\mathcal{A}f \in \dot{\mathcal{C}}$ and so $\partial_t(\mathrm{e}^{-t}M_t f) = M_t(\mathcal{A}f - f)$ is bounded in time in $\dot{\mathcal{C}}$. Since $x\partial_x M_t f(x) = \partial_t M_t f(x) - B(x)(2M_t f(x/2) - M_t f(x))$ and B is locally bounded we deduce that $\mathrm{e}^{-t}\partial_x M_t f$ is locally bounded on $(0,\infty)$ uniformly in $t \geqslant 0$. So the Arzela-Ascoli theorem ensures that there exists a subsequence of $(\mathrm{e}^{-t-n\log 2}M_{t+n\log 2}f(x))_{n\geqslant 0}$ which converges locally uniformly on $[0,\infty) \times (0,\infty)$ to a limit $h(t,x)$. We now use Lemma 9 to identify this limit. The dissipation of entropy for the convex function $H(x) = x^2$, denoted D^2, reads

$$D^2[f] = \int_0^\infty xB(x)\mathcal{U}(x)\left|\frac{f(x/2)}{x/2} - \frac{f(x)}{x}\right|^2 \mathrm{d}x.$$

The general relative entropy inequality in Lemma 9 guarantees that

$$\int_0^\infty D^2[\mathrm{e}^{-t}M_t f]\mathrm{d}t < +\infty$$

and as a consequence, for all $T > 0$,

$$\int_0^T D^2[\mathrm{e}^{-t-n\log 2}M_{t+n\log 2}f]\mathrm{d}t = \int_{n\log 2}^{T+n\log 2} D^2[\mathrm{e}^{-t}M_t f]\mathrm{d}t \xrightarrow[n\to\infty]{} 0.$$

From the Cauchy-Schwarz inequality we deduce that

$$\frac{\mathrm{e}^{-t-n\log 2}M_{t+n\log 2}f(x/2)}{x/2} - \frac{\mathrm{e}^{-t-n\log 2}M_{t+n\log 2}f(x)}{x} \to 0$$

in the distributional sense on $(0,\infty)^2$, and since $\mathrm{e}^{-t-n\log 2}M_{t+n\log 2}f(x)$ converges locally uniformly to $h(t,x)$ we get that for all $t \geqslant 0$ and $x > 0$

$$\frac{h(t,x/2)}{x/2} - \frac{h(t,x)}{x} = 0.$$

This means that $h(t,\cdot) \in X$ for all $t \geqslant 0$, and Lemma 10 then ensures that

$$h(t,\cdot) = \lim_{N\to\infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right)\nu_k(h(t,\cdot))\phi_k.$$

Since by definition of $\mathcal{U}_k$ we have $\nu_k M_t = \mathrm{e}^{\lambda_k t}\nu_k$, the dominated convergence theorem yields

$$\nu_k(h(t,\cdot)) = \lim_{n\to\infty} \mathrm{e}^{-t-n\log 2}(\nu_k M_{t+n\log 2})f = \mathrm{e}^{\frac{2ik\pi}{\log 2}t}\nu_k(f)$$

and so

$$h(t,\cdot) = \lim_{N\to\infty} \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right)\nu_k(f)\mathrm{e}^{\frac{2ik\pi}{\log 2}t}\phi_k = \lim_{N\to\infty} F_N(\mathrm{e}^{-t}M_t f).$$

This guarantees that $(F_N(M_t f))_{N \geqslant 1}$ is convergent in $\dot{\mathcal{C}}$. Its limit denoted by $R_t f$ clearly defines a linear operator $R_t : \mathcal{C}_c^1(0, \infty) \to X \cap \dot{\mathcal{C}}$. Moreover by local uniform convergence of $\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f$ to $R_t f$ we get that

$$\|R_t f\| \leqslant \limsup_{n \to \infty} \|\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f\| \leqslant \|f\|.$$

Thus R_t is bounded and it extends uniquely to a contraction $\dot{\mathcal{C}}_0 \to X \cap \dot{\mathcal{C}}$. The local uniform convergence of $\mathrm{e}^{-t-n\log 2} M_{t+n\log 2} f(x)$ to $R_t f(x)$ for $f \in \mathcal{C}_c^1(0, \infty)$ also guarantees the local uniform convergence of $\mathrm{e}^{-t} M_t f - R_t f$ to zero when $t \to +\infty$. Indeed, letting K be a compact set of $(0, \infty)$ and defining for all $t \geqslant 0$ the integer part $n := \left\lfloor \frac{t}{\log 2} \right\rfloor$, so that $t' := t - k\log 2 \in [0, \log 2]$, one has

$$\sup_{x \in K} |\mathrm{e}^{-t} M_t f(x) - R_t f(x)| = \sup_{x \in K} |\mathrm{e}^{-(n\log 2 + t')} M_{n\log 2 + t'} f(x) - R_{t'} f(x)|$$

$$\leqslant \sup_{x \in K} \sup_{s \in [0, \log 2]} |\mathrm{e}^{-(n\log 2 + s)} M_{n\log 2 + s} f(x) - R_s f(x)|.$$

This convergence extends to any $f \in \dot{\mathcal{C}}_0$ by density. $\qquad\square$

We now give a consequence of Theorem 11 in terms of mean ergodicity. Since the limit is $\log 2$-periodic we expect by taking the mean in time of the semigroup to get alignment on the Perron solution.

Corollary 12. *For any $f \in \dot{\mathcal{C}}_0$ the two mappings*

$$t \mapsto \frac{1}{\log 2} \int_t^{t+\log 2} \mathrm{e}^{-s} M_s f \, \mathrm{d}s \qquad and \qquad t \mapsto \frac{1}{t} \int_0^t \mathrm{e}^{-s} M_s f \, \mathrm{d}s$$

converge locally uniformly to $\nu_0(f)\phi_0$ when t tends to infinity.

Proof. Let $f \in \mathcal{C}_c^1(0, \infty)$. On the one hand, since $\mathrm{e}^{-t} M_t$ and R_t are contractions in $\dot{\mathcal{C}}$ and $\mathrm{e}^{-t} M_t f - R_t f$ tends to zero locally uniformly, we have by dominated convergence theorem the local uniform convergence

$$\frac{1}{\log 2} \int_t^{t+\log 2} \mathrm{e}^{-s} M_s f \, \mathrm{d}s - \frac{1}{\log 2} \int_t^{t+\log 2} R_s f \, \mathrm{d}s \xrightarrow[t \to \infty]{} 0.$$

On the other hand, due to the convergence

$$\left\| R_s f - \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \nu_k(f) \mathrm{e}^{\frac{2i\pi k}{\log 2} s} \phi_k \right\| \xrightarrow[N \to \infty]{} 0$$

we have that for all $t \geqslant 0$

$$\frac{1}{\log 2} \int_t^{t+\log 2} R_s f \mathrm{d}s = \nu_0(f)\phi_0.$$

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This proves the convergence of the first integral of the lemma for $f \in \mathcal{C}_c^1(0, \infty)$, which remains valid for $f \in \dot{\mathcal{C}}_0$ by density. As a consequence the Cesàro means

$$\frac{1}{N} \sum_{n=0}^{N-1} \int_{n \log 2}^{(n+1) \log 2} \mathrm{e}^{-s} M_s f \, \mathrm{d}s = \frac{1}{N \log 2} \int_0^{N \log 2} \mathrm{e}^{-s} M_s f \, \mathrm{d}s$$

also converges to $\nu_0(f)\phi_0$ locally uniformly when $N \to \infty$, and it implies the convergence of the second mapping in the lemma. $\qquad\square$

Finally, we transpose by duality the convergence results of the right semigroup to the left one. Due to the Riesz representation $\dot{\mathcal{M}} \simeq \dot{\mathcal{C}}_0'$, we can define a $\log 2$-periodic contraction semigroup R_t on $\dot{\mathcal{M}}$ by setting for all $\mu \in \dot{\mathcal{M}}$ and all $f \in \dot{\mathcal{C}}_0$

$$(\mu R_t)(f) := \mu(R_t f).$$

Then we readily deduce from Theorem 11 and Corollary 12 the following convergence results.

Corollary 13. *For all $\mu \in \dot{\mathcal{M}}$ we have when $t \to +\infty$*

$$\mathrm{e}^{-t} \mu M_t - \mu R_t \overset{*}{\rightharpoonup} 0,$$

$$\frac{1}{\log 2} \int_t^{t + \log 2} \mathrm{e}^{-s} \mu M_s \, \mathrm{d}s \overset{*}{\rightharpoonup} \mu(\phi_0)\nu_0,$$

$$\frac{1}{t} \int_0^t \mathrm{e}^{-s} \mu M_s \, \mathrm{d}s \overset{*}{\rightharpoonup} \mu(\phi_0)\nu_0.$$

5   Discussion and future work

In this work, we investigated how the cyclic asymptotic behavior of the rescaled solutions of Equation (1) exhibited in [8] is transposed in the measure setting. Despite the absence of Hilbert structure, we managed to build a suitable projection on the boundary spectral subspace by taking advantage of the general relative entropy of the dual equation. It allowed us to obtain the weak-* convergence of the rescaled measure solutions to a periodic behavior. The question whether it can be strengthened into a strong convergence, for the weighted total variation norm or a weaker one such as the bounded Lipschitz norm [34], is a challenging natural fit for a continuation of the present work.

In [38], more general growth rates than linear are considered, namely those satisfying $g(2x) = 2g(x)$. Our method would work in this case, replacing the weight x by the corresponding dual eigenfunction $\phi(x)$ and the space X by the functions such that $f(2x)/\phi(2x) = f(x)/\phi(x)$. However, considering such general coefficients, while interesting from mathematical point of view, is not motivated by modeling concerns, that is why we decided to focus on the linear case. In addition, it makes computations lighter, in particular those of the flow which is explicitly given by an exponential when $g(x) = x$.

Our method would also apply to more sophisticated models of mitosis. For instance the equation considered in [36] exhibits a similar countable family of boundary eigenelements for the singular mitosis kernel. To the prize of additional technicalities, our approach can be used to study its long time behavior.

References

[1] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[2] D. Balagué, J. A. Cañizo, and P. Gabriel. Fine asymptotics of profiles and relaxation to equilibrium for growth-fragmentation equations with variable drift rates. *Kinet. Relat. Models*, 6(2):219–243, 2013.

[3] J. Banasiak, K. Pichór, and R. Rudnicki. Asynchronous exponential growth of a general structured population model. *Acta Appl. Math.*, 119:149–166, 2012.

[4] V. Bansaye, B. Cloez, and P. Gabriel. Ergodic behavior of non-conservative semigroups via generalized doeblin's conditions. *Acta Appl. Math.*, 2019. https://doi.org/10.1007/s10440-019-00253-5.

[5] V. Bansaye, B. Cloez, P. Gabriel, and A. Marguet. A non-conservative Harris' ergodic theorem. Preprint, arXiv:1903.03946.

[6] J.-B. Bardet, A. Christen, A. Guillin, F. Malrieu, and P.-A. Zitt. Total variation estimates for the TCP process. *Electron. J. Probab.*, 18(10):1–21, 2013.

[7] G. I. Bell and E. C. Anderson. Cell growth and division: I. A Mathematical Model with Applications to Cell Volume Distributions in Mammalian Suspension Cultures. *Biophys. J.*, 7(4):329–351, 1967.

[8] E. Bernard, M. Doumic, and P. Gabriel. Cyclic asymptotic behaviour of a population reproducing by fission into two equal parts. *Kinet Relat. Models*, 12(3):551–571, 2019.

[9] E. Bernard and P. Gabriel. Asynchronous exponential growth of the growth-fragmentation equation with unbounded fragmentation rate. *J. Evol. Equ.*, 2019. https://doi.org/10.1007/s00028-019-00526-4.

[10] J. Bertoin. On a feynman-kac approach to growth-fragmentation semigroups and their asymptotic behaviors. *J. Funct. Anal.*, 2019. https://doi.org/10.1016/j.jfa.2019.06.012.

[11] J. Bertoin and A. R. Watson. The strong Malthusian behavior of growth-fragmentation processes. Preprint, arXiv:1901.07251.

[12] J. Bertoin and A. R. Watson. A probabilistic approach to spectral analysis of growth-fragmentation equations. *J. Funct. Anal.*, 274(8):2163–2204, 2018.

[13] F. Bouguet. A probabilistic look at conservative growth-fragmentation equations. In *Séminaire de Probabilités XLIX*, volume 2215 of *Lecture Notes in Math.*, pages 57–74. Springer, Cham, 2018.

[14] J. Broda, A. Grigo, and N. P. Petrov. Convergence rates for semistochastic processes. *Discrete Contin. Dyn. Syst. Ser. B*, 24(1):109–125, 2019.

[15] J. A. Cañizo, J. A. Carrillo, and S. Cuadrado. Measure solutions for some models in population dynamics. *Acta Appl. Math.*, 123:141–156, 2013.

[16] J. A. Cañizo and H. Yoldaş. Asymptotic behaviour of neuron population models structured by elapsed-time. *Nonlinearity*, 32(2):464–495, 2019.

[17] M. J. Cáceres, J. A. Cañizo, and S. Mischler. Rate of convergence to self-similarity for the fragmentation equation in L^1 spaces. *Comm. Appl. Ind. Math.*, 1(2):299–308, 2010.

[18] M. J. Cáceres, J. A. Cañizo, and S. Mischler. Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations. *J. Math. Pures Appl.*, 96(4):334–362, 2011.

[19] J. Carrillo, R. Colombo, P. Gwiazda, and A. Ulikowska. Structured populations, cell growth and measure valued balance laws. *J. Differential Equations*, 252(4):3245–3277, 2012.

[20] B. Cavalli. On a family of critical growth-fragmentation semigroups and refracted Lévy processes. *Acta Appl. Math.*, 2019. https://doi.org/10.1007/s10440-019-00261-5.

[21] D. Chafaï, F. Malrieu, and K. Paroux. On the long time behavior of the TCP window size process. *Stochastic Process. Appl.*, 120(8):1518–1534, 2010.

[22] B. Cloez. Limit theorems for some branching measure-valued processes. *Adv. in Appl. Probab.*, 49(2):549–580, 2017.

[23] T. Debiec, M. Doumic, P. Gwiazda, and E. Wiedemann. Relative entropy method for measure solutions of the growth-fragmentation equation. *SIAM J. Math. Anal.*, 50(6):5811–5824, 2018.

[24] G. Derfel, B. van Brunt, and G. Wake. A cell growth model revisited. *Funct. Differ. Equ.*, 19(1-2):75–85, 2012.

[25] O. Diekmann, H. Heijmans, and H. Thieme. On the stability of the cell size distribution. *J. Math. Biol.*, 19:227–248, 1984.

[26] M. Doumic and M. Escobedo. Time asymptotics for a critical case in fragmentation and growth-fragmentation equations. *Kinet. Relat. Models*, 9(2):251–297, 2016.

[27] M. Doumic and P. Gabriel. Eigenelements of a General Aggregation-Fragmentation Model. *Math. Models Methods Appl. Sci.*, 20(5):757–783, 2010.

[28] M. Doumic, M. Hoffmann, N. Krell, and L. Robert. Statistical estimation of a growth-fragmentation model observed on a genealogical tree. *Bernoulli*, 21:1760–1799, 2015. 46 pages, 4 figures.

[29] M. Doumic and B. van Brunt. Explicit solution and fine asymptotics for a critical growth-fragmentation equation. *ESAIM Proc. Surveys*, 62:30–42, 2018.

[30] G. Dumont and P. Gabriel. The mean-field equation of a leaky integrate-and-fire neural network: measure solutions and steady states. Preprint, arXiv:1710.05596.

[31] M. Escobedo, S. Mischler, and M. Rodriguez Ricard. On self-similarity and stationary problem for fragmentation and coagulation models. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(1):99–125, 2005.

[32] J. H. M. Evers, S. C. Hille, and A. Muntean. Mild solutions to a measure-valued mass evolution problem with flux boundary conditions. *J. Differential Equations*, 259(3):1068–1097, 2015.

[33] J. H. M. Evers, S. C. Hille, and A. Muntean. Measure-valued mass evolution problems with flux boundary conditions and solution-dependent velocities. *SIAM J. Math. Anal.*, 48(3):1929–1953, 2016.

[34] J. Z. Farkas, P. Gwiazda, and A. Marciniak-Czochra. Asymptotic behaviour of a structured population model on a space of measures. Preprint, arXiv:1902.06096.

[35] P. Gabriel. Measure solutions to the conservative renewal equation. *ESAIM Proc. Surveys*, 62:68–78, 2018.

[36] P. Gabriel and H. Martin. Steady distribution of the incremental model for bacteria proliferation. *Netw. Heterog. Media*, 14(1):149–171, 2019.

[37] P. Gabriel and F. Salvarani. Exponential relaxation to self-similarity for the superquadratic fragmentation equation. *Appl. Math. Lett.*, 27:74–78, 2014.

[38] G. Greiner and R. Nagel. Growth of cell populations via one-parameter semigroups of positive operators. In *Mathematics applied to science (New Orleans, La., 1986)*, pages 79–105. Academic Press, Boston, MA, 1988.

[39] P. Gwiazda, T. Lorenz, and A. Marciniak-Czochra. A nonlinear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients. *Journal of Differential Equations*, 248(11):2703–2735, jun 2010.

[40] A. J. Hall and G. C. Wake. Functional-differential equations determining steady size distributions for populations of cells growing exponentially. *J. Austral. Math. Soc. Ser. B*, 31(4):434–453, 1990.

[41] H. J. A. M. Heijmans. On the stable size distribution of populations reproducing by fission into two unequal parts. *Math. Biosci.*, 72(1):19–50, 1984.

[42] H. J. A. M. Heijmans. An eigenvalue problem related to cell growth. *J. Math. Anal. Appl.*, 111(1):253–280, 1985.

[43] P. Laurençot and B. Perthame. Exponential decay for the growth-fragmentation/cell-division equation. *Comm. Math. Sci.*, 7(2):503–510, 2009.

[44] F. Malrieu. Some simple but challenging Markov processes. *Ann. Fac. Sci. Toulouse Math. (6)*, 24(4):857–883, 2015.

[45] A. Marguet. A law of large numbers for branching markov processes by the ergodicity of ancestral lineages. *ESAIM Probab. Stat.*, 2018. https://doi.org/10.1051/ps/2018029.

[46] A. Marguet. Uniform sampling in a structured branching population. *Bernoulli*, to appear.

[47] P. Michel. Existence of a solution to the cell division eigenproblem. *Math. Models Methods Appl. Sci.*, 16(supp01):1125–1153, 2006.

[48] P. Michel, S. Mischler, and B. Perthame. General relative entropy inequality: an illustration on growth models. *J. Math. Pures Appl. (9)*, 84(9):1235 – 1260, 2005.

[49] S. Mischler and J. Scher. Spectral analysis of semigroups and growth-fragmentation equations. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 33(3):849–898, 2016.

[50] P. Monmarché. On $\mathcal{H}^1$ and entropic convergence for contractive PDMP. *Electron. J. Probab.*, 20(128):1–30, 2015.

[51] K. Pakdaman, B. Perthame, and D. Salort. Adaptation and fatigue model for neuron networks and large time asymptotics in a nonlinear fragmentation equation. *J. Math. Neurosci.*, 4(1):14, 2014.

[52] B. Perthame and L. Ryzhik. Exponential decay for the fragmentation or cell-division equation. *J. Differential Equations*, 210(1):155–177, mar 2005.

[53] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[54] R. Rudnicki and K. Pichór. Markov semigroups and stability of the cell maturity distribution. *J. Biol. Syst.*, 08(01):69–94, 2000.

[55] J. Sinko and W. Streifer. A model for populations reproducing by fission. *Ecology*, 52(2):330–335, 1971.

[56] B. van Brunt, A. Almalki, T. Lynch, and A. Zaidi. On a cell division equation with a linear growth rate. *ANZIAM J.*, 59(3):293–312, 2018.

[57] G. F. Webb and A. Grabosch. Asynchronous exponential growth in transition probability models of the cell cycle. *SIAM J. Math. Anal.*, 18(4):897–908, 1987.

[58] A. A. Zaidi, B. Van Brunt, and G. C. Wake. Solutions to an advanced functional partial differential equation of the pantograph type. *Proc. A.*, 471(2179):20140947, 15, 2015.