On a lemma of Jacques-Louis Lions and its relation to other fundamental results

Chérif Amrouche\textsuperscript{a}, Philippe G. Ciarlet\textsuperscript{b,⁎}, Cristinel Mardare\textsuperscript{c}

\textsuperscript{a} Laboratoire de Mathématiques et de leurs Applications – Pau, UMR-CNRS 5142, Université de Pau et des Pays de l’Adour, Avenue de l’Université – BP 1155, 64013 PAU CEDEX, France
\textsuperscript{b} Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kouloon, Hong Kong
\textsuperscript{c} Laboratoire Jacques-Louis Lions, UMR-CNRS 7598, Sorbonne Universités, Université Pierre et Marie Curie, 4 Place Jussieu, 75005 Paris, France

\textbf{A R T I C L E H I S T O R Y}

Received 6 August 2014
Available online 20 November 2014

\textbf{MSC:}
46E05
74B05

\textbf{Keywords:}
J.L. Lions lemma
Poincaré lemma
de Rham theorem
Néčas inequality
Bogovskii’s proof of the surjectivity of the divergence operator

\textbf{A B S T R A C T}

Let \( \Omega \) be a domain in \( \mathbb{R}^N \), i.e., a bounded and connected open subset of \( \mathbb{R}^N \) with a Lipschitz-continuous boundary \( \partial \Omega \), the set \( \Omega \) being locally on the same side of \( \partial \Omega \). A fundamental lemma, due to Jacques-Louis Lions, provides a characterization of the space \( L^2(\Omega) \), as the space of all distributions on \( \Omega \) whose gradient is in the space \( H^{-1}(\Omega) \). This lemma, which provides in particular a short proof of a crucial inequality due to J. Nečas, is also a key for proving other basic results, such as, among others, the surjectivity of the divergence operator acting from \( H^1_0(\Omega) \) into \( L^2(\Omega) \), a “weak” form of the Poincaré lemma or a “simplified version” of de Rham theorem, each of which provides sufficient conditions insuring that a vector field in \( H^{-1}(\Omega) \) is the gradient of a function in \( L^2(\Omega) \).

The main objective of this paper is to establish an “equivalence theorem”, which asserts that J.L. Lions lemma is in effect equivalent to a number of other fundamental properties, which include in particular the ones mentioned above. The key for proving this theorem is a specific “approximation lemma”, itself one of these equivalent results, which appears to be new to the best of our knowledge.

Some of these equivalent properties can be given an independent, i.e., “direct”, proof, such as for instance the constructive proof by M.E. Bogovskii of the surjectivity of the divergence operator. Therefore, the proof of any one of such properties provides, by way of our equivalence theorem, a means of proving J.L. Lions lemma, the known “direct” proofs of which for a general domain are notoriously difficult.

© 2014 Elsevier Masson SAS. All rights reserved.

\textbf{R É S U M É}

Soit \( \Omega \) un domaine de \( \mathbb{R}^N \), c’est-à-dire un ouvert borné et connexe de \( \mathbb{R}^N \), de frontière \( \partial \Omega \) lipschitzienne, l’ouvert \( \Omega \) étant localement d’un même côté de \( \partial \Omega \). Un lemme fondamental, dû à Jacques-Louis Lions, fournit une caractérisation de l’espace \( L^2(\Omega) \) comme l’espace de toutes les distributions sur \( \Omega \) dont le gradient est dans l’espace \( H^{-1}(\Omega) \). Ce lemme, qui fournit en particulier une démonstration courte d’une inégalité cruciale due à J. Nečas, est aussi à la base de la démonstration

⁎ Corresponding author.

E-mail addresses: cherif.amrouche@univ-pau.fr (C. Amrouche), mapgc@cityu.edu.hk (P.G. Ciarlet), mardare@ann.jussieu.fr (C. Mardare).

http://dx.doi.org/10.1016/j.matpur.2014.11.007
0021-7824/© 2014 Elsevier Masson SAS. All rights reserved.
d’autres résultats fondamentaux, tels que, entre autres, la surjectivité de l’opérateur divergence agissant de $H^{1}_0(\Omega)$ dans $L^2_0(\Omega)$, une version “faible” du lemme de Poincaré ou une “version simplifiée” du théorème de de Rham, qui fournissent chacune des conditions suffisantes pour qu’un champ de vecteurs dans $H^{-1}(\Omega)$ soit le gradient d’une fonction de $L^2(\Omega)$.

L’objet principal de cet article est d’établir un “théorème d’équivalence”, qui montre que le lemme de J.L. Lions est en fait équivalent à un certain nombre d’autres propriétés fondamentales, qui incluent en particulier celles mentionnées ci-dessus. La clé pour démontrer ce théorème est un “lemme d’approximation” spécifique, qui constitue lui-même l’un de ces résultats équivalents et qui semble nouveau à notre connaissance.

Certaines de ces propriétés équivalentes peuvent être démontrées indépendamment, c’est-à-dire par une démonstration “directe”, telle que, par exemple, la démonstration constructive par M.E. Bogovskii de la surjectivité de l’opérateur divergence. Par suite, la démonstration de chacune de ces propriétés fournit, par le biais de notre théorème d’équivalence, une démonstration du lemme de J.L. Lions, dont les preuves “directes” connues pour un domaine général sont notoirement difficiles.

© 2014 Elsevier Masson SAS. All rights reserved.

1. Introduction

All the notations not defined in this introduction are defined in Section 2.

A domain $\Omega$ in $\mathbb{R}^N$ is a bounded and connected open subset of $\mathbb{R}^N$ whose boundary $\partial \Omega$ is Lipschitz-continuous, the set $\Omega$ being locally on the same side of $\partial \Omega$, in the sense of Nečas [31] or Adams [1]. This means that, given any point $x \in \partial \Omega$, there exists an open neighborhood $V_x$ of $x$ such that $V_x \cap \Omega$ is congruent to a set that lies below the graph of a Lipschitz-continuous function of $(N - 1)$ variables; since $\partial \Omega$ is compact, there thus exists a finite number of such neighborhoods $V_x$ whose union constitutes an open covering of $\partial \Omega$.

Let $\Omega$ be a domain in $\mathbb{R}^N$. The classical J.L. Lions lemma asserts that any distribution in the space $H^{-1}(\Omega)$ with a gradient (in the sense of distributions) in the space $H^{-1}(\Omega)$ is a function in $L^2(\Omega)$. This result was first established in 1958 for domains with smooth boundaries by Jacques-Louis Lions, as stated in footnote number 27 in a paper by Magenes & Stampacchia [27]; its first published proof appeared in 1972; cf. Theorem 3.2 in Chapter 3 of Duvaut & Lions [20]. Then Tartar [36] gave another proof in 1978, again for domains with smooth boundaries. Note that the proofs of Jacques-Louis Lions and Luc Tartar are “independent” ones, in the sense that they do not rely on any one of the other equivalent results that will be discussed in the present paper.

The first proof of the classical J.L. Lions lemma for a “general” domain, i.e., whose boundary is only Lipschitz-continuous appeared in 1986 in Geymonat & Suquet [21]. Their proof crucially relies on an inequality published in 1965 in a paper by Nečas [30]. This inequality, accordingly referred to as Nečas inequality, asserts the existence of a constant $C_0(\Omega)$ such that

$$\|f\|_{L^2(\Omega)} \leq C_0(\Omega)(\|f\|_{H^{-1}(\Omega)} + \|\text{grad } f\|_{H^{-1}(\Omega)})$$

for all $f \in L^2(\Omega)$.

The proof in [21] also relies on a useful functional analytic result, due to Peetre [32] and Tartar [36], referred to for this reason as the Peetre–Tartar lemma (for the statement and proof of this lemma, see, e.g., Theorem 2.1 in Chapter 1 of Girault & Raviart [22]).

That the assumption $f \in H^{-1}(\Omega)$ in the statement of the classical J.L. Lions lemma can be replaced by the weaker assumption $f \in \mathcal{D}'(\Omega)$ was first established in 1990 by Borchers & Sohr [8] for a general domain $\Omega$; their proof crucially relies on the surjectivity of the operator

$$\text{div} : H^1_0(\Omega) \to L^2_0(\Omega) = \left\{ f \in L^2(\Omega); \int_\Omega f \, dx = 0 \right\},$$
a property first essentially proved in 1969 in Section 2.1 in Chapter 1 of Ladyzhenskaya [24] for domains in $\mathbb{R}^3$ with a boundary of class $W^{2,\infty}$; see also Ladyzhenskaya & Solonnikov [25]. Then Bogovskii [7] gave a constructive proof of this surjectivity that works for a general domain in $\mathbb{R}^N$. When the boundary of $\Omega$ is smooth, a concise and elegant constructive proof was found by Dacorogna [18]; see also Dacorogna [19].

We shall call J.L. Lions lemma this stronger version of the classical J.L. Lions lemma, which thus asserts that any distribution in the space $\mathcal{D}'(\Omega)$ with a gradient in the space $H^{-1}(\Omega)$ is a function in $L^2(\Omega)$.

All the known “direct” proofs of both forms of J.L. Lions lemma, of Nečas inequality, or of the surjectivity of $\text{div} : H^1_0(\Omega) \to L^2_0(\Omega)$, have in common that they are delicate, especially for general domains.

J.L. Lions lemma, whether in its classical or in its general form, is a fundamental result, as it provides a key for proving various other fundamental results. For instance, it is well known that it is the keystone of a particularly elegant and concise proof of Korn’s inequality, as first shown in Theorem 3.1 in Chapter 3 of [20]; as shown in the proof of Theorem 6.14-1 in [15], it also provides a short proof of Nečas inequality, which in turn provides short proofs of the surjectivity of the operator $\text{div} : H^1_0(\Omega) \to L^2_0(\Omega)$ and of a “coarse” version of de Rham’s theorem, asserting that a vector-valued distribution $h \in H^{-1}(\Omega)$ is the gradient of a scalar function if (and clearly only if)

$$H^{-1}(\Omega)[h, v]_{H^1_0(\Omega)} = 0 \quad \text{for all } v \in H^1_0(\Omega) \quad \text{that satisfy } \text{div } v = 0 \quad \text{in } \Omega;$$

as shown in [3], J.L. Lions lemma is the basis for various characterizations of matrix fields as linearized strain tensor fields.

The main objective of this paper is to establish (Theorem 3.1) that all the above properties, viz., the classical and the “general” J.L. Lions lemmas, Nečas inequality, the above “coarse” version of de Rham’s theorem, and the surjectivity of the operator $\text{div} : H^1_0(\Omega) \to L^2_0(\Omega)$, are in effect equivalent properties. The key for establishing this equivalence is an approximation lemma (Theorem 3.1(f)), itself one of these equivalent properties, and which is new to the best of our knowledge.

In addition, we show (Theorem 4.1) that J.L. Lions lemma is also equivalent to a “simplified” version of de Rham’s theorem found in Theorem 2.3 in Chapter 1 of Girault & Raviart [22], asserting that, given a domain $\Omega$ in $\mathbb{R}^N$, a vector-valued distribution $h \in H^{-1}(\Omega)$ is the gradient of a scalar function in the space $L^2(\Omega)$ if (and clearly only if)

$$H^{-1}(\Omega)[h, \varphi]_{H^1_0(\Omega)} = 0 \quad \text{for all } \varphi \in \mathcal{D}(\Omega) \quad \text{that satisfy } \text{div } \varphi = 0 \quad \text{in } \Omega.$$

Finally, we show (Theorem 5.1) that J.L. Lions lemma is also equivalent to a “weak” version of Poincaré’s lemma, due to Ciarlet & Ciarlet Jr. [16], asserting that, given a simply-connected domain $\Omega$ in $\mathbb{R}^N$, a vector-valued distribution $h \in H^{-1}(\Omega)$ is the gradient of a scalar function in the space $L^2(\Omega)$ if (and clearly only if)

$$\text{curl } h = 0 \quad \text{in } H^{-2}(\Omega).$$

The elegant proof of this equivalence, which is due to Kesavan [23], will be included here for the sake of completeness.

Some of these equivalent properties can be given an independent, i.e., “direct”, proof such as for instance Nečas inequality (see Nečas [30,31], or Bramble [11]), or the surjectivity of $\text{div} : H^1_0(\Omega) \to L^2_0(\Omega)$ (see Bogovskii [7]). Therefore, the proof of any such property provides, by way of our equivalence theorem, a means of proving J.L. Lions lemma.

For simplicity, the results of this paper are established with functions and distributions in the spaces $L^2(\Omega), H^1_0(\Omega)$, and $H^{-m}(\Omega), m = 1, 2$, but it should be clear that they hold as well if these spaces are respectively replaced with the spaces $L^p(\Omega), W^{1,p}_0(\Omega)$, and $W^{-m,q}(\Omega)$, for any $1 < p < +\infty$ and $q := \frac{p}{p-1}$; in this direction, see Amrouche & Girault [4], who established the following generalized J.L. Lions lemma,
with a generalized Nečas inequality and the Peetre–Tartar lemma as their points of departure: Let $\Omega$ be a domain in $\mathbb{R}^N$. Then, given any $1 < p < \infty$ and any integer $m \in \mathbb{Z}$,

$$f \in \mathcal{D}'(\Omega) \text{ and } \nabla f \in W^{m-1,p} \implies f \in W^{m,p}(\Omega).$$

Some of the above results were announced in Amrouche, Ciarlet & C. Mardare [2].

2. Notations and preliminaries

This section briefly reviews the notations and the main functional analytic results used in this article. Most of these results are recorded simply for convenience, as they are well-known. But the proofs of some other results, called here “auxiliary lemmas” (cf. Theorems 2.1 to 2.3), do not seem to be easy to locate in the literature; for this reason, their proofs will be briefly sketched.

Throughout this paper, $N$ designates a fixed integer $\geq 2$ and Latin indices range in the set $\{1, 2, \ldots, N\}$ unless otherwise specified, such as for instance when they are used for indexing sequences. All the vector spaces considered are over $\mathbb{R}$.

Given two vector spaces $V$ and $W$ and a linear operator $A : V \to W$, the notation $\text{Ker} A$ designates the kernel of $A$ and the notation $\text{Im} A$ designates the range of $A$, i.e., the image of $V$ under $A$. If $V$ and $W$ are normed vector spaces, $\mathcal{L}(V; W)$ designates the space formed by all continuous linear operators from $V$ to $W$.

The notation $V'$ designates the dual space of a topological vector space $V$ and $\langle \cdot, \cdot \rangle_V$ designates the duality between $V'$ and $V$. Given a subspace $Z$ of a normed vector space $V$,

$$Z^0 := \{ v' \in V' ; \langle v', z \rangle_V = 0 \text{ for all } z \in Z \}$$

designates the polar set of $Z$; if $V$ is a Hilbert space, $Z^\perp$ designates the orthogonal complement of $Z$.

Given two normed vector spaces $V$ and $W$, the dual operator $A' \in \mathcal{L}(W'; V')$ of an operator $A \in \mathcal{L}(V; W)$ satisfies

$$\langle A' w', v \rangle_V = \langle w', Av \rangle_W \text{ for all } v \in V \text{ and all } w' \in W'.$$

Recall that the operators $A$ and $A'$ are related through the well-known Banach closed range theorem (see, e.g., Yosida [39], Rudin [34], Brezis [12], or Ciarlet [15]), which asserts in particular that, given two Banach spaces $V$ and $W$ and an operator $A \in \mathcal{L}(V; W)$, the following three conditions are equivalent:

(a) The operator $A$ has a closed range, i.e., $\text{Im} A$ is a closed subspace of $W$.

(b) The dual operator $A' \in \mathcal{L}(W'; V')$ has a closed range, i.e., $\text{Im} A'$ is a closed subspace of $V'$.

(c) $\text{Im} A' = (\text{Ker} A)^0 = \{ v' \in V' ; \langle v', v \rangle_V = 0 \text{ for all } v \in V \text{ such that } Av = 0 \}$.

Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $x = (x_i)$ be a generic point in $\Omega$. Partial derivative operators of the first order, in the classical sense or in the sense of distributions, are denoted $\partial_i := \partial/\partial x_i$. The space of functions that are indefinitely differentiable in $\Omega$ and have compact supports in $\Omega$ is denoted $\mathcal{D}(\Omega)$ and the space of distributions on $\Omega$ is denoted $\mathcal{D}'(\Omega)$. If $f \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, we also use the shorter notation $\langle f, \varphi \rangle := \mathcal{D}'(\Omega) \langle f, \varphi \rangle_{\mathcal{D}(\Omega)}$.

Recall that the space $\mathcal{D}(\Omega)$ is equipped with its natural inductive limit topology, which makes it a topological vector space. In this topology, a sequence $(\varphi_n)_{n=1}^\infty$ of functions $\varphi_n \in \mathcal{D}(\Omega)$ converges to a function $\varphi \in \mathcal{D}(\Omega)$ if and only if there exists a compact subset $K$ of $\Omega$ such that

$$\text{supp } \varphi \subset K \text{ and } \text{supp } \varphi_n \subset K \text{ for all } n \geq 1,$$

and, for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ with $|\alpha| := \sum_{i=1}^N \alpha_i \geq 0$,

$$\sup_{x \in K} |\partial^\alpha \varphi_n(x) - \partial^\alpha \varphi(x)| \to 0 \text{ as } n \to \infty.$$
Note that these convergences are not meant to be uniform with respect to all multi-indices $|\alpha|$ with $|\alpha| \geq 0$.

The space $\mathcal{D}'(\Omega)$ is then defined as the dual space of $\mathcal{D}(\Omega)$, and it can be shown (see, e.g., Vo Khac Khoan [38] for detailed proofs) that a linear form $f : \mathcal{D}(\Omega) \to \mathbb{R}$ belongs to $\mathcal{D}'(\Omega)$ if and only if one of the two following equivalent conditions is satisfied:

Either, given any compact subset $K$ of $\Omega$, there exist a constant $C(K)$ and an integer $m(K) \geq 0$ such that

$$|f(\varphi)| \leq C(K) \max_{|\alpha| \leq m(K)} \sup_{x \in K} |\partial^\alpha \varphi(x)| \quad \text{for all } \varphi \in \mathcal{D}(\Omega) \text{ with } \text{supp} \varphi \subset K;$$

or, given any sequence $(\varphi_n)_{n=1}^{\infty}$ of functions $\varphi \in \mathcal{D}(\Omega)$ that converges to a function $\varphi \in \mathcal{D}(\Omega)$ in the topology of the space $\mathcal{D}(\Omega)$,

$$f(\varphi_n) \to f(\varphi) \quad \text{as } n \to \infty.$$ 

If, given a distribution $f \in \mathcal{D}'(\Omega)$, there exists a function $v_f \in L^1_{\text{loc}}(\Omega)$ such that

$$f(\varphi) = \int_{\Omega} v_f \varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega),$$

the distribution $f \in \mathcal{D}'(\Omega)$ is then said “to be a function in $L^1_{\text{loc}}(\Omega)$”, in the sense that $f$ is identified with the function $v_f$.

Given a distribution $f \in \mathcal{D}'(\Omega)$ and any multi-index $\alpha$ with $|\alpha| \geq 0$, the distribution $\partial^\alpha f$ is defined by

$$\partial^\alpha f(\varphi) = (-1)^{|\alpha|} f(\partial^\alpha \varphi) \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Spaces of functions and vector fields defined over $\Omega$, or of distributions and vector-valued distributions, are respectively denoted by italic capitals and boldface Roman capitals.

The gradient operator $\text{grad} : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is defined for each $f \in \mathcal{D}'(\Omega)$ by $\text{grad } f = (\partial_i f)$, or equivalently by

$$\mathcal{D}'(\Omega) \langle \text{grad } f, \varphi \rangle_{\mathcal{D}(\Omega)} := \mathcal{D}'(\Omega) \langle f, \text{div } \varphi \rangle_{\mathcal{D}(\Omega)} \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

If the open subset $\Omega$ is connected, any distribution $f \in \mathcal{D}'(\Omega)$ such that $\text{grad } f = 0$ in $\mathcal{D}'(\Omega)$ can be identified with a constant function (for a proof, see Schwartz [35] when $\Omega = \mathbb{R}^N$, or, e.g., Boyer & Fabrie [9] in the general case).

Given any open subset $\Omega$ of $\mathbb{R}^N$, any distribution $f \in L^1_{\text{loc}}(\Omega)$ that satisfies $\Delta f = 0$ in $\mathcal{D}'(\Omega)$ belongs to the space $C^\infty(\Omega)$: This is a special case of the so-called hypo-ellipticity of the Laplace operator (for a proof, see, e.g., Theorem 6.4-2 in [15]).

For any integer $m \geq 1$, the notations $H^m(\Omega)$ and $H^m_0(\Omega)$ designate the usual Sobolev spaces, and the notation $H^{-m}(\Omega)$ designates the dual space of $H^m_0(\Omega)$ endowed with the norm of $H^m(\Omega)$. Finally, we define the space

$$L^2_0(\Omega) := \left\{ f \in L^2(\Omega) : \int_{\Omega} f \, dx = 0 \right\}.$$

The mapping $\text{grad} : L^2_0(\Omega) \to H^{-1}(\Omega)$ satisfies

$$H^{-1}(\Omega) \langle \text{grad } f, v \rangle_{H^1_0(\Omega)} = - \int_{\Omega} f \text{ div } v \, dx \quad \text{for all } f \in L^2_0(\Omega) \text{ and all } v \in H^1_0(\Omega).$$
This shows that the operator $\text{grad} : L^2_0(\Omega) \to H^{-1}(\Omega)$ is the dual operator of $- \text{div} : H^1_0(\Omega) \to L^2_0(\Omega)$. Also note that, if the open set $\Omega$ is connected, the operator $\text{grad} : L^2_0(\Omega) \to H^{-1}(\Omega)$ is one-to-one.

The curl operator $\text{curl} : \mathcal{D}'(\Omega; \mathbb{R}^{N(N-1)/2})$ is defined for each $\mathbf{h} = (h_i) \in \mathcal{D}'(\Omega)$ by

$$(\text{curl} \mathbf{h})_{ij} = \partial_j h_i - \partial_i h_j \quad \text{for each } i < j.$$ 

Finally, we will also use the classical Poincaré lemma, which asserts that, given a simply-connected open subset $\Omega$ of $\mathbb{R}^N$ and a vector field $\mathbf{h} \in \mathcal{C}^1(\Omega)$, there exists a function $p \in \mathcal{C}^2(\Omega)$ such that

$$\text{grad} p = \mathbf{h} \quad \text{in } \Omega$$

if (and clearly only if)

$$\text{curl} \mathbf{h} = 0 \quad \text{in } \Omega$$

(for a proof, see, e.g., Theorem 6.17-2 in [15]).

All the above results are well-known and their proofs found at many places. By contrast, three specific properties of domains in $\mathbb{R}^N$, also crucially needed in the sequel, seem to be often taken for granted and accordingly given without proof in the literature. These results are assembled below as three “auxiliary lemmas”. For brevity, their sometimes lengthy proofs are only sketched.

Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$ and, given $x \in \mathbb{R}^N$ and $r > 0$, let $B(x; r) := \{y \in \mathbb{R}^N; |y-x| < r\}$. An open subset of $\mathbb{R}^N$ is starlike with respect to an open ball $B(x; r)$ if, for each $z \in \Omega$, the convex hull of the set $\{z\} \cup B(x; r)$ is contained in the set $\Omega$.

**Theorem 2.1** (first auxiliary lemma). Let $\Omega$ be a domain in $\mathbb{R}^N$. Then there exists a finite number of domains $\Omega_i$ in $\mathbb{R}^N$, $i \in I$, each one contained in $\Omega$ and starlike with respect to an open ball, such that $\Omega = \bigcup_{i \in I} \Omega_i$.

**Sketch of proof.** A domain $\Omega$ in $\mathbb{R}^N$ (the definition of a domain is given at the beginning of the introduction) possesses the cone property, i.e., each point $z \in \Omega$ is the vertex of a cone, which is the convex hull of a set of the form $\{z\} \cup B(x; r)$ for some ball $B(x; r) \subset \Omega$, and all these cones are congruent to each other. In particular, the radius $r > 0$ can be chosen independently of $z \in \Omega$. The conclusion then follows from Lemma 2 in Section 1.1.9 of Maz’ya [29]; see also Costabel & McIntosh [17] for a similar approach. \[\square\]

The detailed proof of the next result is due to Boyer & Fabrie [10]; see also Ball & Zarnescu [6] for closely related ideas. Note that the class $\mathcal{C}^\infty$ of the boundaries $\partial \Omega_j$, $j \geq 1$, found in the next theorem will not be used here; that they be Lipschitz-continuous is sufficient for our subsequent purposes.

**Theorem 2.2** (second auxiliary lemma). Let $\Omega$ be a domain in $\mathbb{R}^N$. Then there exists domains $\Omega_j$ in $\mathbb{R}^N$, $j \geq 1$, with the following properties:

$$\partial \Omega_j \text{ is of class } \mathcal{C}^\infty \text{ and } \Omega_j \subset \Omega_{j+1} \subset \Omega \text{ for each } j \geq 1, \text{ and } \Omega = \bigcup_{j=1}^\infty \Omega_j.$$ 

**Sketch of proof.** The main idea consists in using the regularized signed distance $\rho(\cdot, \partial \Omega) : \mathbb{R}^N \to \mathbb{R}$ to the boundary $\partial \Omega$ introduced by Lieberman [26] as the implicit function $\rho(\cdot, \partial \Omega) : \mathbb{R}^N \to \mathbb{R}$ defined by the relation

$$\rho(x, \partial \Omega) = \int_{B(0; 1)} d\left(x + \frac{\rho(x, \partial \Omega)}{2} z, \partial \Omega\right) \omega(z) \, dz, \quad x \in \mathbb{R}^N,$$
where $\omega : \mathbb{R}^N \to \mathbb{R}$ is any function possessing the following properties:

$$
\omega(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N, \quad \text{supp} \omega = B(0,1), \quad \text{and} \quad \int_{\mathbb{R}^N} \omega \, dx = 1,
$$

and $d(\cdot, \partial \Omega) : \mathbb{R}^N \to \mathbb{R}$ is the signed distance to $\partial \Omega$, i.e.,

$$
d(x, \partial \Omega) := \text{dist}(x, \partial \Omega) := \inf_{y \in \partial \Omega} |x - y| \quad \text{if } x \in \Omega, \quad \text{and} \quad d(x, \partial \Omega) := -\text{dist}(x, \partial \Omega) \quad \text{if } x \notin \Omega.
$$

It can then be shown that

$$
\rho(\cdot, \partial \Omega) \in C^\infty(\mathbb{R}^N - \partial \Omega),
$$

and that there exist constants $C_i > 0$, $1 \leq i \leq 4$, such that

$$
C_1 \leq \frac{d(x, \partial \Omega)}{\rho(x, \partial \Omega)} \leq C_2 \quad \text{for all } x \in \mathbb{R}^N - \partial \Omega,
$$

$$
|\nabla \rho(x, \partial \Omega)| \geq C_3 \quad \text{for all } x \in \{y \in \Omega; d(y, \partial \Omega) < C_4\}.
$$

One then defines the open sets

$$
\Omega_j := \left\{ x \in \Omega; \rho(x, \partial \Omega) > \frac{1}{j} \right\}
$$

for each integer $j \geq 1$. While it easily follows that $\partial \Omega_j$ is of class $C^\infty$, that $\overline{\Omega}_j \subset \Omega_{j+1} \subset \Omega$ for each $j \geq 1$, and that the set $\Omega_j$ is locally on the same side of $\partial \Omega_j$, the connectedness of $\Omega_j$ is more delicate to verify; in fact, it holds only for $j$ large enough, as can be seen by analyzing the flow along the gradient of the regularized distance to $\partial \Omega$. □

**Theorem 2.3** (third auxiliary lemma). Let $\Omega$ be a domain in $\mathbb{R}^N$. Then there exists a finite number of simply-connected domains $\Omega_i$ in $\mathbb{R}^N$, $i \in I$, such that $\Omega = \bigcup_{i \in I} \Omega_i$.

**Sketch of proof.** Given any point $x \in \Omega$, there exists a simply-connected domain $\Omega_x$ such that $x \subset \Omega_x \subset \Omega$, namely an open ball $B(x; r)$ for $r > 0$ small enough. Given any point $y \in \partial \Omega$, there exists an open neighborhood $U_y$ of $y$ such that $U_y \cap \Omega$ is a simply-connected domain (for instance, $U_y \cap \Omega$ can be defined as the set of points of $\Omega$ that lie below the graph of the Lipschitz-continuous function that defines $\partial \Omega \cap U_y$; then $U_y \cap \Omega$ is simply-connected as it is homeomorphic to the unit cube). It then suffices to take a finite subcovering of $\overline{\Omega} \subset (\bigcup_{x \in \Omega} \Omega_x) \cup (\bigcup_{y \in \partial \Omega} U_y)$. □

3. The equivalence theorem

We now establish an “equivalence theorem” (cf. Theorem 3.1 below), which shows that seven different results, called (a), (b), . . . , (g), where (a) stands for the classical J.L. Lions lemma and (g) stands for the “general” J.L. Lions lemma, are in effect all equivalent. With the classical J.L. Lions Lemma as its point of departure, our proof consists in establishing a series of implications, such as “(a) implies (b)”, etc., the collection of which constitutes the equivalence theorem. Naturally, special care must be exercised so as not to use at some part of the proof any result not yet established!

The proofs of the implications “(a) implies (b)”, . . . , “(d) implies (e)” essentially follow those of Theorems 6.14-1 and 6.14-2 in [15], which altogether avoid the use of the Peetre–Tartar lemma. The implications
“(e) implies the approximation lemma (f)” and “(f) implies (g)” are new to the best of our knowledge. For clarity, the proofs of some implications are broken into several parts, numbered (i), (ii), etc.

**Theorem 3.1.** Let \( \Omega \) be a domain in \( \mathbb{R}^N \). The following statements (a), (b), \ldots, (g) are equivalent:

(a) **Classical J.L. Lions lemma:**

\[
  f \in H^{-1}(\Omega) \quad \text{and} \quad \nabla f \in H^{-1}(\Omega) \quad \text{implies} \quad f \in L^2(\Omega).
\]

(b) **J. Nečas inequality:** There exists a constant \( C_0(\Omega) \) such that

\[
  \|f\|_{L^2(\Omega)} \leq C_0(\Omega)(\|f\|_{H^{-1}(\Omega)} + \|\nabla f\|_{H^{-1}(\Omega)}) \quad \text{for all} \quad f \in L^2(\Omega).
\]

(c) **The operator \( \nabla \) has a closed range:** The image of the space

\[
  L^2_0(\Omega) := \left\{ f \in L^2(\Omega); \int_{\Omega} f \, dx = 0 \right\}
\]

under the operator \( \nabla \) is a closed subspace of \( H^{-1}(\Omega) \).

(d) **Coarse version of de Rham’s theorem:** Given a vector-valued distribution \( h \in H^{-1}(\Omega) \), there exists a function \( p \in L^2_0(\Omega) \) such that

\[
  \nabla p = h \quad \text{in} \quad H^{-1}(\Omega)
\]

if and only if

\[
  H^{-1}(\Omega) \langle h, v \rangle_{H^1_0(\Omega)} = 0 \quad \text{for all} \quad v \in H^1_0(\Omega) \quad \text{that satisfy} \quad \text{div} \, v = 0 \quad \text{in} \quad \Omega.
\]

If this is the case, the function \( p \in L^2_0(\Omega) \) is uniquely determined.

(e) **The operator \( \text{div} \) is onto:** The operator

\[
  \text{div} : H^1_0(\Omega) \to L^2_0(\Omega)
\]

is onto. Consequently, for each \( f \in L^2_0(\Omega) \), there exists a unique element \( u_f \in (\text{Ker} \, \text{div})^\perp \subset H^1_0(\Omega) \) such that

\[
  \text{div} \, u_f = f,
\]

and there exists a constant \( C_1(\Omega) \) such that the linear operator \( f \in L^2_0(\Omega) \to u_f \in (\text{Ker} \, \text{div})^\perp \) defined in this fashion satisfies

\[
  \|u_f\|_{H^1(\Omega)} \leq C_1(\Omega)\|f\|_{L^2(\Omega)} \quad \text{for all} \quad f \in L^2_0(\Omega).
\]

(f) **Approximation lemma:** Assume that the domain \( \Omega \) is starlike with respect to an open ball. Then there exists a constant \( C_2(\Omega) \) such that, given any function \( \varphi \) in the space

\[
  \mathcal{D}_0(\Omega) := \left\{ \varphi \in \mathcal{D}(\Omega); \int_{\Omega} \varphi \, dx = 0 \right\} \subset L^2_0(\Omega),
\]

there exist vector fields \( v_n = v_n(\varphi) \in \mathcal{D}(\Omega), n \geq 1, \) such that
\[ \|v_n\|_{H^1(\Omega)} \leq C_2(\Omega)\|\varphi\|_{L^2(\Omega)} \quad \text{for all } n \geq 1 \quad \text{and} \quad \text{div}\, v_n \to \varphi \quad \text{in } D(\Omega) \quad \text{as } n \to \infty. \]

(g) J.L. Lions lemma:

\[ f \in D'(\Omega) \quad \text{and} \quad \text{grad} \, f \in H^{-1}(\Omega) \quad \text{implies} \quad f \in L^2(\Omega). \]

Proof. (a) implies (b): To begin with, we show that the space

\[ V(\Omega) := \{ f \in H^{-1}(\Omega); \quad \text{grad} \, f \in H^{-1}(\Omega) \}, \]

equipped with the norm

\[ f \in V(\Omega) \to (\|f\|_{H^{-1}(\Omega)} + \|\text{grad} \, f\|_{H^{-1}(\Omega)}), \]

is complete. Let \( (f_k) \) be a Cauchy sequence in this space; then there exist \( f \in H^{-1}(\Omega) \) and \( h \in H^{-1}(\Omega) \) such that

\[ f_k \to f \quad \text{in } H^{-1}(\Omega) \quad \text{and} \quad \text{grad} \, f_k \to h \quad \text{in } H^{-1}(\Omega), \]

since, as dual spaces, the spaces \( H^{-1}(\Omega) \) and \( H^{-1}(\Omega) \) are complete.

Given any vector field \( \varphi \in D(\Omega), \)

\[ H^{-1}(\Omega) \langle \text{grad} \, f_k, \varphi \rangle_{H^1(\Omega)} = -H^{-1}(\Omega) \langle f_k, \text{div} \, \varphi \rangle_{H^1(\Omega)} \quad \text{for each } k \geq 1 \]

so that letting \( k \to \infty \) gives

\[ H^{-1}(\Omega) \langle h, \varphi \rangle_{H^1(\Omega)} = -H^{-1}(\Omega) \langle f, \text{div} \, \varphi \rangle_{H^1(\Omega)}, \]

which shows that \( h = \text{grad} \, f \). Hence \( V(\Omega) \) is complete.

The canonical injection \( i : L^2(\Omega) \to V(\Omega) \) is one-to-one, continuous (there clearly exists a constant \( c(\Omega) \) such that \( \|f\|_{H^{-1}(\Omega)} + \|\text{grad} \, f\|_{H^{-1}(\Omega)} \leq c(\Omega) \|f\|_{L^2(\Omega)} \) for all \( f \in L^2(\Omega) \)), and onto by the classical J.L. Lions lemma. Hence the inverse mapping \( i^{-1} : V(\Omega) \to L^2(\Omega) \) is also continuous by Banach open mapping theorem; there thus exists a constant \( C_0(\Omega) \) such that

\[ \|f\|_{L^2(\Omega)} \leq C_0(\Omega)(\|f\|_{H^{-1}(\Omega)} + \|\text{grad} \, f\|_{H^{-1}(\Omega)}), \]

i.e., Nečas inequality holds.

(b) implies (c): To show that the image of the space \( L^2_0(\Omega) \) under the operator \( \text{grad} \) is closed in \( H^{-1}(\Omega) \), it suffices to show that there exists a constant \( C(\Omega) \) such that

\[ \|f\|_{L^2(\Omega)} \leq C(\Omega) \|\text{grad} \, f\|_{H^{-1}(\Omega)} \quad \text{for all } f \in L^2_0(\Omega). \]

Assume that this property does not hold. Then there exist functions \( f_k \in L^2_0(\Omega) \) such that

\[ \|f_k\|_{L^2(\Omega)} = 1 \quad \text{for all } k, \quad \text{and} \quad \|\text{grad} \, f_k\|_{H^{-1}(\Omega)} \to 0 \quad \text{as } k \to \infty. \]

Since the sequence \( (f_k) \) is then bounded in \( L^2(\Omega) \), there exists a subsequence \( (f_\ell) \) that converges in \( H^{-1}(\Omega) \) (the injection from \( L^2(\Omega) \) into \( H^{-1}(\Omega) \) is compact; cf., e.g., Theorem 6.11-3 in [15]); therefore, \( (f_\ell) \) is a Cauchy sequence in \( H^{-1}(\Omega) \). Besides, \( \langle \text{grad} \, f_\ell \rangle \) is also a Cauchy sequence, this time in \( H^{-1}(\Omega) \), since \( \langle \text{grad} \, f_k \rangle \) is a convergent sequence (to \( 0 \)) in this space.
Něcas inequality then implies that \((f_\ell)\) is a Cauchy sequence in \(L^2(\Omega)\). So, let \(f \in L^2(\Omega)\) be such that
\[
 f_\ell \to f \quad \text{in } L^2(\Omega) \text{ as } \ell \to \infty.
\]
Since the mapping \(f \in L^2(\Omega) \to \text{grad } f \in H^{-1}(\Omega)\) is continuous,
\[
 \text{grad } f_\ell \to \text{grad } f = 0 \quad \text{in } H^{-1}(\Omega) \text{ as } \ell \to \infty.
\]
Hence the function \(f\) is a constant (the open set \(\Omega\) is connected by assumption), and this constant is 0 since \(f_\ell \in L^2(\Omega)\), \(\ell \geq 1\), implies that \(f \in L^1(\Omega)\). But this contradicts the relation \(\|f_\ell\|_{L^2(\Omega)} = 1\) for all \(\ell\). Hence the operator \(\text{grad } : L^2(\Omega) \to H^{-1}(\Omega)\) has a closed range.

(c) is equivalent to (d): Since \(\text{grad } : L^2(\Omega) \to H^{-1}(\Omega)\) is the dual operator of \(-\text{div } : H^1_0(\Omega) \to L^2(\Omega)\), Banach closed range theorem asserts that \(\text{Im } \text{grad}\) is a closed subspace of \(H^{-1}(\Omega)\) if and only if
\[
 \text{Im } \text{grad} = (\ker \text{div})^0 = \{ h \in H^{-1}(\Omega); \langle h, v \rangle_{\Omega} = 0 \text{ for all } v \in H^1_0(\Omega) \text{ that satisfy } \text{div } v = 0 \text{ in } \Omega \},
\]
which is exactly what the coarse version of de Rham’s theorem asserts.

(d) implies (e): Again by Banach closed range theorem,
\[
 \text{Im } \text{div} = (\ker \text{grad})^0,
\]
and \(\ker \text{grad} = \{0\}\) since the mapping \(\text{grad } : L^2(\Omega) \to H^{-1}(\Omega)\) is one-to-one. Hence \(\text{Im } \text{div} = L^2(\Omega)\). In other words, the mapping \(\text{div } : (\ker \text{div})^\perp \to L^2(\Omega)\), which is clearly linear, continuous, and one-to-one, is onto, and thus its inverse is also continuous by Banach open mapping theorem. Hence (e) is proved.

(e) implies (f): (i) We are now given a domain \(\Omega\) that is starlike with respect to a ball of radius \(r\), the center of which will be assumed without loss of generality to be at the origin, and we are given a function \(\varphi\) in the space
\[
 \mathcal{D}_0(\Omega) := \{ \varphi \in \mathcal{D}(\Omega); \int_{\Omega} \varphi \, dx = 0 \} \subset L^2(\Omega).
\]
The objective is to construct a sequence \((v_n)\) of vector fields \(v_n = v_n(\varphi) \in \mathcal{D}(\Omega)\) such that
\[
 \|v_n\|_{H^1(\Omega)} \leq C_2(\Omega)\|\varphi\|_{L^2(\Omega)} \quad \text{for all } n \geq 1, \quad \text{and } \quad \text{div } v_n \to \varphi \quad \text{in } \mathcal{D}(\Omega) \text{ as } n \to \infty,
\]
with a constant \(C_2(\Omega)\) independent of this function \(\varphi \in \mathcal{D}_0(\Omega)\).

(i) Definition of the auxiliary vector fields \(u_n = u_n(\varphi) \in H^1(\mathbb{R}^N), n \geq n_0, \) where \(n_0\) denotes the smallest integer \(\geq 1\) that satisfies \(n_0 > \frac{r}{\delta}\). That the operator \(\text{div } : H^1_0(\Omega) \to L^2(\Omega)\) is onto implies that there exists a vector field \(u = u(\varphi) \in (\ker \text{div})^\perp \subset H^1_0(\Omega)\) such that
\[
 \text{div } u = \varphi \quad \text{in } \Omega \quad \text{and } \quad \|u\|_{H^1(\Omega)} \leq C_1(\Omega)\|\varphi\|_{L^2(\Omega)},
\]
where the constant \(C_1(\Omega)\) is independent of the function \(\varphi \in \mathcal{D}_0(\Omega)\). Let \(w = w(\varphi)\) denote the extension of \(u\) by \(0\) on \(\mathbb{R}^N - \Omega\), so that
\[
 w \in H^1(\mathbb{R}^N), \quad \|w\|_{H^1(\mathbb{R}^N)} = \|u\|_{H^1(\Omega)} \leq C_1(\Omega)\|\varphi\|_{L^2(\Omega)},
\]
\[
 \text{div } w = \varphi \quad \text{in } \Omega \quad \text{and } \quad \text{div } w = 0 \quad \text{in } \mathbb{R}^N - \Omega.
\]
Let $n_0 \geq 1$ denote the smallest integer that satisfies $n_0 > \frac{2}{r}$, and for each integer $n \geq n_0$, let

$$\lambda_n := 1 - \frac{2}{nr} \quad \text{and} \quad \Omega_n := \{ \lambda_n x \in \mathbb{R}^N; \ x \in \Omega \} \subset \Omega.$$ 

Then the assumption that $\Omega$ is starlike with respect to the ball $B(0; r)$ combined with Thales theorem implies that

for each $n \geq n_0$, $\text{dist}(x, \partial \Omega) > \frac{2}{n}$ for all $x \in \Omega_n$.

For each $n \geq n_0$, let

$$u_n := \lambda_n w\left(\frac{\cdot}{\lambda_n}\right) \in H^1(\mathbb{R}^N),$$

where, for each $n \geq n_0$, the notation $w(\frac{\cdot}{\lambda_n})$ designates the vector field $x \in \mathbb{R}^N \to w\left(\frac{x}{\lambda_n}\right)$. We thus have, for each $n \geq n_0$,

$$u_n \in H^1(\mathbb{R}^N), \quad u_n = 0 \quad \text{on} \quad \mathbb{R}^N - \Omega_n, \quad \text{and} \quad \text{div} \ u_n = \varphi\left(\frac{\cdot}{\lambda_n}\right) \quad \text{in} \quad \mathbb{R}^N,$$

where $\varphi(\frac{\cdot}{\lambda_n})$ designates the function defined by $\varphi\left(\frac{x}{\lambda_n}\right)$ if $x \in \Omega_n$ and by $0$ if $x \in \mathbb{R}^N - \Omega_n$.

(ii) Definition of the vector fields $v_n = v_n(\varphi) \in D(\Omega)$, $n \geq n_0$. Given a function $\rho \in C^\infty(\mathbb{R}^N)$ that satisfies

$$\rho \geq 0 \quad \text{in} \quad \mathbb{R}^N, \quad \rho(x) = 0 \quad \text{if} \ |x| \geq 1, \quad \int_{\mathbb{R}^N} \rho(x) \, dx = 1,$$

define for each $n \geq 1$ the mollifier $\rho_n \in C^\infty(\mathbb{R}^N)$ by

$$\rho_n : x \in \mathbb{R}^N \to \rho_n(x) := n^N \rho(nx).$$

Since, for each $n \geq n_0$,

$$\text{dist}(x, \partial \Omega) > \frac{2}{n} \quad \text{for all} \ x \in \Omega_n \quad \text{and} \quad \text{supp} \ \rho_n \subset B\left(0; \frac{1}{n}\right),$$

the vector fields $w_n \in C^\infty(\mathbb{R}^N)$ defined for each $n \geq n_0$ as the convolution product

$$w_n := u_n * \rho_n,$$

i.e., by

$$w_n(x) := \int_{B(x; \frac{1}{n})} \rho_n(x - y)u_n(y) \, dy = \int_{B(0; \frac{1}{n})} \rho_n(z)u_n(x - z) \, dz \quad \text{at each} \ x \in \mathbb{R}^N,$$

are such that

$$\text{supp} \ w_n \subset \left\{ x \in \Omega; \ \text{dist}(x, \partial \Omega) > \frac{1}{n} \right\}.$$ 

This shows in particular that the vector fields $v_n : \Omega \to \mathbb{R}^N$ defined for each $n \geq n_0$ by
\[ v_n := w_n |_\Omega \]

satisfy

\[ v_n \in \mathcal{D}(\Omega). \]

Notice that the assumption that \( \Omega \) is starlike with respect to a ball is crucially used here, as it implies that, for each \( n \geq n_0 \),

\[ \text{dist}(x, \partial \Omega) > \frac{2}{n} \text{ for all } x \in \Omega_n. \]

(iii) The vector fields \( v_n \in \mathcal{D}(\Omega) \) satisfy

\[ \| v_n \|_{H^1(\Omega)} \leq C_1(\Omega) \| \varphi \|_{L^2(\Omega)} \text{ for all } n \geq n_0. \]

It is known that, if \( f \in L^2(\mathbb{R}^N) \), then \( f * \rho_n \in L^2(\mathbb{R}^N) \) and \( \| f * \rho_n \|_{L^2(\mathbb{R}^N)} \leq \| f \|_{L^2(\mathbb{R}^N)} \) and that, if \( g \in H^1(\mathbb{R}^N) \), then \( \partial_i (g * \rho_n) = (\partial_i g) * \rho_n \). It therefore follows that

\[ \| u_n \|_{H^1(\Omega)} \leq \| w_n \|_{H^1(\mathbb{R}^N)} \leq \| u_n \|_{H^1(\mathbb{R}^N)} \text{ for all } n \geq n_0. \]

Taking \( y := \frac{x}{\lambda_n} \) as the new variable in the integrals below shows that

\[ \| u_n \|_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \left| \lambda_n \left( \frac{x}{\lambda_n} \right) \right|^2 \, dx + \sum_{i,j} \int_{\mathbb{R}^N} \left| \partial_i w_j \left( \frac{x}{\lambda_n} \right) \right|^2 \, dx \]
\[ = \lambda_n^{N+2} \int_{\mathbb{R}^N} \left| w(y) \right|^2 \, dy + \sum_{i,j} \lambda_n^N \int_{\mathbb{R}^N} \left| \partial_i w_j (y) \right|^2 \, dy \]
\[ \leq \| w \|_{H^1(\mathbb{R}^N)}^2 = \| u \|_{H^1(\Omega)}^2, \]

so that, by (i),

\[ \| v_n \|_{H^1(\Omega)} \leq \| u_n \|_{H^1(\mathbb{R}^N)} \leq \| u \|_{H^1(\Omega)} \leq C_1(\Omega) \| \varphi \|_{L^2(\Omega)} \text{ for all } n \geq n_0. \]

(iv) The vector fields \( v_n \in \mathcal{D}(\Omega), n \geq n_0, \) satisfy

\[ \text{div } v_n \to \varphi \text{ in } \mathcal{D}(\Omega) \text{ as } n \to \infty. \]

The definitions of the vector fields \( u_n \) and \( v_n \) (cf. (i) and (ii)) show that, for each \( n \geq n_0 \),

\[ \text{div } v_n = \text{div } (u_n * \rho_n) = (\text{div } u_n) * \rho_n = \varphi \left( \frac{\cdot}{\lambda_n} \right) * \rho_n \text{ in } \Omega. \]

Since there exists a constant \( \alpha \) such that

\[ d(x, \partial \Omega) \geq \alpha > 0 \text{ for all } x \in \text{supp } \varphi, \]

and since \( \text{supp } \rho_n \subset \overline{B(0, \frac{1}{n})} \), \( n \geq 1 \), it follows that there exist a constant \( \beta \) and an integer \( n_1 \geq n_0 \) such that

\[ d(x, \partial \Omega) \geq \beta > 0 \text{ for all } x \in \text{supp } \left( \varphi \left( \frac{\cdot}{\lambda_n} \right) * \rho_n \right) \cup \text{supp } \varphi \text{ and all } n \geq n_1. \]
Therefore,

$$\text{supp} (\text{div} \nu_n) \cup \text{supp} \varphi \subset K := \{ x \in \Omega; \text{dist}(x, \partial \Omega) \geq \beta \} \quad \text{for all } n \geq n_1.$$ 

A well-known property of mollifiers asserts that, given any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, i.e., with arbitrary integers $\alpha_i \geq 0$,

$$\partial^\alpha \left( \varphi \left( \frac{x}{\lambda_n} \right) \ast \rho_n \right) = \left( \partial^\alpha \varphi \left( \frac{x}{\lambda_n} \right) \right) \ast \rho_n \quad \text{in } \mathbb{R}^N.$$ 

For each $n \geq n_1$, we thus have:

$$\partial^\alpha (\text{div} \nu_n)(x) - \partial^\alpha \varphi(x) = \partial^\alpha \left( \varphi \left( \frac{x}{\lambda_n} \right) \ast \rho_n \right)(x) - \partial^\alpha \varphi(x)$$

$$= \int_{\mathbb{R}^N} \left( \frac{1}{\lambda_n^{|\alpha|}} \partial^\alpha \varphi \left( \frac{x-y}{\lambda_n} \right) - \partial^\alpha \varphi(x) \right) \rho_n(y) \, dy \quad \text{at each } x \in \Omega,$$

which in turn implies that

$$\sup_{x \in K} |\partial^\alpha (\text{div} \nu_n)(x) - \partial^\alpha \varphi(x)| = \sup_{x \in K} \left| \int_{\mathbb{R}^N} \left[ \left( \frac{1}{\lambda_n^{|\alpha|}} - 1 \right) \partial^\alpha \varphi \left( \frac{x-y}{\lambda_n} \right) \rho_n(y) \right] \, dy \right|$$

$$\leq \sup_{x \in \mathbb{R}^N} |\partial^\alpha \varphi(z)| \left( \frac{1}{\lambda_n^{|\alpha|}} - 1 \right)$$

$$+ \sup_{x \in K} \left| \int_{B(0; \frac{1}{n})} (\partial^\alpha \varphi(x + \delta_n(x, y)) - \partial^\alpha \varphi(x)) \rho_n(y) \, dy \right|,$$

where $\delta_n(x, y) := \left( \frac{1 - \lambda_n}{\lambda_n} \right)x - \frac{y}{\lambda_n}$. Hence

$$\sup_{x \in K} \sup_{y \in B(0; \frac{1}{n})} |\delta_n(x, y)| \leq \left| \frac{1 - \lambda_n}{\lambda_n} \right| \sup_{x \in K} |x| + \frac{1}{n |\lambda_n|},$$

so that $\sup_{x \in K} \sup_{y \in B(0; \frac{1}{n})} |\delta_n(x, y)|$ can be made arbitrarily small if $n$ is large enough. Consequently, for each multi-index $\alpha$,

$$\sup_{x \in K} |\partial^\alpha (\text{div} \nu_n)(x) - \partial^\alpha \varphi(x)| \to 0 \quad \text{as } n \to \infty,$$

since the function $\partial^\alpha \varphi$ is uniformly continuous and bounded. Together with the relation $\text{supp}(\text{div} \nu_n) \subset K$ for all $n \geq n_1$, this last relation shows that $\text{div} \nu_n \to \varphi$ in $D'(\Omega)$ as $n \to \infty$.

(f) implies (g): (i) Assume first that the domain $\Omega$ is starlike with respect to an open ball, and let a distribution $f \in D'(\Omega)$ be such that $\text{grad} f \in H^{-1}(\Omega)$. To show that $f \in L^2(\Omega)$, it is enough to show that there exists a constant $C_0(f, \Omega)$ such that

$$|D'(\Omega) \langle f, \varphi \rangle_{D(\Omega)}| \leq C_0(f, \Omega) \| \varphi \|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(\Omega),$$

thanks to the F. Riesz representation theorem and to the density of $D(\Omega)$ in $L^2(\Omega)$. 
Let a function $\varphi_1 \in D(\Omega)$ such that $\int_\Omega \varphi_1 \, dx = 1$ be chosen once and for all. Hence, given any function $\varphi \in D(\Omega)$, the function $\varphi_0 = \varphi_0(\varphi)$ defined by

$$\varphi_0 := \varphi - \left( \int_\Omega \varphi \, dx \right) \varphi_1$$

belongs to the space $D_0(\Omega)$; besides, there clearly exists a constant $C(\Omega, \varphi_1)$ independent of $\varphi$ such that

$$\|\varphi_0\|_{L^2(\Omega)} \leq C(\Omega, \varphi_1) \|\varphi\|_{L^2(\Omega)}.$$

By assumption, there exists a constant $C_1(f, \Omega)$ such that

$$|D'(\Omega)\langle f, \text{div} \psi \rangle_{D(\Omega)}| = |D'(\Omega)\langle \text{grad} f, \psi \rangle_{D(\Omega)}| \leq C_1(f, \Omega)\|\psi\|_{H^1(\Omega)} \quad \text{for all } \psi \in D(\Omega),$$

and by the approximation lemma, there exists a constant $C_2(\Omega)$ and there exist vector fields $\mathbf{v}_n = \mathbf{v}_n(\varphi_0) = \mathbf{v}_n(\varphi) \in D(\Omega)$, $n \geq 1$, such that

$$\text{div} \mathbf{v}_n \to \varphi_0 \quad \text{in } D(\Omega) \quad \text{as } n \to \infty \quad \text{and} \quad \|\mathbf{v}_n\|_{H^1(\Omega)} \leq C_2(\Omega)\|\varphi_0\|_{L^2(\Omega)} \quad \text{for all } n \geq 1.$$

Combining the relations

$$D'(\Omega)\langle f, \varphi \rangle_{D(\Omega)} = D'(\Omega)\langle f, \varphi_0 \rangle_{D(\Omega)} + \left( \int_\Omega \varphi \, dx \right) D'(\Omega)\langle f, \varphi_1 \rangle_{D(\Omega)},$$

$$D'(\Omega)\langle f, \varphi_0 \rangle_{D(\Omega)} = \lim_{n \to \infty} D'(\Omega)\langle f, \text{div} \mathbf{v}_n \rangle_{D(\Omega)},$$

$$|D'(\Omega)\langle f, \text{div} \mathbf{v}_n \rangle_{D(\Omega)}| \leq C_1(f, \Omega)\|\mathbf{v}_n\|_{H^1(\Omega)} \leq C_1(f, \Omega)C_2(\Omega)\|\varphi_0\|_{L^2(\Omega)} \quad \text{for all } n \geq 1,$$

we thus infer that

$$|D'(\Omega)\langle f, \varphi \rangle_{D(\Omega)}| \leq C_0(f, \Omega)\|\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in D(\Omega),$$

where

$$C_0(f, \Omega) := C(\Omega, \varphi_1)C_1(f, \Omega)C_2(\Omega) + (\text{meas } \Omega)^{1/2}|D'(\Omega)\langle f, \varphi_1 \rangle_{D(\Omega)}|,$$

which proves J.L. Lions lemma for domains that are starlike with respect to an open ball.

(ii) Assume next that $\Omega$ is a general domain. Then, by the first auxiliary lemma (Theorem 2.1), there exists a finite number of domains $\Omega_i$, $i \in I$, each one contained in $\Omega$ and starlike with respect to an open ball, such that $\Omega = \bigcup_{i \in I} \Omega_i$.

Let a distribution $f \in D'(\Omega)$ be such that $\text{grad} f \in H^{-1}(\Omega)$, i.e., such that there exists a constant $C_1(f, \Omega)$ such that

$$|D'(\Omega)\langle \text{grad} f, \psi \rangle_{D(\Omega)}| \leq C_1(f, \Omega)\|\psi\|_{H^1(\Omega)} \quad \text{for all } \psi \in D(\Omega).$$

Let $\bar{\theta} \in D(\Omega)$, resp. $\bar{\theta}$, denote the extension of any function $\theta \in D(\Omega_i)$, resp. any vector field $\theta \in D(\Omega_i)$ by 0, resp. 0, on $\Omega - \Omega_i$. For each $i \in I$, define a linear form $f_i : D(\Omega_i) \to \mathbb{R}$ by

$$f_i(\varphi) := D'(\Omega)\langle f, \bar{\varphi} \rangle_{D(\Omega)} \quad \text{for each } \varphi \in D(\Omega_i).$$
Since then $f_i \in \mathcal{D}'(\Omega_i)$, (to see this, apply either one of the two equivalent definitions of a distribution recalled in Section 2), the previous relation can be re-written as

$$\mathcal{D}'(\Omega_i) \langle f_i, \varphi \rangle_{\mathcal{D}(\Omega_i)} := \mathcal{D}'(\Omega) \langle f, \varphi \rangle_{\mathcal{D}(\Omega)} \text{ for each } \varphi \in \mathcal{D}(\Omega_i).$$

Then, for each $\varphi \in \mathcal{D}(\Omega_i)$,

$$\mathcal{D}'(\Omega_i) \langle \text{grad } f_i, \varphi \rangle_{\mathcal{D}(\Omega_i)} := -\mathcal{D}'(\Omega_i) \langle f_i, \text{div } \varphi \rangle_{\mathcal{D}(\Omega_i)} = -\mathcal{D}'(\Omega) \langle f, \text{div } \varphi \rangle_{\mathcal{D}(\Omega)} = \mathcal{D}'(\Omega) \langle \text{grad } f, \varphi \rangle_{\mathcal{D}(\Omega)}.$$

Therefore,

$$\left| \mathcal{D}'(\Omega_i) \langle \text{grad } f_i, \varphi \rangle_{\mathcal{D}(\Omega_i)} \right| \leq C_1(f, \Omega) \| \varphi \|_{H^1(\Omega)} = C_1(f, \Omega) \| \varphi \|_{H^1(\Omega_i)},$$

which shows that $\text{grad } f_i \in H^{-1}(\Omega_i)$.

Then J.L. Lions for a domain that is starlike with respect to an open ball shows that there exist functions $\hat{f}_i \in L^2(\Omega_i)$, $i \in I$, such that

$$\mathcal{D}'(\Omega_i) \langle f_i, \varphi \rangle_{\mathcal{D}(\Omega_i)} = \int_{\Omega_i} \hat{f}_i \varphi \, dx \text{ for all } \varphi \in \mathcal{D}(\Omega_i).$$

Let $i \in I$ and $j \in I$ be given such that $\Omega_i \cap \Omega_j \neq \emptyset$. For each $\varphi \in \mathcal{D}(\Omega_i \cap \Omega_j)$, let $\bar{\varphi} \in \mathcal{D}(\Omega)$ denote the extension of $\varphi$ by 0 on $\Omega - (\Omega_i \cap \Omega_j)$. Then

$$\int_{\Omega_i \cap \Omega_j} (\hat{f}_i - \hat{f}_j) \varphi \, dx = \mathcal{D}'(\Omega_i) \langle f_i, \varphi \rangle_{\mathcal{D}(\Omega_i)} - \mathcal{D}'(\Omega_j) \langle f_j, \varphi \rangle_{\mathcal{D}(\Omega_j)} = \mathcal{D}'(\Omega) \langle f_i, \varphi \rangle_{\mathcal{D}(\Omega)} - \mathcal{D}'(\Omega) \langle f_j, \varphi \rangle_{\mathcal{D}(\Omega)} = 0,$$

which shows that $\hat{f}_i = \hat{f}_j$ on $\Omega_i \cap \Omega_j$. Therefore the function $\hat{f} : \Omega \to \mathbb{R}$ defined by $\hat{f}|_{\Omega_i} = \hat{f}_i$ is well-defined and belongs to the space $L^2(\Omega)$.

Given any function $\varphi \in \mathcal{D}(\Omega)$, let $(\alpha_i)_{i \in I}$ be a partition of unity associated with the open cover $\bigcup_{i \in I} \Omega_i$ of the compact set

$$K := \text{supp } \varphi,$$

i.e., $\alpha_i \in \mathcal{D}(\Omega)$, $\text{supp } \alpha_i \subset \Omega_i$, and $\sum_{i \in I} \alpha_i(x) = 1$ for all $x \in K$. Then

$$\mathcal{D}'(\Omega) \langle f, \varphi \rangle_{\mathcal{D}(\Omega)} = \sum_{i \in I} \mathcal{D}'(\Omega) \langle f, \alpha_i \varphi \rangle_{\mathcal{D}(\Omega)} = \sum_{i \in I} \mathcal{D}'(\Omega_i) \langle f_i, (\alpha_i \varphi)|_{\Omega_i} \rangle_{\mathcal{D}(\Omega_i)}$$

$$= \sum_{i \in I} \int_{\Omega_i} \hat{f}_i(\alpha_i \varphi)|_{\Omega_i} \, dx = \sum_{i \in I} \int_K \hat{f}_i \alpha_i \varphi \, dx = \int_K \left( \sum_{i \in I} \alpha_i \right) \hat{f} \varphi \, dx = \int_{\Omega} \hat{f} \varphi \, dx.$$

Hence the distribution $f \in \mathcal{D}'(\Omega)$ is equal to the function $\hat{f}$, which shows that J.L. Lions lemma holds. (g) implies (a): Clear. □
4. Relation between J.L. Lions lemma and a simplified version of de Rham theorem

We showed in Theorem 3.1 that J.L. Lions lemma is equivalent to, among other properties, a “coarse” version of de Rham theorem; cf. part (d) in ibid. We now show how J.L. Lions lemma is also closely related to a less coarse version of de Rham theorem (“less coarse” in the sense that the divergence-free trial vector fields now belong to the space $\mathcal{D}'(\Omega)$, instead of the larger space $H_0^1(\Omega)$ as in Theorem 3.1(d)), accordingly named “simplified version”. The following proof is in part modeled after that of Theorem 2.3 in Chapter 1 of Girault & Raviart [22] (see also Theorem 3.1.18 in Boyer & Fabrie [9]). Note, however, that the usage of the “general” J.L. Lions lemma considerably shortens the argument.

The original de Rham theorem (cf. [33]) is a deep, and difficult to prove, result in distribution theory, asserting that any vector-valued distribution $h \in \mathcal{D}'(\Omega)$ such that

$$\mathcal{D}'(\Omega)\langle h, \varphi \rangle_{\mathcal{D}(\Omega)} = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega) \text{ that satisfy } \text{div } \varphi = 0 \text{ in } \Omega$$

is the gradient of a distribution; see also [28], where de Rham’s theorem is also analyzed in terms of its “Sobolev space” versions found here.

**Theorem 4.1.** Let $\Omega$ be a domain in $\mathbb{R}^N$. Then J.L. Lions lemma together with the coarse version of de Rham theorem (itself a consequence of J.L. Lions lemma; cf. Theorem 3.1) imply that the following simplified version of de Rham theorem holds: Let there be given a vector field $h \in H^{-1}(\Omega)$ that satisfies

$$H^{-1}(\Omega)\langle h, \varphi \rangle_{H_0^1(\Omega)} = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega) \text{ that satisfy } \text{div } \varphi = 0 \text{ in } \Omega.$$ 

Then there exists a uniquely determined function $p \in L^2_0(\Omega)$ such that

$$\text{grad } p = h \text{ in } H^{-1}(\Omega).$$

Conversely, the simplified version of de Rham theorem implies that J.L. Lions lemma holds.

**Proof.** In what follows, $h \in H^{-1}(\Omega)$ is a given vector-valued distribution such that $H^{-1}(\Omega)\langle h, \varphi \rangle_{H_0^1(\Omega)} = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ that satisfy $\text{div } \varphi = 0$ in $\Omega$.

By the second auxiliary lemma (Theorem 2.2), there exist domains $\Omega_j \subset \Omega$, $j \geq 1$, such that

$$\Omega_j \subset \Omega_{j+1} \text{ and } \overline{\Omega}_j \subset \Omega \text{ for each } j \geq 1, \text{ and } \Omega = \bigcup_{j=1}^\infty \Omega_j.$$ 

For each fixed integer $j \geq 1$, let $v_j$ be any vector field in the space $H_0^1(\Omega_j)$ that satisfies $\text{div } v_j = 0$ in $\Omega_j$, and let $\tilde{v}_j : \mathbb{R}^N \to \mathbb{R}^N$ denote its extension by 0 on $\mathbb{R}^N - \Omega_j$.

Let $(\rho_n)_{n \geq 1}$ be the same family of mollifiers as that constructed in the proof of the implication “(e) implies (f)” in Theorem 3.1. Since $\overline{\Omega}_j \subset \Omega$, there exists an integer $n_0(j)$ and a compact subset $K_j$ of $\Omega$ such that

$$\text{supp}(\tilde{v}_j * \rho_n) \subset K_j, \text{ and therefore } \tilde{v}_j * \rho_n|_{\Omega} \in \mathcal{D}(\Omega), \text{ for all } n \geq n_0(j),$$

$$\text{div}(\tilde{v}_j * \rho_n) = (\text{div } \tilde{v}_j) * \rho_n = 0 \text{ in } \mathbb{R}^N \text{ for all } n \geq n_0(j),$$

$$\lim_{n \to \infty} \|\tilde{v}_j * \rho_n - \tilde{v}_j\|_{H^1(\mathbb{R}^N)} = 0.$$ 

Again for each fixed integer $j \geq 1$, let the vector-valued distribution $h_j \in H^{-1}(\Omega_j)$ be defined as the restriction of $h$ to the space $H_0^1(\Omega_j)$, identified here with a subspace of $H_0^1(\Omega)$. Consequently,
\[ H^{-1}(\Omega) \langle h_j, v_j \rangle_{H^b_0(\Omega)} = H^{-1}(\Omega) \langle h, \tilde{v}_j \rangle_{H^b_0(\Omega)} = \lim_{n \to \infty} H^{-1}(\Omega) \langle h, \tilde{v}_j * \rho_n \rangle_{H^b_0(\Omega)} = 0 \] for all \( v_j \in H^1_0(\Omega) \),

since \( \tilde{v}_j * \rho_n |_{\Omega} \in D(\Omega) \) and \( \text{div} (\tilde{v}_j * \rho_n |_{\Omega}) = 0 \) in \( \Omega \) for all \( n \geq n_0(j) \).

The assumed coarse version of de Rham theorem therefore implies that, for each integer \( j \geq 1 \), there exists a uniquely determined function \( q_j \in L^2_0(\Omega_j) \) such that

\[ \text{grad} q_j = h_j \quad \text{in} \quad H^{-1}(\Omega_j). \]

For each \( j \geq 1 \), let

\[ p_j := q_j - \frac{1}{\text{meas } \Omega_j} \int_{\Omega_j} q_j(x) \, dx. \]

Then

\[ p_j \in L^2(\Omega_j) \quad \text{and} \quad \text{grad} p_j = h_j \quad \text{in} \quad H^{-1}(\Omega_j). \]

Besides, the relations \( \text{grad} p_j = h_j \) in \( H^{-1}(\Omega_j) \) and \( \text{grad} p_{j+1} = h_{j+1} \) in \( H^{-1}(\Omega_{j+1}) \) together imply that \( (p_{j+1} |_{\Omega_j} - p_j) : \Omega_j \to \mathbb{R} \) is a constant function since the open set \( \Omega_j \) is connected. Since \( \Omega_1 \subset \Omega_j \) and \( \int_{\Omega_j} p_j \, dx = 0 \), this constant function vanishes; hence

\[ p_j = p_{j+1} \quad \text{a.e. in } \Omega_j. \]

Given any \( x \in \Omega \), there exists an integer \( j(x) \geq 1 \) such that \( x \in \Omega_j \) for all \( j \geq j(x) \) since \( \Omega_j \subset \Omega_{j+1} \) for all \( j \geq 1 \) and \( \Omega = \bigcup_{j=1}^{\infty} \Omega_j \). Consequently, the function \( \tilde{p} : \Omega \to \mathbb{R} \) unambiguously defined for almost all \( x \in \Omega \) by \( \tilde{p}(x) = p_j(x) \) for any \( j \geq j(x) \) is in the space \( L^2_\text{loc}(\Omega) \).

Given any vector field \( \varphi \in D(\Omega) \), there exists an integer \( j = j(\varphi) \geq 1 \) such that \( \text{supp} \varphi \subset \Omega_j \). Since \( \tilde{p} \in L^2_\text{loc}(\Omega) \) and \( \text{div} \varphi \in D(\Omega) \), we then have

\[ D'(\Omega) \langle \text{grad} \tilde{p}, \varphi \rangle_{D(\Omega)} := -D'(\Omega) \langle \tilde{p}, \text{div} \varphi \rangle_{D(\Omega)} = - \int_{\Omega} \tilde{p} \text{div} \varphi \, dx \]

\[ = - \int_{\Omega_j} p_j (\text{div} \varphi |_{\Omega_j}) \, dx = D'(\Omega_j) \langle \text{grad} p_j, \varphi |_{\Omega_j} \rangle_{D(\Omega_j)} \]

\[ = H^{-1}(\Omega_j) \langle h_j, \varphi |_{\Omega_j} \rangle_{H^b_0(\Omega_j)} = H^{-1}(\Omega) \langle h, \varphi \rangle_{H^b_0(\Omega)} = D'(\Omega) \langle h, \varphi \rangle_{D(\Omega)}, \]

which shows that

\[ \text{grad} \tilde{p} \in H^{-1}(\Omega) \quad \text{and} \quad h = \text{grad} \tilde{p} \quad \text{in} \quad H^{-1}(\Omega), \]

since \( \varphi \in D(\Omega) \) is arbitrary. Finally, J.L. Lions lemma shows that in fact \( \tilde{p} \in L^2(\Omega) \), which in turn implies that

\[ p := \left( \tilde{p} - \frac{1}{\text{meas } \Omega} \int_{\Omega} \tilde{p}(x) \, dx \right) \in L^2_0(\Omega) \quad \text{and} \quad \text{grad} p = h \quad \text{in} \quad H^{-1}(\Omega). \]

Conversely, the simplified version of de Rham theorem obviously implies that its coarse version a fortiori holds, which in turn implies, by way of Theorem 3.1, that J.L. Lions lemma holds. □
5. Relation between J.L. Lions lemma and a weak version of Poincaré’s lemma

Ciarlet & Ciarlet Jr. [16] have shown that the “classical” Poincaré lemma holds as well with substantial weaker assumed regularity on the data, in the sense that the given vector field, denoted $h$ in the next theorem, need only be in the space $H^{-1}(\Omega)$ instead of the space $C^1(\Omega)$ in the classical version. The proof of [16] was then (almost immediately) simplified by Kesavan [23], whose particularly elegant argument is reproduced in the first part of the next proof.

**Theorem 5.1.** Let $\Omega$ be a simply-connected domain in $\mathbb{R}^N$. Then the classical J.L. Lions lemma together with the surjectivity of the operator $\text{div} : H_0^1(\Omega) \rightarrow L_0^2(\Omega)$ (itself a consequence of J.L. Lions lemma; cf. Theorem 3.1) imply that the following weak Poincaré lemma holds: Let there be given a vector field $h \in H^{-1}(\Omega)$ that satisfies

$$\text{curl} \, h = 0 \quad \text{in} \; H^{-2}(\Omega).$$

Then there exists a function $p \in L^2(\Omega)$, uniquely determined up to the addition of a constant, such that

$$\text{grad} \, p = h \quad \text{in} \; H^{-1}(\Omega).$$

Conversely, the weak Poincaré lemma on any simply-connected domain in $\mathbb{R}^N$ implies that J.L. Lions lemma holds on any domain in $\mathbb{R}^N$.

**Proof.** Let a vector-valued distribution $h \in H^{-1}(\Omega)$ be given. Then, thanks to the Babuška–Brezzi inf–sup theorem (cf. Babuška [5] and Brezzi [13]) and to the surjectivity of $\text{div} : H_0^1(\Omega) \rightarrow L_0^2(\Omega)$, the Stokes equations

$$-\Delta u + \text{grad} \lambda = h \quad \text{in} \; H^{-1}(\Omega),$$

$$\text{div} \, u = 0 \quad \text{in} \; L^2(\Omega),$$

possess a solution $(u, \lambda) \in H_0^1(\Omega) \times L^2(\Omega)$ (see, e.g., Chapter 1 in Temam [37], Theorem 5.1 in Chapter 1 of Girault & Raviart [22], Chapter 1 in Brezzi & Fortin [14], or Theorem 6.14-3 in [15]).

The assumption that $\text{curl} \, h = 0$ then implies that

$$\Delta(\text{curl} \, u) = \text{curl} \,(\Delta u) = \text{curl} \, \text{grad} \, \lambda - \text{curl} \, h = 0.$$

Hence $\partial_j u_i - \partial_i u_j = (\text{curl} \, u)_{ij} \in C^\infty(\Omega)$ by the hypo-ellipticity of the Laplace operator (cf. Section 2). Consequently,

$$\sum_j \partial_j(\partial_j u_i - \partial_i u_j) = \Delta u_i - \partial_i(\text{div} \, u) = \Delta u_i \in C^\infty(\Omega).$$

Since $\Delta u \in C^\infty(\Omega)$, $\text{curl} \, \Delta u = 0$ in $\Omega$, and the open set $\Omega$ is simply-connected, the classical Poincaré lemma (cf. Section 2) can be applied, showing that there exists a function $\tilde{p} \in C^\infty(\Omega) \subset \mathcal{D}'(\Omega)$ such that

$$\text{grad} \, \tilde{p} = \Delta u = \text{grad} \, \lambda - h \quad \text{in} \; H^{-1}(\Omega).$$

The distribution $p \in \mathcal{D}'(\Omega)$ defined by

$$p := \lambda - \tilde{p}$$
therefore satisfies

$$\nabla p = \nabla \lambda - \nabla \hat{p} = h \in H^{-1}(\Omega).$$

Consequently, $p \in L^2(\Omega)$ by J.L. Lions lemma.

To prove the converse implication, assume first that the domain $\Omega$ is simply-connected, and let $f \in \mathcal{D}'(\Omega)$ be such that $\nabla f \in H^{-1}(\Omega)$. Since then $\text{curl} \nabla f = 0$, the weak Poincaré lemma implies that there exists a function $p \in L^2(\Omega)$ such that

$$\nabla p = \nabla f.$$

Hence there exists a constant $C$ such that

$$f = p + C \quad \text{in} \ \mathcal{D}'(\Omega),$$

which shows that $f \in L^2(\Omega)$, i.e., that J.L. Lions lemma holds if the domain $\Omega$ is simply-connected.

Assume next that $\Omega$ is a general domain, and let $f \in \mathcal{D}'(\Omega)$ be such that $\nabla f \in H^{-1}(\Omega)$. By the third auxiliary lemma (Theorem 2.3), there exists a finite number of simply-connected domains $\Omega_i$, $i \in I$, such that $\Omega = \bigcup_{i \in I} \Omega_i$. The rest of the proof is then similar to part (ii) of the proof of the implication “(f) implies (g)” in Theorem 3.1.

Acknowledgement

This work was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No. 9041637-CityU 100711].

References