Min-max Theory, Willmore conjecture, and Energy of links

André Neves

(Joint with Fernando Marques)
Q: What is the best way of immersing a sphere in space?

A: The one that minimizes the *bending energy*

\[ \int_{\Sigma} \bar{H}^2 = \int_{\Sigma} \left( \frac{k_1 + k_2}{2} \right)^2. \]
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\[
\int_{\Sigma} \tilde{H}^2 = \int_{\Sigma} \left( \frac{k_1 + k_2}{2} \right)^2.
\]

- bending energy is conformally invariant;

- every compact surface has \( \int_{\Sigma} \tilde{H}^2 \geq 4\pi \) with equality only for round sphere.
What is the best way of immersing a torus in space?
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Conjecture (Willmore, '65)

For every torus $\Sigma$ immersed in $\mathbb{R}^3$, 

$$\int_{\Sigma} \bar{H}^2 \geq \frac{2\pi}{2}.$$
What is the best way of immersing a torus in space?

Conjecture (Willmore, ’65)
For every torus $\Sigma$ immersed in $\mathbb{R}^3$

$$\int_{\Sigma} \bar{H}^2 \geq 2\pi^2.$$
Consider stereographic projection
\[ \pi : S^3 - \{ \text{north pole} \} \longrightarrow \mathbb{R}^3. \]

- If \( \Sigma \subset S^3 \) then
  \[ \int_{\Sigma} (1 + H^2) = \int_{\pi(\Sigma)} \bar{H}^2. \]

- the Willmore energy of \( \Sigma \subset S^3 \) is defined to be
  \[ \mathcal{W}(\Sigma) = \int_{\Sigma} (1 + H^2). \]
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- the Willmore energy of \( \Sigma \subset S^3 \) is defined to be
  \[ W(\Sigma) = \int_{\Sigma} (1 + H^2). \]

**Willmore Conjecture - ’65**

Every torus \( \Sigma \) immersed in \( S^3 \) has \( W(\Sigma) \geq 2\pi^2. \)
Theorem (Marques-N., ’12)

For every embedded surface $\Sigma \subset S^3$ with genus $g \geq 1$

$$\mathcal{W}(\Sigma) \geq 2\pi^2$$

with equality if and only if $\Sigma$ is conformal to $S^1 \left( \frac{1}{\sqrt{2}} \right) \times S^1 \left( \frac{1}{\sqrt{2}} \right)$.

Corollary A

Willmore conjecture holds.

Corollary B

The only two minimal surfaces in $S^3$ with area $\leq 2\pi^2$ are great spheres and Clifford torus.
Links in space

- $\gamma_1$ and $\gamma_2$ two linked curves in $\mathbb{R}^3$. 
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- $\gamma_1$ and $\gamma_2$ two linked curves in $\mathbb{R}^3$.

- $\text{lk}(\gamma_1, \gamma_2)$ is the linking number

  $\text{lk} = 1$

  $\text{lk} = 3$
Links in space

- Möbius cross energy of a link \((\gamma_1, \gamma_2)\) is
  \[
  E(\gamma_1, \gamma_2) = \int_{S^1 \times S^1} \frac{|\gamma_1'(s)||\gamma_2'(t)|}{|\gamma_1(s) - \gamma_2(t)|^2} \, ds \, dt.
  \]

- \(E(\gamma_1, \gamma_2)\) is conformally invariant and
  \[
  E(\gamma_1, \gamma_2) \geq 4\pi |\text{lk} (\gamma_1, \gamma_2)|
  \]
Links in space

Q: What is the best way of immersing a non-trivial link $(\gamma_1, \gamma_2)$ in $\mathbb{R}^3$?

A: The one that minimizes the Möbius energy.
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A: The one that minimizes the Möbius energy.

Conjecture (Freedman-He-Wang, ’94)

If \(\text{lk}(\gamma_1, \gamma_2) = \pm 1\), then \(E(\gamma_1, \gamma_2) \geq 2\pi^2\).

- If true, every non-trivial link has \(E(\gamma_1, \gamma_2) \geq 2\pi^2\).
Theorem (Agol-Marques-N., ’12)

If \( \text{lk}(\gamma_1, \gamma_2) = \pm 1 \), then \( E(\gamma_1, \gamma_2) \geq 2\pi^2 \).
Theorem (Agol-Marques-N., ’12)

If \( \text{lk}(\gamma_1, \gamma_2) = \pm 1 \), then \( E(\gamma_1, \gamma_2) \geq 2\pi^2 \).

If equality then \((\gamma_1, \gamma_2)\) is conformal to the standard Hopf link in \( S^3 \)

\[
\beta_1(t) = (\cos t, \sin t, 0, 0) \quad \text{and} \quad \beta_2(s) = (0, 0, \cos s, \sin s).
\]
Partial results...

- (Willmore ’71, Shioama-Takagi ’70): Conjecture is true for tubes of constant radius around a space curve.
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• (Langevin-Rosenberg ’76): If \( \Sigma \subset \mathbb{R}^3 \) is a knotted torus then \( \mathcal{W}(\Sigma) \geq 8\pi \).

• (Chen, ’83): Conjecture is true for flat tori in 3-sphere.

• (Langer-Singer ’84): Conjecture is true for tori of revolution.
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- (Li-Yau '82): Conjecture is true for a set of conformal classes which contains the conformal class of the Clifford torus.

- (Montiel-Ros '86): Conjecture is true for a larger set of conformal classes than the set given by Li-Yau.
Partial results...

(Benisom-Mutz ’91)

Conjecture is true for observed toroidal vesicles in some cells.

\[ v_{\text{cliff}} = 0.71 \]

\[ v_{\text{red}} = 0.80 \]
Partial results...

- (Simon ’93): There is a torus which has least Willmore energy among all tori.
- (Ros ’99, Topping ’00): Conjecture is true for tori symmetric under the antipodal map.
- (Ros ’00): Conjecture is true for tori symmetric with respect to a point.
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Min-max Theory

- $Z_2(S^3) = \{ \text{integral 2-currents with no boundary in } S^3 \}.$
- $M : Z_2(S^3) \to \mathbb{R}, \quad M(\Sigma) = \text{mass of } \Sigma.$
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$\bullet$ $\phi : I^n \to \mathcal{Z}_2(S^3)$ continuous in flat norm, where $I^n = k$-cube.
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- $[\phi] = \{ \psi \text{ homotopic to } \phi \text{ relative to the boundary (i.e. } \psi|_{\partial l^n} = \phi|_{\partial l^n} ) \}$. 
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- $[\phi] = \{ \psi \text{ homotopic to } \phi \text{ relative to the boundary (i.e. } \psi|_{\partial l^n} = \phi|_{\partial l^n}) \}$
- $L([\phi]) = \inf_{\psi \in [\phi]} \sup_{x \in l^n} M(\psi(x))$. 
Theorem (Pitts, ’81)

Assume

\[ L(\lbrack \phi \rbrack) = \inf_{\psi \in \lbrack \phi \rbrack} \sup_{x \in I^k} M(\psi(x)) > \sup_{x \in \partial I^k} M(\phi(x)), \]

(this means \( \lbrack \phi \rbrack \neq 0 \in \pi_k(\mathbb{Z}_2(S^3), \phi|_{\partial I^k}). \))

There is \( \Sigma \subset S^3 \) smooth embedded minimal surface (with multiplicities) so that

\[ L(\lbrack \phi \rbrack) = M(\Sigma). \]
## Min-max Theory

### Theorem (Pitts, ’81)

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**Conjecture:** \(\text{index}(\Sigma) \leq k.\)
Min-max Theory

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Theorem (Almgren, ’62)

- \(\pi_k(\mathbb{Z}_2(S^3), 0) = 0\) if \(k \neq 1\);
- \(\pi_1(\mathbb{Z}_2(S^3), 0) = \mathbb{Z}\)
Min-max Theory

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Conjecture: \( \text{index}(\Sigma) \leq k. \)

Theorem (Almgren, ’62)
• \( \pi_k(\mathbb{Z}_2(S^3), 0) = 0 \) if \( k \neq 1; \)
• \( \pi_1(\mathbb{Z}_2(S^3), 0) = \mathbb{Z} \) and given \( \Sigma = \partial A \) surface then
\[ \phi_0 : [-\pi, \pi] \to \mathbb{Z}_2(S^3), \quad \phi_0(t) = \partial\{x : \text{dist}(x, \Sigma) < t\} \]
has \( \pi_1(\mathbb{Z}_2(S^3), 0) = \mathbb{Z}[\phi_0]. \)
Min-max Theory

Theorem (Urbano, ’90)

\( \Sigma \subset S^3 \) compact embedded minimal surface with \( \text{index}(\Sigma) \leq 5 \). Then either

- \( \Sigma \) is a great sphere (\( \text{index}(\Sigma) = 1 \) and \( \text{area}(\Sigma) = 4\pi \)), or
- \( \Sigma \) is the Clifford torus (\( \text{index}(\Sigma) = 5 \) and \( \text{area}(\Sigma) = 2\pi^2 \)).
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- \( L([\phi_0]) = 4\pi \) where \( \phi_0 \) = Almgren’s sweepout.
- How to obtain \( \phi \) with \( L([\phi]) = 2\pi^2 \)?
Min-max Theory

- $\mathcal{T} = \{\text{oriented great spheres}\} \approx S^3$
- $\mathcal{R} = \{\text{oriented round spheres}\} \approx S^3 \times [0, \pi]$
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Consider $\phi : I^5 \to \mathbb{Z}_2(S^3)$ continuous with
**Min-max Theory**

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Consider $\phi : I^5 \to \mathbb{Z}_2(S^3)$ continuous with

1. $\phi(I^4 \times \{0\}) = \phi(I^4 \times \{1\}) = 0$;
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1. $\phi(I^4 \times \{0\}) = \phi(I^4 \times \{1\}) = 0$;
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3. the sweepout $t \mapsto \phi(1/2, 1/2, 1/2, 1/2, 1/2, t)$ is non-trivial.
Min-max Theory

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If \( \phi : \partial I^4 \times \{1/2\} \approx S^3 \to \mathcal{T} \approx S^3 \) has non-zero degree then

\[
L([\phi]) > 4\pi = \max_{x \in \partial I^5} \mathbf{M}(\phi(x)).
\]
Corollary

For $\phi$ as in previous theorem, $\sup_{x \in I} M(\phi(x)) \geq 2\pi^2$. 
Min-max Theory

Corollary

For \( \phi \) as in previous theorem, \( \sup_{x \in I} M(\phi(x)) \geq 2\pi^2 \).

Idea:

\[
(\text{Previous Thm}) \implies L([\phi]) > \sup_{x \in \partial I} M(\phi(x)) = 4\pi
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(Previous Thm) $\implies L([\phi]) > \sup_{x \in \partial I^5} M(\phi(x)) = 4\pi$

(Pitts’ Thm) $\implies \exists \Sigma$ minimal: $L([\phi]) = M(\Sigma) > 4\pi$ and $\text{index}(\Sigma) \leq 5$
Min-max Theory

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For $\phi$ as in previous theorem, $\sup_{x \in I^5} M(\phi(x)) \geq 2\pi^2$.

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(Previous Thm) $\implies$ $L([\phi]) > \sup_{x \in \partial I^5} M(\phi(x)) = 4\pi$

(Pitts’ Thm) $\implies$ $\exists \Sigma$ minimal: $L([\phi]) = M(\Sigma) > 4\pi$ and $\text{index}(\Sigma) \leq 5$

(Urbano’s Thm) $\implies$ $\Sigma = \text{Clifford torus} \implies \sup_{x \in I^5} M(\phi(x)) \geq L([\phi]) = 2\pi^2$. 
Sketch of proof of Theorem:

- Assume $L([\phi]) = 4\pi = \sup_{x \in I} M(\psi(x))$ for some $\psi \in [\phi]$. 
Sketch of proof of Theorem:

• Assume $L([\phi]) = 4\pi = \sup_{x \in I^k} M(\psi(x))$ for some $\psi \in [\phi]$.
• Set $K = \psi^{-1}(T)$, where $\partial K \subset \partial I^4 \times \{1/2\}$.

(A) \quad (B)
Sketch of proof of Theorem:

- Assume $L([\phi]) = 4\pi = \sup_{x \in l^k} M(\psi(x))$ for some $\psi \in [\phi]$.
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\[\begin{align*}
\text{(A)} \quad & [\partial K] = [\partial l^4 \times \{1/2\}] \quad \text{in} \quad H_3(\partial l^4 \times \{1/2\}, \mathbb{Z})
\end{align*}\]
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(A) \([\partial K] = [\partial I^4 \times \{1/2\}]\) in \( H_3(\partial I^4 \times \{1/2\}, \mathbb{Z})\)

\[ \phi = \psi \text{ on } \partial K \text{ and } \psi : K \to T \approx S^3 \]

\[ \Rightarrow \text{degree}(\phi|_{\partial I^4 \times \{1/2\}}) = \text{degree}(\psi|_{\partial K}) = 0 \]
Sketch of proof of Theorem:

- Assume \( L(\phi) = 4\pi = \sup_{x \in I^k} M(\psi(x)) \) for some \( \psi \in [\phi] \).
- Set \( K = \psi^{-1}(T) \), where \( \partial K \subset \partial I^4 \times \{1/2\} \).

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\begin{align*}
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\phi &= \psi \text{ on } \partial K \text{ and } \psi : K \to T \approx S^3 \\
\implies \text{degree}(\phi|_{\partial I^4 \times \{1/2\}}) &= \text{degree}(\psi|_{\partial K}) = 0
\end{align*}
\]

\[
\begin{align*}
\text{(B)} \quad [\partial K] &= 0 \text{ in } H_3(\partial I^4 \times \{1/2\}, \mathbb{Z})
\end{align*}
\]
Sketch of proof of Theorem:

• Assume $L([\phi]) = 4\pi = \sup_{x \in I} M(\psi(x))$ for some $\psi \in [\phi]$.
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(A) $[\partial K] = [\partial I^4 \times \{1/2\}]$ in $H_3(\partial I^4 \times \{1/2\}, \mathbb{Z})$

\[ \phi = \psi \text{ on } \partial K \text{ and } \psi : K \to T \approx S^3 \]

\[ \implies \deg(\phi|_{\partial I^4 \times \{1/2\}}) = \deg(\psi|_{\partial K}) = 0 \]

(B) $[\partial K] = 0$ in $H_3(\partial I^4 \times \{1/2\}, \mathbb{Z})$

$\psi \circ c$ non-trivial sweepout with $\sup_{t \in I} M(\psi \circ c(t)) \leq \sup_{x \in I^k} M(\psi(x)) = 4\pi$

\[ \implies \psi \circ c(I) \cap T \neq \emptyset \implies c(I) \cap K \neq \emptyset. \]
Theorem (Marques-N.)

If $\Sigma \subset S^3$ has positive genus then $\mathcal{W}(\Sigma) \geq 2\pi^2$. 

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Corollary to our Min-max Thm can be applied and so $2\pi^2 \leq \sup_{(v, t) \in B_4 \times [-\pi, \pi]} M(\phi(x)) = \sup_{(v, t) \in B_4 \times [-\pi, \pi]} M(C(v, t)) \leq \mathcal{W}(\Sigma)$.
Theorem (Marques-N.)

If $\Sigma \subset S^3$ has positive genus then $\mathcal{W}(\Sigma) \geq 2\pi^2$.

1. Given $v \in B^4$ consider $F_v \in \text{Conf}(S^3)$, $x \mapsto \frac{1-|v|^2}{|x - v|^2}(x - v) - v$
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1. Given $v \in B^4$ consider $F_v \in \text{Conf}(S^3)$, $x \mapsto \frac{1-|v|^2}{|x-v|^2}(x-v) - v$

2. $C : B^4 \times [-\pi, \pi] \to \mathbb{Z}_2(S^3)$, $C(v, t) = \partial \{ x : \text{dist}(x, F_v(\Sigma)) < t \}$
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3. (Ros, 99) $\mathbf{M}(C(v, t)) \leq \mathcal{W}(C(v, 0)) = \mathcal{W}(F_v(\Sigma)) = \mathcal{W}(\Sigma)$
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3. (Ros, 99) $M(C(v, t)) \leq \mathcal{W}(C(v, 0)) = \mathcal{W}(F_v(\Sigma)) = \mathcal{W}(\Sigma)$

4. $C|_{S^3 \times [-\pi, \pi]}$ is not continuous
   $v_i \in B^4 \to p \notin \Sigma \implies F_{v_i}(\Sigma) \to \{ -p \}$
   $v_i \in B^4 \to p \in \Sigma$ with slope $\theta \implies F_{v_i}(\Sigma) \to \partial B_{\frac{\pi}{2} - \theta} ( - \cos \theta N(p) - \sin \theta p)$.
Theorem (Marques-N.)

If \( \Sigma \subset S^3 \) has positive genus then \( \mathcal{W}(\Sigma) \geq 2\pi^2 \).

1. Given \( \nu \in B^4 \) consider \( F_\nu \in \text{Conf}(S^3) \), \( x \mapsto \frac{1-|\nu|^2}{|x-\nu|^2} (x - \nu) - \nu \)

2. \( C : B^4 \times [-\pi, \pi] \rightarrow \mathcal{Z}_2(S^3) \), \( C(\nu, t) = \partial\{x : \text{dist}(x, F_\nu(\Sigma)) < t\} \)

3. (Ros, 99) \( M(C(\nu, t)) \leq \mathcal{W}(C(\nu, 0)) = \mathcal{W}(F_\nu(\Sigma)) = \mathcal{W}(\Sigma) \)

4. \( C|_{S^3 \times [-\pi, \pi]} \) is not continuous
   - \( \nu_i \in B^4 \rightarrow p \notin \Sigma \implies F_{\nu_i}(\Sigma) \rightarrow \{-p\} \)
   - \( \nu_i \in B^4 \rightarrow p \in \Sigma \) with slope \( \theta \implies F_{\nu_i}(\Sigma) \rightarrow \partial B_{\frac{\pi}{2}} - \theta(-\cos \theta N(p) - \sin \theta p) \).

5. Reparametrize \( C \) to obtain \( \phi : \overline{B^4} \times [-\pi, \pi] \rightarrow \mathcal{Z}_2(S^3) \) continuous with
   \[ \phi(S^3 \times [-\pi, \pi]) \subset \mathcal{R} \text{ and } \phi(x, t) \in \mathcal{T} \iff t = 0; \]

6. \( \text{degree}(\phi|_{S^3 \times \{0\}}) = \text{genus of } \Sigma! \)
Theorem (Marques-N.)

If \( \Sigma \subset S^3 \) has positive genus then \( \mathcal{W}(\Sigma) \geq 2\pi^2 \).

1. Given \( \nu \in B^4 \) consider \( F_\nu \in \text{Conf}(S^3) \), \( x \mapsto \frac{1-|\nu|^2}{|x-\nu|^2} (x - \nu) - \nu \)

2. \( C : B^4 \times [-\pi, \pi] \to \mathbb{Z}_2(S^3), \quad C(\nu, t) = \partial \{x : \text{dist}(x, F_\nu(\Sigma)) < t\} \)

3. (Ros, 99) \( \mathbf{M}(C(\nu, t)) \leq \mathcal{W}(C(\nu, 0)) = \mathcal{W}(F_\nu(\Sigma)) = \mathcal{W}(\Sigma) \)

4. \( C|_{S^3 \times [-\pi, \pi]} \) is not continuous

   \( \nu_i \in B^4 \to p \notin \Sigma \implies F_{\nu_i}(\Sigma) \to \{-p\} \)

   \( \nu_i \in B^4 \to p \in \Sigma \) with slope \( \theta \implies F_{\nu_i}(\Sigma) \to \partial B_{\frac{\pi}{2} - \theta}(-\cos \theta N(p) - \sin \theta p) \).

5. Reparametrize \( C \) to obtain \( \phi : \overline{B}^4 \times [-\pi, \pi] \to \mathbb{Z}_2(S^3) \) continuous with

   \[ \phi(S^3 \times [-\pi, \pi]) \subset R \text{ and } \phi(x, t) \in T \iff t = 0; \]

6. \( \text{degree}(\phi|_{S^3 \times \{0\}}) = \text{genus of } \Sigma! \)

7. Corollary to our Min-max Thm can be applied and so

\[ 2\pi^2 \leq \sup_{(\nu, t) \in \overline{B}^4 \times [-\pi, \pi]} \mathbf{M}(\phi(\nu)) = \sup_{(\nu, t) \in B^4 \times [-\pi, \pi]} \mathbf{M}(C(\nu, t)) \leq \mathcal{W}(\Sigma) \]
Min-max Theory

Continuous

- $I^n = n$-cube;

Discrete

- $I(n, k)$ vertices of $3^{nk}$ subdivisions of $I^n$;
Min-max Theory

Continuous

- \( l^n = n\) cube;
- \( \phi : l^n \to \mathbb{Z}_2(S^3) \) continuous in the flat topology;

Discrete

- \( l(n, k) \) vertices of \( 3^{nk} \) subdivisions of \( l^n \);
- sequence \( S = \{\phi_i\}_{i \in \mathbb{N}} \) with \( \phi_i : l(n, k_i) \to \mathbb{Z}_2(S^3) \), where \( x, y \) adjacent vertices \( \implies \text{M}(\phi_i(x) - \phi_i(y)) \leq \delta_i \to 0; \)
Min-max Theory

Continuous

- \( I^n = n\)-cube;
- \( \phi : I^n \to \mathbb{Z}_2(S^3) \) continuous in the flat topology;
- \( \sup\{M(\phi(x)) : x \in I^n\} \);

Discrete

- \( I(n, k) \) vertices of \( 3^{nk} \) subdivisions of \( I^n \);
- sequence \( S = \{\phi_i\}_{i \in \mathbb{N}} \) with \( \phi_i : I(n, k_i) \to \mathbb{Z}_2(S^3) \), where \( x, y \) adjacent vertices \( \implies M(\phi_i(x) - \phi_i(y)) \leq \delta_i \to 0 \);
- \( L(S) = \lim_{i \to \infty} \sup\{M(\phi_i(x)) : x \in I(n, k_i)\} \).
Min-max Theory

Continuous

- $l^n = n$-cube;
- $\phi : l^n \to \mathbb{Z}_2(S^3)$ continuous in the flat topology;
- $\sup\{M(\phi(x)) : x \in l^n\}$;
- define $[\phi] \in \pi_n(\mathbb{Z}_2(S^3), \phi|_{\partial l^n})$;

Discrete

- $l(n, k)$ vertices of $3^{nk}$ subdivisions of $l^n$;
- sequence $S = \{\phi_i\}_{i \in \mathbb{N}}$ with $\phi_i : l(n, k_i) \to \mathbb{Z}_2(S^3)$, where $x, y$ adjacent vertices $\implies M(\phi_i(x) - \phi_i(y)) \leq \delta_i \to 0$;
- $L(S) = \lim_{i \to \infty} \sup\{M(\phi_i(x)) : x \in l(n, k_i)\}$;
- define $\Pi = [S] \in \pi_n^\#(\mathbb{Z}_2(S^3), \phi|_{\partial l^n})$ (need $\phi|_{\partial l^n}$ continuous in varifold norm);
Min-max Theory

Continuous

- \( I^n = n\)-cube;
- \( \phi : I^n \to \mathbb{Z}_2(S^3) \) continuous in the flat topology;
- \( \sup \{ M(\phi(x)) : x \in I^n \} \);
- define \([\phi] \in \pi_n(\mathbb{Z}_2(S^3), \phi|_{\partial I^n})\);
- \( L([\phi]) = \inf_{\psi \in [\phi]} \sup \{ M(\psi(x)) : x \in I^n \} \).

Discrete

- \( I(n, k) \) vertices of \( 3^{nk} \) subdivisions of \( I^n \);
- sequence \( S = \{\phi_i\}_{i \in \mathbb{N}} \) with \( \phi_i : I(n, k_i) \to \mathbb{Z}_2(S^3) \), where \( x, y \) adjacent vertices \( \implies M(\phi_i(x) - \phi_i(y)) \leq \delta_i \to 0 \);
- \( L(S) = \lim_{i \to \infty} \sup \{ M(\phi_i(x)) : x \in I(n, k_i) \} \);
- define \( \Pi = [S] \in \pi_n^\#(\mathbb{Z}_2(S^3), \phi|_{\partial I^n}) \) (need \( \phi|_{\partial I^n} \) continuous in varifold norm);
- \( L([S]) = \inf_{S' \in [S]} L(S) \).
Min-Max Theory

Pitts’ Theorem

Consider $[S] \in \pi_n^\#(\mathcal{Z}_2(S^3), \phi|_{\partial I^n})$ with $L([S]) > \sup\{M(\phi(x)) : x \in \partial I^n\}$.
Min-Max Theory

Pitts’ Theorem

Consider $[S] \in \pi_n^#(\mathbb{Z}_2(S^3), \phi|_{\partial I^n})$ with $L([S]) > \sup\{M(\phi(x)) : x \in \partial I^n\}$.

There is $\Sigma$ a smooth embedded minimal surface (with multiplicities) so that

$$L([S]) = \inf_{S' \in [S]} L(S) = M(\Sigma).$$
Min-Max Theory

Pitts’ Theorem

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Interpolation Theorem

Given $\phi : I^n \to \mathbb{Z}_2(S^3)$ continuous in the flat topology, there exists $k_i \in \mathbb{N}$ and $\phi_i : I(n, k_i) \to \mathbb{Z}_2(S^3)$ so that:

(a) $S = [\{\phi_i\}_{i \in \mathbb{N}}] \in \pi_n^\#(\mathbb{Z}_2(S^3), \phi|_{\partial I^n})$, 

Min-Max Theory

**Pitts’ Theorem**

Consider $[S] \in \pi_n^\#(\mathcal{Z}_2(S^3), \phi_{|\partial I^n})$ with $L([S]) > \sup \{M(\phi(x)) : x \in \partial I^n\}$.

There is $\Sigma$ a smooth embedded minimal surface (with multiplicities) so that

$$L([S]) = \inf_{S' \in [S]} L(S) = M(\Sigma).$$

**Interpolation Theorem**

Given $\phi : I^n \to \mathcal{Z}_2(S^3)$ continuous in the flat topology, there exists $k_i \in \mathbb{N}$ and $\phi_i : I(n, k_i)_0 \to \mathcal{Z}_2(S^3)$ so that:

- (a) $S = \{\{\phi_i\}_{i \in \mathbb{N}} \} \in \pi_n^\#(\mathcal{Z}_2(S^3), \phi_{|\partial I^n})$,
- (b) $L(\{\phi_i\}) \leq \sup \{M(\phi(x)) : x \in I^n\}$,
Min-Max Theory

Pitts’ Theorem

Consider \([S] \in \pi^n_\#(\mathbb{Z}_2(S^3), \phi|_{\partial I^n})\) with \(L([S]) > \sup \{M(\phi(x)) : x \in \partial I^n\}\).

There is \(\Sigma\) a smooth embedded minimal surface (with multiplicities) so that

\[
L([S]) = \inf_{S' \in \{S\}} L(S) = M(\Sigma).
\]

Interpolation Theorem

Given \(\phi : I^n \to \mathbb{Z}_2(S^3)\) continuous in the flat topology, there exists \(k_i \in \mathbb{N}\) and \(\phi_i : I(n, k_i)_0 \to \mathbb{Z}_2(S^3)\) so that:

(a) \(S = \{\phi_i\}_{i \in \mathbb{N}} \in \pi^n_\#(\mathbb{Z}_2(S^3), \phi|_{\partial I^n})\),

(b) \(L(\{\phi_i\}) \leq \sup \{M(\phi(x)) : x \in I^n\}\),

(c) \(\lim_{i \to \infty} \sup \{F(\phi_i(x) - \phi(x)) : x \in I(n, k_i)_0\} = 0\).

Moreover, \(L([S]) = L([\phi])\).
Theorem (Agol-Marques-N.)

If $(\gamma_1, \gamma_2)$ link in $S^3$ has $lk(\gamma_1, \gamma_2) = \pm 1$, then $E(\gamma_1, \gamma_2) \geq 2\pi^2$. 
Theorem (Agol-Marques-N.)

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- Given \((\sigma_1, \sigma_2) \subset \mathbb{R}^4\) consider \(G(\sigma_1, \sigma_2) : S^1 \times S^1 \rightarrow S^3\)
  \[
  G(\sigma_1, \sigma_2)(s, t) = \frac{(\sigma_1(s) - \sigma_2(t))}{|\sigma_1(s) - \sigma_2(t)|}.
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\[\phi : B^4 \times [0, +\infty] \to \mathbb{Z}_2(S^3),\]
\[
\phi(v, \lambda) = G(F_v(\gamma_1), \lambda(F_v(\gamma_2) - c(v)) + c(v)).
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1. \(\phi : \overline{B}^4 \times [0, +\infty] \to Z_2(S^3)\),
   \[
   \phi(v, \lambda) = G(F_v(\gamma_1), \lambda(F_v(\gamma_2) - c(v)) + c(v)).
   \]

2. \(\text{area}(\phi(v, \lambda)) \leq E(\phi(v, 1)) = E(F_v(\gamma_1), F_v(\gamma_2)) = E(\gamma_1, \gamma_2)\).

3. Given \(v \in S^3\),
   \[
   \phi(v, 1) = \text{lk}(\gamma_1, \gamma_2) \partial B_{\pi/2}(-v) \implies \text{degree}(\phi|_{S^3 \times \{1\}}) = \text{lk}(\gamma_1, \gamma_2) \neq 0
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• \(\text{area}(G(\sigma_1, \sigma_2)) \leq E(\gamma_1, \gamma_2)\).

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3. Given \(v \in S^3\),

\[
\phi(v, 1) = lk(\gamma_1, \gamma_2)\partial B_{\pi/2}(-v) \implies \text{degree}(\phi|_{S^3 \times \{1\}}) = lk(\gamma_1, \gamma_2) \neq 0
\]

4. Can apply our Min-max Thm to conclude

\[
2\pi^2 \leq \max_{(v,t) \in \overline{B^4} \times [0, +\infty]} \text{area}(\phi(v, t)) \leq E(\gamma_1, \gamma_2).
\]