Non absolutely convergent integrals with respect to distributions

Jan MALÝ

Charles University, Prague

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Nonabsolutely convergent integrals

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\int_X f \, d\mu \text{ converges } \iff \int_X |f| \, d\mu \text{ converges}.
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Is this a desirable feature? Not always!

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\int_0^\infty \frac{\sin x}{x} \, dx = \frac{\pi}{2}.
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Nonabsolutely convergent integral in Newton’s sense:

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a), \]

where \( F \) is the antiderivative of \( f \), this means, \( F' = f \). This integrates some Lebesgue nonintegrable functions, like

\[ f(x) = \begin{cases} \frac{1}{x} \cos \frac{1}{x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases} \]
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Professional nonabsolutely convergent integrals: include the Lebesgue integral and integrate all derivatives.
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- requires an additional structure.

For example, to give a sense to the nonabsolutely convergent sum
\[ \sum_{n=1}^{\infty} (-1)^n n \]
we need the ordering of the set of natural numbers, a permutation of the summands can lead to a different result.

Similarly, to give a sense to the integral
\[ \int_0^{\infty} \sin x \, dx \]
and to others nonabsolutely convergent integrals, we need the ordering of the real line.
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Early history:
A. Denjoy 1912, N. Luzin 1912, O. Perron 1914, Khintchine 1916


Definition (H. Bendová and J.M. 2011)
Let $I = (a, b) \subset \mathbb{R}$ be an interval. We say that $F: I \to \mathbb{R}$ is an indefinite MC-integral of $f: I \to \mathbb{R}$ if there is a strictly increasing function $\xi: I \to \mathbb{R}$ (the so-called control function for the pair $(F, f)$) such that
\[
\lim_{y \to x} F(y) - F(x) - f(x)(y - x) \xi(y) - \xi(x) = 0
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for each $x \in I$.

This integrates the same class of functions as the Denjoy-Perron (or Henstock-Kurzweil) integral.

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For abstract Lebesgue integration the indefinite integral is just the function

\[ E \mapsto \int_{E} f \, d\mu, \quad E \text{ measurable}. \]
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If we want to gain some nonabsolutely convergent integrands, we need to restrict the family of sets in accordance with the additional structure.

One-dimensional case: the most natural system of sets is the family \( I \) of all bounded intervals. We can represent the indefinite integral \( F: I \mapsto \int_I f(x) \, dx, \quad I \in I \) by a function \( F: \mathbb{R} \rightarrow \mathbb{R} \) with the property that

\[ F(b) - F(a) = F([a, b]), \quad 0 < a \leq b < \infty. \]
Indefinite integral

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- the integral integrates all derivatives of differentiable functions but not both! (example J. Jarník, J. Kurzweil and Š. Schwabik 1983).

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Consider class $\mathcal{A}$ of subsets of $\mathbb{R}^n$ and a function $F : \mathcal{A} \to \mathbb{R}$, where $F(\mathcal{A})$ has the meaning of a generalized integral $\int_{\mathcal{A}} f(x) \, dx$. 

Choose first $\mathcal{A}$ to be the family of all bounded $n$-dimensional intervals. Advantage: It is a simple class, admits partitions. Disadvantage: Not invariant with respect to changes of coordinates (the less for a nonlinear change of variables).
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Disadvantage: Not invariant with respect to changes of coordinates (the less for a nonlinear change of variables).
Other versions for $\mathcal{A}$: simplices, polyhedra, convex sets.
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**Advantages:** bilipschitz change of variables, integrates all derivatives.
Idea: Indefinite integral is not 
\[ A \mapsto \int_A f(x) \, dx \]
where \( A \) runs over a family of sets, but
\[ \varphi \mapsto \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx \]
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Functional approach to indefinite integral

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where \( \varphi \) runs over a family of functions.

**Warning:** It is customary to identify a locally integrable function \( f \) with the distribution

\[ \varphi \mapsto \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \]

Then, the indefinite integral of \( f \) becomes the function \( f \) itself!!

However, the goal of integration is just to give a sense to expressions like \( \int f \varphi \, dx \) ! Now, nonabsolutely convergent integration means that we can assign a distribution to a function which is not locally Lebesgue integrable.
A version of the functional approach was \textbf{PU integral}: Instead of partition into intervals we can use partitions of unity into functions. This idea is due to J. Jarník and J. Kurzweil (1985, 1988), improved by J. Kurzweil, J. Mawhin, W. Pfeffer (1991).
Integrands

So far we integrated expressions with density with respect to the Lebesgue measure, \( f(x) \, dx \). There are two next degrees of generality: Stieltjes integration and general integrands.
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We will illustrate this on the real line:

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**General integrands** \( DU \):
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F(y) - F(x) \sim U(x, y) - U(x, x).
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In the functional approach, the general integrand is a variable \( x \mapsto G(x) : \Omega \to D'(\Omega) \). The indefinite integral is a distribution \( \mathcal{F} \) such that

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\mathcal{F}(\varphi) \sim G(x)(\varphi)
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**Examples**

A Stieltjes integrand

$$G(x)(\varphi) = \int_{\Omega} f(x)\varphi(y) \, d\mu(y).$$

We can replace the Radon measure $\mu$ by a distribution.
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A varifold type structure:

\( M \) a \( k \)-dimensional countably rectifiable set

\( T_x(M) \) tangent spaces to \( M \) (affine, \( x \in T_x(M) \)).

\[
G(x)(\varphi) = \int_{T_x(M)} f(x)\varphi(y) \, dy.
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Definitions

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We consider a family \((p_{x,r})_{x,r}\) of norms on \(\mathcal{D} = \mathcal{D}(\Omega)\) related to balls \(B(x, r)\), we skip precise assumptions, the model example is

\[
p_{x,r}(\varphi) = r^k \|D^{(k)}\varphi\|_{\infty},
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\[\varphi \in \mathcal{D}, \ spt \varphi \subset B(x, r)\).

Here \(k\) is a given order of differentiation and \(|D^{(k)}\varphi(x)|\) is the \(k\)-th order total differential of \(\varphi\).
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Here \(k\) is a given order of differentiation and \(|D^{(k)} \varphi(x)|\) is the \(k\)-th order total differential of \(\varphi\). The dual norm is

\[
p^*_{x,r}(\mathcal{T}) = \sup \left\{ \langle \mathcal{T}, \varphi \rangle : \varphi \in \mathcal{D}(\mathbb{R}^n), \ spt \varphi \subset B(x, r), \ p_{x,r}(\varphi) \leq 1 \right\},
\]

\[
\mathcal{T} \in \mathcal{D}'(\mathbb{R}^n).
\]
A function $\delta : \Omega \to (0, \infty)$ is termed a *gauge*. A finite system $(B(x_i, r_i))_{i=1}^m$ of balls is called an $\alpha$-*packing* in $\Omega$ if the balls $B(x_i, \alpha r_i)$ are pairwise disjoint and contained in $\Omega$, $i = 1, \ldots, m$. Given a gauge $\delta$, we say that the $\alpha$-packing is $\delta$-*fine* if $r_i < \delta(x_i)$, $i = 1, \ldots, m$. 
A function \( \delta : \Omega \to (0, \infty) \) is termed a \textit{gauge}. A finite system \( (B(x_i, r_i))_{i=1}^m \) of balls is called an \( \alpha \)-\textit{packing} in \( \Omega \) if the balls \( B(x_i, \alpha r_i) \) are pairwise disjoint and contained in \( \Omega \), \( i = 1, \ldots, m \).

Given a gauge \( \delta \), we say that the \( \alpha \)-packing is \( \delta \)-\textit{fine} if \( r_i < \delta(x_i) \), \( i = 1, \ldots, m \).

The purpose of the scaling factor \( \alpha \) is to overturn the dependence of the integral integral on the geometry of balls and to enable bilipschitz change of variables.
Definition (J.M; P. Honzík and J.M.)

Let $F$ be a distribution on $\Omega$ and $(G(x))_{x \in \Omega}$ be a system of distributions on $\Omega$ (standard choice $G(x) = f(x)G$). We say that $F$ is an indefinite packing integral of $(G(x))_{x \in \Omega}$ (with respect to the family $(p_{x,r})_{x,r}$, if there exist $\alpha \geq 1$ such that for each $\varepsilon > 0$ there exists a gauge $\delta : I \to (0, \infty)$ such that for each $\delta$-fine $\alpha$-packing $(B(x_i, r_i)_{i=1}^m$ in $\Omega$ we have

$$\sum_{i=1}^m p_{x_i,r_i}^*(F - G(x_i)) < \varepsilon.$$ 

If $G(x)$ has the form $f(x)G$ where $f$ is a function and $G$ a distribution, we say that $F$ is and indefinite packing integral of the function $f$ w.r.t. the distribution $G$. Then we denote the integral by $\int f \, dG$. We also use the multiplier notation $f \bullet G = \int f \, dG$. 
• The packing integral is well defined (this means unique, see the next page).
Features

- The packing integral is well defined (this means unique, see the next page).
- The Lebesgue integral with respect to a Radon measure $\mu$ is included.

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The Henstock-Kurzweil integral on the real line is also included. Further comparison of various related integrals on the real line brings numerous open questions.
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The packing integral integrates all derivatives

The packing integral allows for a bilipschitz change of variables.
Theorem

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- Given and \( \varepsilon > 0 \), we find the corresponding gage \( \delta \).
- We use a telescopic argument to find good \( \delta \)-fine radii, on which the dual norm is almost doubling.
- We use Vitali type covering argument to find a covering by balls \( B(x, r) \) such that \( B(x, r/5) \) are disjointed.
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- Given and $\varepsilon > 0$, we find the corresponding gage $\delta$.
- We use a telescopic argument to find good $\delta$-fine radii, on which the dual norm is almost doubling.
- We use Vitali type covering argument to find a covering by balls $B(x, r)$ such that $B(x, r/5)$ are disjointed.
- Then we construct a partition of unity related to the selected balls.
Theorem

\( \mathcal{F} \) is an indefinite packing integral of \((G(x))_{x \in \Omega}\) w.r.t. \((p_{x,r})_{x,r}\) if and only if there exist a Radon measure \(\mu\) on \(\Omega\) and \(\alpha \geq 1\) such that

\[
\lim_{r \to 0^+} \frac{p_{x,r}^*(\mathcal{F} - G(x))}{\mu(B(x, \alpha r))} = 0.
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Theorem

$F$ is an indefinite packing integral of $(G(x))_{x \in \Omega}$ w.r.t. $(p_{x,r})_{x,r}$ if and only if there exist a Radon measure $\mu$ on $\Omega$ and $\alpha \geq 1$ such that

$$\lim_{r \to 0^+} \frac{p_{x,r}^*(F - G(x))}{\mu(B(x, \alpha r))} = 0.$$ 

Remarks

- The proof use a method invented by M. Csörnyei in a different context.
Theorem

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- If we integrate a function with respect to a distribution, the inverse process is a kind of Radon-Nikodým differentiation of a distribution of respect to a distribution.
In a series of papers and books, W. Pfeffer proves a general Gauss-Green theorem

$$\int_{\Omega} \text{div} \, f(x) \, dx = \int_{\partial \Omega} f \cdot d\nu.$$  

The indefinite integral in his work is a function of BV set, a charge. Functional approach to charges: T. De Pauw and W. Pfeffer 2008, T. De Pauw, R. Hardt,...
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The boundary integral is with respect to the distributional derivative $-\nu$ of the characteristic function $\chi_\Omega$ of $\Omega$. 

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Now, we know how to integrate with respect to distributions and can to handle integrals with respect to $\nu$ even if $\nu$ is not a measure.
Our aim is to prove that

\[ \int f \, d(D\chi_{\Omega}) + \int \chi_{\Omega} \, d(\text{Div} \, f) = \text{Div} \int f\chi_{\Omega} \]

or in the multiplier notation

\[ f \cdot D\chi_{\Omega} + \chi_{\Omega} \cdot \text{Div} \, f = f\chi_{\Omega} \cdot \mathcal{L} \]

where \( \mathcal{L} \) is the lebesgue measure regarded as distribution. The most powerful versions of the Gauss-Green theorem are fairly complicated. We present here a simple corollary, in which \( \Omega \) need not be of finite perimeter..
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**Theorem (J.M.)**

Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded open set with a countably \((n-1)\)-rectifiable boundary. Let \( f \in C(\overline{\Omega}) \cap W^{1,1}(\Omega) \) be a vector field. Then
\[
\int_\Omega \text{div } f(x) \, dx = \int_{\partial \Omega} f \cdot d\nu,
\]
where the integral on the right is understood as the packing integral of \( f \) with respect to \( \nu = -D\chi_\Omega \).
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\[ f \cdot D\chi_{\Omega} + \chi_{\Omega} \cdot \text{Div} f = f \chi_{\Omega} \cdot \mathcal{L} \]
where \( \mathcal{L} \) is the Lebesgue measure regarded as distribution.

The most powerful versions of the Gauss-Green theorem are fairly complicated. We present here a simple corollary, in which \( \Omega \) need not be of finite perimeter.

**Theorem (J.M.)**

Suppose that \( \Omega \subset \mathbb{R}^n \) is a bounded open set with a countably \((n-1)\)-rectifiable boundary. Let \( f \in C(\overline{\Omega}) \cap W^{1,1}(\Omega) \) be a vector field. Then
\[ \int_{\Omega} \text{div} f(x) \, dx = \int_{\partial\Omega} f \cdot d\nu, \]
where the integral on the right is understood as the packing integral of \( f \) with respect to \( \nu = -D\chi_{\Omega} \).
Stokes theorem?
Stokes theorem?
- in the euclidean setting, a notable version is not yet established.
- may build on recent developments in GMT of currents, chains, cochains, charges,.. T. De Pauw, L. Moonens and W. Pfeffer 2009; T. De Pauw and R. Hardt (new)
The generalization to metric spaces is a joint work with Kristýna Kuncová. Instead of smooth test functions we use Lipschitz test functions and norms of order at most 1. The Lipschitz norm seems to be the most natural choice, so that we set

$$p_{x,r}(\varphi) = \|\varphi\|_{\infty} + r\|\varphi\|_{\text{Lip}}.$$  

(One can regard this as a model case and generalize to other norms. e.g. Sobolev type norms on metric spaces.)
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Dual objects to Lipschitz functions (weak* continuous functionals on compactly supported Lipschitz functions) are called metric distributions. We integrate families of metric distributions and the indefinite integral is a metric distribution.
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The result on the measure control is also available.
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The packing integral allows for a bilipschitz change of variables.
We need the setting of currents on metric spaces: L. Ambrosio and B. Kirchheim 2000, related work T. De Pauw, R. Hardt,...

Roughly, a metric 1-current is a bilinear weak* continuous functional \( T \) on “forms” \( \varphi \, d\psi \), where \((\varphi, \psi)\) is a pair of Lipschitz functions with compact spt \( \varphi \psi \). The definition also requires: if \( \psi \) is constant on \( \{\varphi \neq 0\} \), then \( T(\varphi \, d\psi) = 0 \).

The boundary of a metric 1-current \( T \) is the metric distribution

\[
\partial T : \varphi \mapsto T(1 \, d\varphi).
\]
We are interested in validity of the product rule

\[ \partial(fg \cdot T) = f \cdot \partial(g \cdot T) + g \cdot \partial(f \cdot T). \]

Here the currents can be vector valued. For \( g = \chi_\Omega \) we obtain the Gauss-Green-Stokes formula. Indeed, let us observe the “translation table’, on the right we have the classical case of the divergence theorem.

<table>
<thead>
<tr>
<th>A 1-current ( T ) is given</th>
<th>( T(\varphi , d\psi) = \int_{\mathbb{R}^n} \varphi \nabla \psi , dx )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f \cdot T(\varphi , d\psi) )</td>
<td>( \int_{\mathbb{R}^n} f \varphi \nabla \psi , dx )</td>
</tr>
<tr>
<td>( \partial(f \cdot T)(\varphi) )</td>
<td>( \int_{\mathbb{R}^n} f \nabla \varphi = -\int_{\mathbb{R}^n} \varphi \div f , dx )</td>
</tr>
<tr>
<td>( \chi_\Omega \cdot \partial(f \cdot T)(\varphi) )</td>
<td>( -\int_\Omega \varphi \div f , dx )</td>
</tr>
<tr>
<td>( \chi_\Omega \cdot T(\varphi , d\psi) )</td>
<td>( \int_\Omega \varphi \nabla \psi , dx )</td>
</tr>
<tr>
<td>( \partial(\chi_\Omega \cdot T)(\varphi) )</td>
<td>( \int_\Omega \nabla \varphi , dx = \int_{\partial \Omega} \varphi \cdot \nu , d\mathcal{H}^{n-1} )</td>
</tr>
<tr>
<td>( f \cdot \partial(\chi_\Omega \cdot T)(\varphi) )</td>
<td>( \int_{\partial \Omega} f \varphi \cdot \nu , d\mathcal{H}^{n-1} )</td>
</tr>
<tr>
<td>( \partial(f \chi_\Omega \cdot T)(\varphi) )</td>
<td>( -\int_\Omega f \cdot \nabla \varphi , dx )</td>
</tr>
</tbody>
</table>
At the end, in the classical model the formula

\[ \partial(f\chi_\Omega \cdot T) = f \cdot \partial(\chi_\Omega \cdot T) + \chi_\Omega \cdot \partial(f \cdot T) \]

gives

\[ -\int_\Omega f \cdot \nabla \varphi \, dx = \int_{\partial \Omega} \varphi f \cdot d\nu - \int_\Omega \varphi \text{div} f \, dx. \]

If the test function equals 1 on \( \overline{\Omega} \), we obtain

\[ \int_{\partial \Omega} f \cdot d\nu = \int_\Omega \text{div} f \, dx. \]

We are able to prove the product rule

\[ \partial(fg \cdot T) = f \cdot \partial(g \cdot T) + g \cdot \partial(f \cdot T) \]

under some complicated technical assumptions, which however are not as restrictive as they look like.