Asymptotic modeling of thin curved martensitic films

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Abstract

We consider a thin curved film made of a martensitic material. The behavior of the film is governed by a free energy composed of a bulk energy term and an interfacial energy term. We show that the minimizers of the free energy converge to the minimizers of an energy depending on a two-dimensional deformation and one Cosserat vector field when the thickness of the curved film goes to zero, using $\Gamma$-convergence arguments.

1 Introduction

We study here thin curved films made of a martensitic material, in the spirit of [3] and [18] in the case of planar films. Martensitic materials belong to the class of shape memory alloys. Starting from a high temperature phase called austenite, such materials undergo upon cooling a reversible transformation called the martensitic transformation, and settle in a low temperature phase called martensite. The phases correspond to different crystalline lattice structures.

During this practically instantaneous, diffusionless transformation, a body that is made of such a material suffers a slight volume variation and significant shear in a certain direction. Thanks to the reversibility of the martensitic transformation, the body is able to return to its initial shape simply by reheating. Thus, in the

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absence of loading, the body has several preferred geometric shapes, depending on the temperature.

The shape memory property is of great importance in various fields of applications such as aeronautics, electronics and the medical field. Let us mention for example certain prostheses, the shape of which makes for easier implantation at room temperature and which assume their functional form at body temperature after martensitic transformation.

In the present work, we are interested in the behavior of a thin film of martensitic material at the critical temperature when different phases may coexist. This configuration may be of interest in designing microactuators, see [3].

We thus consider a martensitic film of thickness \( h \) occupying an open subset \( \tilde{\Omega}_h \) of \( \mathbb{R}^3 \), which is a tubular neighborhood of a curved midsurface for \( h \) small enough. The deformations of the film \( \tilde{\phi} \) minimize an energy \( \tilde{e}_h \) subjected to given boundary conditions. This energy consists of two competing parts: a nonlinearly hyperelastic energy term and an interfacial energy term of the Van der Waals type

\[
\tilde{e}_h(\tilde{\phi}) = \int_{\tilde{\Omega}_h} \left[ \kappa |\nabla^2 \tilde{\phi}|^2 + W(\nabla \tilde{\phi}) \right] dx,
\]

where \( \kappa \) is a strictly positive constant, \( \nabla^2 \tilde{\phi} \) is the \( 3 \times 3 \times 3 \) tensor of the second derivatives of the deformation and \( W(\nabla \tilde{\phi}) \) represents the elastic energy density per unit volume. At the critical temperature, this density has a multiwell structure corresponding to the martensite and austenite variants. It is also assumed to be frame-indifferent. The interfacial energy term penalizes sudden spatial changes in the deformation gradient, hence tends to inhibit the coexistence of different phases. We are interested in the asymptotic behavior of this energy and its minimizers when the thickness of the film goes to zero.

We start by flattening the surface and rescaling the energy in order to work on a cylindrical domain that is independent of the thickness \( h \). Then, we use \( \Gamma \)-convergence arguments to study the behavior of energy minimizers when the thickness goes to zero. In order to interpret the obtained limit model in intrinsic terms, we rewrite the two-dimensional limit model on the curved surface. We show that the thin film limit model remains frame-indifferent, but that three-dimensional material symmetries may be lost as a result of the presence of curvature. Finally, we discuss the case of vanishing interfacial energy in the limit model.

As was said above, our work is mostly influenced by [3], see also [18]. We adapt arguments found in [14] to deal with the curved aspect of the problem. Let us mention that the case of martensitic wires is treated in [11, 12]. The results of this article were announced in [15]. The use of \( \Gamma \)-convergence arguments in the derivation of lower-dimensional models in nonlinear elasticity was pioneered by [1] for strings, then by [13] for membranes. Such arguments were subsequently
applied in various more general contexts, see for example [4, 8], among others. Higher order, nonlinear bending models where also obtained by Γ-convergence arguments in [10] for plates, [9] for the curved case, i.e., shells in this context. There is also quite a bit of literature concerning the various energy scalings corresponding to diverse situations, in particular for a multiwell elastic problem, see [5] for membranes and [16] for rods.

2 Notation and geometrical preliminaries

Throughout this article, we assume the summation convention unless otherwise specified. Greek indices take their values in the set \{1, 2\} and Latin indices in the set \{1, 2, 3\}.

Let \((e_1, e_2, e_3)\) be the canonical orthonormal basis of the Euclidean space \(\mathbb{R}^3\). We denote by \(|v|\) the norm of a vector \(v\) in \(\mathbb{R}^3\), by \(u \cdot v\) the scalar product of two vectors in \(\mathbb{R}^3\), by \(u \wedge v\) their vector product and by \(u \otimes v\) their tensor product. Let \(\mathbb{M}_{33}\) be the space of \(3 \times 3\) real matrices endowed with the usual norm \(|F| = \sqrt{\text{tr}(F^T F)}\). This is a matrix norm in the sense that \(|AB| \leq |A||B|\). We denote by \(A = (a_1|a_2|a_3)\) the matrix in \(\mathbb{M}_{33}\) whose \(i\)th column is \(a_i\).

We denote by \(\mathbb{M}_{333}\) the space of real \(3 \times 3 \times 3\) tensors. Let \(P = p_{ijk}e_i \otimes e_j \otimes e_k\) be a third order tensor. The tensor \(P^T\) with components \(q_{ijk} = p_{ikj}\) is the transpose of \(P\) with respect to the second and third indices. Let \(P\) be a tensor of order \(p\) and \(Q\) a tensor of order \(q\). The contracted tensor product of \(P\) and \(Q\), denoted by \(P \otimes Q\), is the tensor of order \(p+q-2\) whose components are obtained by contracting the last index of \(P\) with the first index of \(Q\). For example, if \(p = 3\) and \(q = 2\) we have

\[P \otimes Q = P_{ijk}Q_{km}(e_i \otimes e_j \otimes e_m) \in \mathbb{M}_{333}.\]

The canonical scalar product of two third order tensors \(P\) and \(Q\) with components in the canonical basis \((p_{ijk})\) and \((q_{ijk})\) respectively, is defined by the relation \(P \cdot Q = p_{ijk}q_{ijk}\) and the norm of such a tensor by \(|P| = \sqrt{P \cdot P}\). This tensor norm is also multiplicative with respect to the contracted tensor product for all \(P\) of order 3 and \(Q\) of order 2, as we have

\[|P \otimes Q| \leq |P||Q|.\]  \hfill (2.1)

Indeed, if \(P\) denotes the second order tensor with components \(p_{ijk}\) (\(i\) fixed), then

\[|P \otimes Q|^2 = \sum_i |P_i Q|^2 \leq \sum_i |P_i|^2 |Q|^2 = |P|^2 |Q|^2.\]

A deformation of an open subset \(\Omega\) of \(\mathbb{R}^3\) is a (sufficiently regular) mapping from \(\Omega\) into \(\mathbb{R}^3\). The deformation gradient \(\nabla \varphi\) is the \(\mathbb{M}_{33}\)-valued tensor of the first derivatives and \(\nabla^2 \varphi\) is the \(\mathbb{M}_{333}\)-valued tensor of the second derivatives of \(\varphi\).
We consider a thin curved film of thickness $h > 0$ occupying at rest an open domain $\bar{\Omega}_h$. This reference configuration of the film is described as follows. We are thus given a surface $S$, called the midsurface of the film. It is a bounded two-dimensional $C^3$-submanifold of $\mathbb{R}^3$ and we assume for simplicity that it admits an atlas consisting of one chart. Let $\psi$ be this chart, i.e. a $C^3$-diffeomorphism from a bounded open subset $\omega$ of $\mathbb{R}^2$ onto $S$. We assume that $\omega$ has a Lipschitz boundary $\partial \omega$ and that $\psi$ can be extended into a $C^3(\bar{\omega}, \mathbb{R}^3)$ mapping.

Let $a_{\alpha}(x) = \psi_{,\alpha}(x)$ be the vectors of the covariant basis of the tangent plane $T_{\psi(x)}\bar{S}$ associated with the chart $\psi$, where $\psi_{,\alpha}$ denotes the partial derivative of $\psi$ with respect to $x_{\alpha}$. We assume that there exists $\delta > 0$ such that $|a_1(x) \wedge a_2(x)| \geq \delta$ on $\bar{\omega}$ and we define the unit normal vector $a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|}$, which belongs to $C^2(\bar{\omega}, \mathbb{R}^3)$. The vectors $a_1(x)$, $a_2(x)$ and $a_3(x)$ constitute the covariant basis at point $x$. We define the contravariant basis by the relations $a^i(x) \cdot a_j(x) = \delta^i_j$, so that $a^3(x) \in T_{\psi(x)}\bar{S}$ and $a^3(x) = a_3(x)$.

Next, we define a mapping $\Psi : \omega \times \mathbb{R} \rightarrow \mathbb{R}^3$ by

$$\Psi(x_1, x_2, x_3) = \psi(x_1, x_2) + x_3 a_3(x_1, x_2).$$

It is well known that there exists $h^* > 0$ such that for all $0 < h < h^*$, the restriction of $\Psi$ to $\Omega_h = \omega \times [-h/2, h/2]$ is a $C^2$-diffeomorphism on its image by the tubular neighborhood theorem. For such values of $h$, we set

$$\tilde{\Omega}_h = \Psi(\Omega_h).$$

Alternatively, we can write

$$\tilde{\Omega}_h = \left\{ \tilde{x} \in \mathbb{R}^3, \exists \tilde{\pi}(\tilde{x}) \in \tilde{S}, \tilde{x} = \tilde{\pi}(\tilde{x}) + \eta a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))) \text{ with } \frac{-h}{2} < \eta < \frac{h}{2} \right\},$$

where $\tilde{\pi}$ denotes the orthogonal projection from $\tilde{\Omega}_h$ onto $\tilde{S}$, which is well defined and of class $C^2$ for $h < h^*$. Equivalently, every $\tilde{x} \in \tilde{\Omega}_h$ can be written as

$$\tilde{x} = \tilde{\pi}(\tilde{x}) + \left[ (\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))) \right] \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))).$$

Thus, we have a curvilinear coordinate system in $\tilde{\Omega}_h$ naturally associated with the chart $\psi$ by

$$(x_1, x_2) = \psi^{-1}(\tilde{\pi}(\tilde{x})) \text{ and } x_3 = (\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))).$$

For all $x \in \bar{\omega}$, we let $A(x) = (a_1(x) | a_2(x) | a_3(x))$. We note that $A(x)$ is an invertible matrix on $\bar{\omega}$, and that its inverse is given by $A(x)^{-1} = (a^1(x) | a^2(x) | a^3(x))^T$. 
We also note that \( \det A(x) = |\text{cof} A(x) \cdot e_3| = |a_1(x) \wedge a_2(x)| \geq \delta > 0 \) on \( \tilde{\omega} \). We clearly have

\[
\nabla \Psi(x_1, x_2, x_3) = A(x_1, x_2) + x_3 (a_{3,1}(x_1, x_2) | a_{3,2}(x_1, x_2) | 0).
\]

The matrix \( \nabla \Psi(x_1, x_2, x_3) \) is thus everywhere invertible in \( \tilde{\Omega}_h \) and its determinant is strictly positive, and therefore equal to the Jacobian of the change of variables, for \( h \) small enough.

In the following, \( h \) denotes a generic sequence of real numbers in \( ]0, h^*[ \) that tends to zero. The next convergences are easily established.

**Lemma 2.1** We have

\[
\begin{align*}
\nabla \Psi^{-1} \circ \Psi(x_1, x_2, hx_3) & \to A(x)^{-1}, \\
\nabla^2 \Psi^{-1} \circ \Psi(x_1, x_2, hx_3) & \to \nabla^2 \Psi^{-1} \circ \Psi(x_1, x_2, 0), \\
\det \nabla \Psi(x_1, x_2, hx_3) & \to \det A(x),
\end{align*}
\]

uniformly on \( \tilde{\Omega}_1 \) when \( h \to 0 \). In particular, \( \inf_{\tilde{\Omega}_1} \det \nabla \Psi(x_1, x_2, hx_3) \geq \delta/2 > 0 \) for \( h \) small enough.

**3 The three-dimensional and rescaled problems**

When submitted to boundary conditions, and more generally loadings, the thin film undergoes a deformation \( \tilde{\phi}_h : \tilde{\Omega}_h \to \mathbb{R}^3 \) and we are interested in the asymptotic behavior of such deformations when \( h \to 0 \). We refer to [3] and the references therein for material science background information about martensitic materials. Let it just be said here that the deformations of the thin film are assumed to minimize an energy \( \tilde{e}_h \) which is composed of an interfacial energy \( \tilde{K}_h \) and an elastic energy \( \tilde{I}_h \) of the form

\[
\tilde{e}_h(\tilde{\phi}) = \tilde{K}_h(\tilde{\phi}) + \tilde{I}_h(\tilde{\phi}),
\]

with

\[
\tilde{K}_h(\tilde{\phi}) = \int_{\tilde{\Omega}_h} \kappa |\nabla^2 \tilde{\phi}|^2 d\tilde{x},
\]

\( \kappa > 0 \), and

\[
\tilde{I}_h(\tilde{\phi}) = \int_{\tilde{\Omega}_h} W(\nabla \tilde{\phi}) d\tilde{x}.
\]

The interfacial energy term plays the role of a penalization term for spatial changes in deformation gradient. The fact that \( \kappa > 0 \) ensures that creating an interface layer between two different phases will cost some energy.
The hyperelastic energy density $W: \mathbb{M}_{33} \to \mathbb{R}$ satisfies the following growth and coercivity assumptions: There exists $c_1, c_2 > 0$ such that

$$c_1(|A|^2 - 1) \leq W(A) \leq c_2(|A|^q + 1) \quad \text{with} \quad 2 \leq q < 6,$$

and the axiom of frame indifference which states that for every matrix $A \in \mathbb{M}_{33}$ and every rotation $R$ in $SO(3)$, we have

$$W(RA) = W(A),$$

where $SO(3) = \{ R \in \mathbb{M}_{33}; R^T R = R R^T = I, \det R = 1 \}$.

In the specific context of martensites, we also suppose that there exists a finite number of symmetric positive definite matrices $U_0 = I, U_1, U_2, \ldots, U_n$ such that $W$ reaches its minimum, which we may assume to vanish, on $\bigcup_{i=0}^n SO(3) U_i$. Each of these energy wells correspond respectively to the austenite for $U_0$ and the different martensite variants for $U_i, i = 1, \ldots, n$. The corresponding elastic energy functional may be minimized pointwise by choosing gradients with values in rank-1 compatible wells. This term is responsible for the appearance of fine phase mixtures, see [2]. Of course, there is a competition between the interfacial energy and the hyperelastic energy. Note that the particular martensitic structure of the hyperelastic stored energy function does not plays any role in the ensuing asymptotic analysis.

For simplicity, we will only consider homogeneous boundary conditions of place imposed on the lateral boundary of the film. We thus introduce the space of admissible deformations

$$\tilde{V}_h = \{ \tilde{\phi} \in H^2(\tilde{\Omega}_h; \mathbb{R}^3); \tilde{\phi}(\tilde{x}) = \tilde{A}\tilde{x} \text{ on } \tilde{\Gamma}_h \},$$

where $\tilde{A} = (\tilde{a}_1 | \tilde{a}_2 | \tilde{a}_3) \in \mathbb{M}_{33}$ is a given constant matrix and $\tilde{\Gamma}_h = \Psi(\partial \omega \times [-h/2, h/2])$ is the lateral surface of $\tilde{\Omega}_h$. Note that due to the growth condition satisfied by $W$ and the Sobolev embedding theorem, the energy functional $\tilde{e}_h$ is well defined and takes its values in $\mathbb{R}$ for $\tilde{\phi} \in \tilde{V}_h$.

The minimization problem consists in finding $\tilde{\phi}_h \in \tilde{V}_h$ such that

$$\tilde{e}_h(\tilde{\phi}_h) = \min_{\tilde{\phi} \in \tilde{V}_h} \tilde{e}_h(\tilde{\phi}).$$

It is fairly obvious that such a minimizer exists since $\kappa > 0$ and weak convergence in $H^2$ entails strong convergence in $W^{1,q}$ for $q < 6$.

To study the behavior of this energy and its minimizers, we begin by flattening and rescaling the minimizing problem in order to work on a fixed cylindrical domain. For clarity, we proceed in two steps, flatten first and then rescale.
If \( \tilde{\phi}_h \) is a deformation of the film in its reference configuration, we define \( \phi_h : \Omega_h \rightarrow \mathbb{R}^3 \) by setting \( \phi_h(x) = \tilde{\phi}_h(\Psi(x)) \) for all \( x \in \Omega_h \). Since \( \Psi \) is a \( C^2 \)-diffeomorphism, \( \phi_h \) is in \( H^2 \) whenever \( \tilde{\phi}_h \) is in \( H^2 \). For every such deformation \( \phi_h \), we thus set \( e_h(\phi_h) = \tilde{e}_h(\phi_h \circ \Psi^{-1}) \) and we obtain

\[
e_h(\phi_h) = \int_{\Omega_h} \left\{ \kappa \left| \left( \nabla^2 \phi_h \otimes \left( \nabla \Psi^{-1} \circ \Psi \right) \right) \right| T \otimes \left( \nabla \Psi^{-1} \circ \Psi \right) + \nabla \phi_h \otimes \left( \nabla^2 \Psi^{-1} \circ \Psi \right) \right|^2
+ W(\nabla \phi_h (\nabla \Psi^{-1} \circ \Psi)) \right\} \det \nabla \Psi \, dx.
\]

\[
= \int_{\Omega_h} \left\{ \kappa \sum_{i,j,k=1}^3 \left| (\phi_h)_{i,lm}(x) \Psi^{-1}_{m,k}(\Psi(x)) \Psi_{j,l}^{-1}(\Psi(x)) + (\phi_h)_{i,l}(x) \Psi_{i,j,k}^{-1}(\Psi(x)) \right|^2
+ W(\nabla \phi_h (x) \nabla \Psi^{-1}(\Psi(x))) \right\} \det \nabla \Psi \, dx.
\]

This completes the flattening step.

Let us now turn to the rescaling step. First of all, since the total energy in the limit membrane deformation regime is of order \( h \), we are actually interested in the asymptotic behavior of the energy per unit thickness \( \frac{1}{h} e_h \).

Then, we define \( z_h : \Omega_1 \rightarrow \Omega_h \) by setting

\[ z_h(x_1, x_2, x_3) = (x_1, x_2, hx_3). \]

With every deformation \( \phi_h \) on \( \Omega_h \), we associate the deformation \( \phi(h) \) on \( \Omega_1 \) defined by \( \phi(h)(x) = \phi_h(z_h(x)) \) and its rescaled deformation gradient

\[ \nabla_p \phi(h) = \begin{pmatrix} \phi(h),1 & \phi(h),2 & \frac{1}{h} \phi(h),3 \end{pmatrix} = \nabla_p \phi(h) + \frac{1}{h} \phi(h),3 \otimes e_3, \]

and rescaled tensor of second derivatives

\[ \nabla_p^2 \phi(h) = \nabla_p^2 \phi(h) + \frac{1}{h} (\nabla_p \phi(h),3 \otimes e_3 + (\nabla_p \phi(h),3 \otimes e_3)^T) + \frac{1}{h^2} \phi(h),33 \otimes e_3 \otimes e_3, \]

where the notations \( \nabla_p \phi = \phi_{,\alpha} \otimes e_\alpha \) and \( \nabla_p^2 \phi = \phi_{,\alpha \beta} \otimes e_\alpha \otimes e_\beta \) stand for in-plane first and second gradients. Note that \( \nabla_p (\phi_{,\beta}) = (\nabla_p \phi),3 \), hence the unambiguous notation \( \nabla_p \phi_{,3} \). Note also that \( \nabla_p \phi \) is \( \mathbb{M}_{33} \)-valued and \( \nabla_p^2 \phi \) is \( \mathbb{M}_{3333} \)-valued.

Finally, we set \( e(h)(\phi(h)) = \frac{1}{h} e_h(\phi_h) \). Thus, we obtain a rescaled energy of the form

\[ e(h)(\phi(h)) = K(h)(\phi(h)) + I(h)(\phi(h)), \]

with

\[ I(h)(\phi) = \int_{\Omega_1} W(\nabla_p \phi A_h) d_h \, dx. \]
and
\[ K(h)(\phi) = \int_{\Omega_1} \kappa \left( \nabla^2_x \phi \otimes A_h \right)^T \otimes A_h + \nabla_h \phi \otimes B_h \right) \, \, d_h \, dx, \]
where
\[ d_h(x) = \det \nabla \psi(z_h(x)), \, A_h(x) = \nabla \psi^{-1}(\psi(z_h(x))) \, \text{and} \, B_h(x) = \nabla^2 \psi^{-1}(\psi(z_h(x))). \]
The rescaled interfacial energy \( K(h) \) also reads in components
\[
K(h)(\phi) = \int_{\Omega_1} \kappa \sum_{i,j,k=1}^3 \phi_{i,j,k}(\psi^{-1}_{\alpha,\beta}(\psi(z_h(x)))) \psi^{-1}_{\alpha,j}(\psi(z_h(x))) \psi^{-1}_{\beta,k}(\psi(z_h(x))) \\
+ \frac{1}{h} \phi_{i,\alpha\beta}(x) \left[ \psi^{-1}_{\alpha,k}(\psi(z_h(x))) \psi^{-1}_{3,j}(\psi(z_h(x))) \right] \\
+ \frac{1}{h^2} \phi_i,\alpha(\psi(z_h(x))) \psi^{-1}_{3,k}(\psi(z_h(x))) \\
+ \phi_i,\alpha(\psi(z_h(x))) \psi^{-1}_{\alpha,j}(\psi(z_h(x))) + \frac{1}{h} \phi_{i,3}(\psi(z_h(x))) \psi^{-1}_{3,j}(\psi(z_h(x))) \right) \, d_h(x) dx.
\]
Problem (3.1) then becomes: Find \( \phi(h) \in V_h \) such that
\[
e(h)(\phi(h)) = \min_{\phi \in V_h} e(h)(\phi), \quad (3.2)
\]
with \( V_h = \{ \phi \in H^2(\Omega_1; \mathbb{R}^3); \phi(x) = \tilde{A}\psi(z_h(x)) \text{ on } \partial \omega \times ]-\frac{1}{2}, \frac{1}{2}[, \} \).

Since problems (3.1) and (3.2) are equivalent, and that the existence of solutions to problem (3.1) is obvious, there exists a solution \( \phi(h) \in V_h \) of problem (3.2) for all \( 0 < h < h^* \).

In order to study the behavior of this solution when \( h \to 0 \) we will use \( \Gamma \)-convergence theory, a notion introduced by De Giorgi [6, 7] and successfully used in one form or another, in several studies of the behavior of thin films, see [3, 9, 10, 11, 12, 13, 14, 17, 18].

\section{\( \Gamma \)-convergence and behavior of the minimizers}

Let us begin with the boundedness properties of deformations with bounded energies. In the sequel, \( C \) denotes a generic strictly positive constant.

\begin{lemma}
Let \( \phi(h) \in V_h \) be a sequence verifying \( e(h)(\phi(h)) \leq C \). Then we have
\[
\| \nabla_p \phi(h) \|_{L^2} \leq C, \quad \frac{1}{h} \| \phi(h),3 \|_{L^2} \leq C, \quad (4.1)
\]
and
\[
\| \nabla^2_p \phi(h) \|_{L^2} \leq C, \quad \left\| \frac{1}{h} \nabla_p \phi(h),3 \right\|_{L^2}^2 \leq C \quad \text{and} \quad \left\| \frac{1}{h^2} \phi(h),33 \right\|_{L^2}^2 \leq C. \quad (4.2)
\]
\end{lemma}
Since \( e(h)(\phi(h)) = I(h)(\phi(h)) + K(h)(\phi(h)) \) and both terms are positive (recall that we have assumed \( W \geq 0 \)), we get two estimates
\[
I(h)(\phi(h)) \leq C \quad \text{and} \quad K(h)(\phi(h)) \leq C. \tag{4.3}
\]

The first estimate in (4.3) and the assumed coercivity of the stored energy function \( W \) imply at once that
\[
\left| \nabla_h \phi(h) \right|^2 \leq C.
\]

We then use Lemma 2.1 and the fact that the norm \( |\cdot| \) is a matrix norm to estimate
\[
\left| \nabla_h \phi(h) \right|^2 \leq 2 \sup_{\Omega_1} |A_h^{-1}|^2 \left| \nabla_h \phi(h) \right|^2 dh dx.
\]

We thus obtain estimate (4.1).

Next we use the second estimate in (4.3), which reads
\[
\int_{\Omega_1} \left| (\nabla^2_h \phi(h) \otimes A_h) + \nabla_h \phi(h) \otimes B_h \right|^2 \, dx \leq C,
\]
since \( \kappa > 0 \). Let us first estimate the last term in the left-hand side of the previous inequality. We have
\[
\int_{\Omega_1} \left| \nabla_h \phi(h) \otimes B_h \right|^2 \, dx = \int_{\Omega_1} \sum_{i,j,k=1}^3 \left| (\nabla_h \phi(h))_{i,j,k} \right|^2 \, dx
\]
\[
\leq 27 \max_{m,j,k} \left| (B_h(x))_{m,j,k} \right|^2 \int_{\Omega_1} \sum_{i,m} \left| (\nabla_h \phi(h))_{i,m} \right|^2 \, dx
\]
\[
\leq C, \tag{4.4}
\]
due to estimate (4.1) and Lemma 2.1. Then, using Young’s inequality with estimates (4.3) and (4.4), we obtain
\[
\int_{\Omega_1} \left| (\nabla^2_h \phi(h) \otimes A_h) + \nabla_h \phi(h) \otimes B_h \right|^2 \, dx
\]
\[
\leq 2 \int_{\Omega_1} \left| \nabla_h \phi(h) \otimes B_h \right|^2 \, dx + \frac{2}{\kappa} K(h)(\phi(h)) \leq C.
\]

We then write \( \nabla^2_h \phi(h) = \nabla^2_h \phi(h) \otimes (A_hA_h^{-1}) \) and use the multiplicative tensor norm property (2.1) twice to conclude that
\[
\int_{\Omega_1} \left| \nabla^2_h \phi(h) \right|^2 \, dx \leq C,
\]
which is nothing but estimate (4.2), much in the same way as we did for estimate (4.1). \( \square \)
The following is a direct consequence of the previous lemma.

**Lemma 4.2** Let \( \phi(h) \in L^2(\Omega_1; \mathbb{R}^3) \) be a sequence verifying \( e^*(h)(\phi(h)) \leq C \). Then there exists a subsequence still denoted \( \phi(h) \) and \( \phi \in H^2(\Omega_1; \mathbb{R}^3) \), \( b \in H^1(\Omega_1; \mathbb{R}^3) \) and \( c \in L^2(\Omega_1; \mathbb{R}^3) \) such that

\[
\begin{align*}
\phi(h) & \to \phi \quad \text{weakly in } H^2(\Omega_1; \mathbb{R}^3), \\
\frac{1}{h}\phi(h),3 & \to b \quad \text{weakly in } H^1(\Omega_1; \mathbb{R}^3), \\
\frac{1}{h^2}\phi(h),33 & \to c \quad \text{weakly in } L^2(\Omega_1; \mathbb{R}^3),
\end{align*}
\]  

(4.5)

with \( (\phi, b, c) \in W_0 \), where

\[
W_0 = \{ (\phi, b, c) \in H^2(\Omega_1; \mathbb{R}^3) \times H^1(\Omega_1; \mathbb{R}^3) \times L^2(\Omega_1; \mathbb{R}^3) \text{ such that} \\
\phi,3 = 0, b,3 = 0 \text{ in } \Omega_1 \text{ and } \phi(x) = \bar{A}\Psi(x_1, x_2, 0), \\
b(x) = \bar{A}\Psi,3(x_1, x_2, 0) = \bar{Aa}3(x_1, x_2) \text{ on } \partial\omega \times ]-1/2, 1/2[ \}.
\]

**Proof** The Poincaré inequality and estimates (4.1) and (4.2) imply that

\[
\|\phi(h)\|_{H^2} \leq C, \quad \frac{1}{h}\|\phi(h),3\|_{H^1} \leq C \quad \text{and} \quad \frac{1}{h^2}\|\phi(h),33\|_{L^2} \leq C.
\]

We can thus extract a weakly convergent subsequence as in (4.5). On the other hand, the second part of estimate (4.1) implies that \( \phi,3 = 0 \) and the third part of estimate (4.2) that \( b,3 = 0 \). The boundary conditions follow from the trace theorem.

**Remark 4.1** The conditions \( \phi,3 = 0 \) and \( b,3 = 0 \) can be used to identify the functions \( \phi \) and \( b \) with functions in two variables \( \bar{\phi} \in H^2(\omega; \mathbb{R}^3) \) and \( \bar{b} \in H^1(\omega; \mathbb{R}^3) \) verifying the boundary conditions \( \bar{\phi}(x) = \bar{A}\Psi(x_1, x_2), \bar{b}(x) = \bar{Aa}3(x_1, x_2) \text{ on } \partial\omega \). We will retain the three-dimensional notation in the sequel. There is no such identification for \( c \) at this stage and there is no boundary condition for \( c \) either, since there is no trace on \( L^2(\Omega_1; \mathbb{R}^3) \).

The previous lemmas show that three-dimensional deformations of finite energy *a priori* decouple into a two-dimensional deformation and two vector fields in the zero thickness limit. It is thus convenient to reflect this in the energy by introducing a new energy functional defined on \( L^2(\Omega_1; \mathbb{R}^3)^3 \) by

\[
E(h)(\phi, b, c) = \begin{cases} 
\frac{e(h)(\phi)}{h^2} & \text{if } \phi \in V_h, b = \frac{1}{h}\phi,3 \text{ and } c = \frac{1}{h^2}\phi,33, \\
+\infty & \text{otherwise}.
\end{cases}
\]
It is clearly entirely equivalent to study the asymptotic behavior of the minimizers $\phi(h)$ of $e(h)$ and those $(\phi(h),b(h),c(h))$ of $E(h)$. It follows moreover from the same previous lemmas that these minimizers and their weak limits remain in an open ball $B$ of $L^2(\Omega_1;\mathbb{R}^3)^3$, which is metrizable for the weak topology of $L^2(\Omega_1;\mathbb{R}^3)^3$.

In order to characterize the asymptotic behavior of the minimizers, we are thus going to compute the $\Gamma$-limit of the sequence $E(h)$ on $B$ for the weak topology of $L^2(\Omega_1;\mathbb{R}^3)^3$. Since this topology is metrizable on $B$, the usual characterizations of $\Gamma$-convergence using sequences apply. Let us thus define

$$E(0)(\phi,b,c) = \begin{cases} \int_{\Omega_1} \left\{ \sum_{\iota} \left[ \left( \nabla^2 \phi + \nabla b \otimes e_3 \right) \otimes A_0 \right] + \nabla \left( \psi \phi + b \otimes e_3 \right) \otimes B_0 \right\}^2 \right. \\ + W \left[ \nabla \phi + b \otimes e_3 \right] A_0 \} d_0 dx \\ \infty \quad \text{if } (\phi,b,c) \in W_0, \\ \text{otherwise.} \end{cases}$$

where

$$A_0(x) = \nabla \psi^{-1}(\Psi(x_1,x_2,0)) = A(x)^{-1}, B_0(x) = \nabla^2 \psi^{-1}(\Psi(x_1,x_2,0))$$

and $d_0(x) = \det \nabla \psi(x_1,x_2,0) = \det A(x)$.

We obtain the following theorem.

**Theorem 4.1** Let $B$ be an open ball of $L^2(\Omega_1;\mathbb{R}^3)^3$. The sequence $E(h)$ $\Gamma$-converges on $B$ to $E(0)$ for the weak topology of $L^2(\Omega_1;\mathbb{R}^3)^3$ when $h$ goes to zero.

The proof of Theorem 4.1 is a consequence of the following two propositions.

**Proposition 4.1** We have that

$$E(0) \leq \Gamma\text{-lim inf } E(h)$$

on $B$.

**Proof** Let $(\phi,b,c) \in B$. We have to show that for every sequence $(\phi(h),b(h),c(h)) \in B$ verifying

$$\begin{cases} \phi(h) \rightharpoonup \phi & \text{in } L^2(\Omega_1;\mathbb{R}^3), \\ b(h) \rightharpoonup b & \text{in } L^2(\Omega_1;\mathbb{R}^3), \\ c(h) \rightharpoonup c & \text{in } L^2(\Omega_1;\mathbb{R}^3), \end{cases} \quad \text{(4.6)}$$

we have

$$\liminf E(h)(\phi(h),b(h),c(h)) \geq E(0)(\phi,b,c).$$
Let us first consider the case \((\phi, b, c) \not\in W_0\) for which \(E(0)(\phi, b, c) = +\infty\). By Lemma 4.2, no sequence satisfying (4.6) can have bounded energies, consequently \(\liminf E(h)(\phi(h), b(h), c(h)) = +\infty\) as well.

Next we take \((\phi, b, c) \in W_0 \cap \mathcal{B}\) and \((\phi(h), b(h), c(h)) \in \mathcal{B}\) verifying (4.6). If we assume that \(\liminf E(h)(\phi(h), b(h), c(h)) = +\infty\), then we trivially have the result in this case. From now on, we may thus assume that there exists a subsequence such that \(b(h) = \frac{1}{h}\phi(h), 3\). \(c(h) = \frac{1}{h}\phi(h), 33\). \(\phi(h) \in \mathcal{V}_h\) and such that \(E(h)(\phi(h), b(h), c(h))\) tends to the finite inferior limit. In this case, we have that \(E(h)(\phi(h), b(h), c(h)) = e(h)(\phi(h))\). Using Lemma 4.2, we see that

\[
\begin{align*}
\phi(h) \to \phi & \quad \text{in } H^2(\Omega_1; \mathbb{R}^3), \\
\frac{1}{h}\phi(h), 3 \to b & \quad \text{in } H^1(\Omega_1; \mathbb{R}^3), \\
\frac{1}{h}\phi(h), 33 \to c & \quad \text{in } L^2(\Omega_1; \mathbb{R}^3).
\end{align*}
\]

By the Rellich-Sobolev embeddings, this implies that

\[
\begin{align*}
\nabla_p \phi(h) \to \nabla_p \phi & \quad \text{in } L^p(\Omega_1; \mathcal{M}_{33}), \\
\frac{1}{h}\phi(h), 3 \to b & \quad \text{in } L^p(\Omega_1; \mathbb{R}^3),
\end{align*}
\]

for all \(1 \leq p < 6\) and we may extract a further subsequence such that the above convergences also hold almost everywhere in \(\Omega_1\) and are dominated. Then, the growth property of \(W\) and the Lebesgue dominated convergence theorem plus Lemma 2.1 imply that

\[
I(h)(\phi(h)) = \int_{\Omega_1} W\left[\left(\nabla_p \phi(h) + \frac{1}{h}\phi(h), 3 \otimes e_3\right) A_h\right] d_h dx
\]

\[
\to \int_{\Omega_1} W\left[\left(\nabla_p \phi + b \otimes e_3\right) A_0\right] d_0 dx = I(0)(\phi, b). \tag{4.8}
\]

Next, we consider the interfacial energy terms and introduce the third order tensors

\[
\Xi(h)(\phi(h)) = \sqrt{d_h}\left[\left(\nabla^2_h \phi(h)\right) \otimes A_h\right]^T \otimes A_h
\]

and

\[
\Phi(h)(\phi(h)) = \sqrt{d_h}\nabla_h \phi(h) \otimes B_h.
\]

With this notation, we see that

\[
K(h)(\phi(h)) = \kappa \int_{\Omega_1} \left\{ |\Xi(h)(\phi(h))|^2 + |\Phi(h)(\phi(h))|^2
\right.
\]

\[
+ 2\Xi(h)(\phi(h)) \cdot \Phi(h)(\phi(h))\} dx.
\]

Let us also define

\[
\Xi(0) = \sqrt{d_0}\left[\left(\nabla^2_0 \phi + \nabla_p b \otimes e_3 + \left(\nabla_p b \otimes e_3\right)^T + c \otimes e_3 \otimes e_3\right) \otimes A_0\right]^T \otimes A_0
\]
and
\[ \Phi(0) = \sqrt{d_0(\nabla_p + b \otimes e_3) \otimes B_0}. \]

The convergences (4.7) and Lemma 2.1 imply that \( \Phi(h)(\phi(h)) \) converges strongly to \( \Phi(0) \) in \( L^2(\Omega_1; M_{333}) \) so that, in particular,
\[ \int_{\Omega_1} |\Phi(h)(\phi(h))|^2 \, dx \to \int_{\Omega_1} |\Phi(0)|^2 \, dx. \] (4.9)

Consider then the dot product term. Since \( \phi(h) \to \phi \) weakly in \( H^2(\Omega_1; \mathbb{R}^3) \), \( \frac{1}{h} \phi(h),_3 \to b \) weakly in \( H^1(\Omega_1; \mathbb{R}^3) \), \( \frac{1}{h^2} \phi(h),_33 \to c \) weakly in \( L^2(\Omega_1; \mathbb{R}^3) \), and \( A_h \) and \( d_h \) converge uniformly in \( \bar{\Omega}_1 \), we see that \( \Xi(h)(\phi(h)) \) converges weakly to \( \Xi(0) \) in \( L^2(\Omega_1; M_{333}) \) and therefore
\[ \int_{\Omega_1} \Xi(h)(\phi(h)) \cdot \Phi(h)(\phi(h)) \, dx \to \int_{\Omega_1} \Xi(0) \cdot \Phi(0) \, dx. \] (4.10)

Finally, since \( \int_{\Omega_1} |\Xi(\phi(h))|^2 \, dx \) is convex with respect to \( \Xi(\phi(h)) \), hence weakly lower semicontinuous on \( L^2(\Omega_1; M_{333}) \) we have
\[ \liminf \int_{\Omega_1} |\Xi(h)(\phi(h))|^2 \, dx \geq \int_{\Omega_1} |\Xi(0)|^2 \, dx. \] (4.11)

Putting (4.9), (4.10) and (4.11) together, we obtain
\[ \liminf K(h)(\phi(h)) \geq K(0)(\phi, b, c) = \kappa \int_{\Omega_1} |\Xi(0) + \Phi(0)|^2 \, dx, \]
which gives, together with (4.8)
\[ \liminf e(h)(\phi(h)) \geq K(0)(\phi, b, c) + I(0)(\phi, b) = E(0)(\phi, b, c), \]
and thus the Proposition. \( \square \)

After having obtained the lower bound for the \( \Gamma \)-limit, we next establish the upper bound.

**Proposition 4.2** We have that
\[ \Gamma \text{-lim sup} E(h) \leq E(0) \]
on \( \mathcal{B} \).

**Proof** To prove this result, we need to show that, for every \( (\phi, b, c) \in \mathcal{B} \) there exists \( (\phi(h), b(h), c(h)) \in \mathcal{B} \) such that
\[ \begin{cases} \phi(h) \rightharpoonup \phi & \text{in } L^2(\Omega_1; \mathbb{R}^3), \\
 b(h) \rightharpoonup b & \text{in } L^2(\Omega_1; \mathbb{R}^3), \\
 c(h) \rightharpoonup c & \text{in } L^2(\Omega_1; \mathbb{R}^3), \end{cases} \] (4.12)
and
\[ E(h)(\phi(h), b(h), c(h)) \to E(0)(\phi, b, c), \]  
(4.13)
in other words, construct a recovery sequence.

If \((\phi, b, c) \notin W_0\), then no sequence satisfying (4.12) can have bounded energies, due to Lemma 4.2. Hence, there exists a subsequence such that \(E(h)(\phi, b, c) \to +\infty\), which gives (4.13) in this case.

Let us thus consider \((\phi, b, c) \in W_0 \cap B\), so that \(E(0)(\phi, b, c) = e(0)(\phi, b, c)\). Actually, since \(B\) is an open ball, it is enough to construct a recovery sequence such that the convergences in (4.12) are strong, since it will then eventually end up in \(B\) and thus be an admissible sequence for the computation of the \(\Gamma\)-limsup on \(B\).

Let us first assume that \(b\) and \(c\) are of class \(C^2\) on \(\bar{\Omega}_1\) (this is because the boundary value for \(b\) involves \(a_3\), which is of class \(C^2\)) and that \(c\) has compact support in \(\Omega_1\). The set of such functions is clearly dense in \(W_0\), see Remark 4.1. We then set
\[ \phi(h)(x) = \phi(x) + h x_3 b(x) + h^2 \int_0^x \int_0^t c(x_1, x_2, s) \, ds \, dt. \]
This function is in \(H^2\), satisfies the boundary conditions and therefore belongs to \(V_h\). Moreover, it is clear by construction that
\[
\begin{align*}
\phi(h) \to \phi & \quad \text{strongly in } H^2(\Omega_1; \mathbb{R}^3), \\
\frac{1}{h} \phi(h)_{,3} \to b & \quad \text{strongly in } H^1(\Omega_1; \mathbb{R}^3), \\
\frac{1}{h^2} \phi(h)_{,33} = c,
\end{align*}
\]
when \(h\) goes to zero. In addition, it is obvious that
\[ E(h)\left(\phi(h), \frac{1}{h} \phi(h)_{,3}, \frac{1}{h^2} \phi(h)_{,33}\right) \to E(0)(\phi, b, c), \]
so that a strongly converging recovery sequence is easily constructed in this case, which proves that
\[ \Gamma\text{-lim sup} E(h)(\phi, b, c) \leq E(0)(\phi, b, c), \]
for all \((\phi, b, c) \in W_0 \cap B\) such that \(b\) and \(c\) are \(C^2\) and \(c\) has compact support. Since this set is dense in \(W_0 \cap B\), \(E(0)\) is strongly continuous, and a \(\Gamma\)-upper limit is automatically lower semicontinuous, here for the weak topology of \(L^2(\Omega_1; \mathbb{R}^3)^3\), hence for the strong topology, we conclude by density.

\[ \square \]

The conjunction of Propositions 4.1 and 4.2 gives Theorem 4.1.
Remark 4.2 Note that there is no need to relax the elastic energy density by quasiconvexification in order to compute the \( \Gamma \)-limit of the energy. This is due to the presence of the interfacial term, which provides strong convergence in \( W^{1,q}(\Omega_1;\mathbb{R}^3) \). This is in sharp contrast with the purely hyperelastic case, see [13, 14]. □

In order for a \( \Gamma \)-limit computation to be meaningful, it is necessary that minimizers of the energies remain in a compact set for the chosen topology.

Proposition 4.3 There exists an open ball \( B \) of \( L^2(\Omega_1;\mathbb{R}^3)^3 \) such that the minimizers \( \Phi(h) \) of problem (3.2) are such that \( (\Phi(h), \frac{1}{h}\Phi(h),3, \frac{1}{h^2}\Phi(h),33) \) and their weak limit points remain in \( B \).

Proof Let \( \Phi(h)(x) = \tilde{\Phi}(z_h(x)) \in V_h \). Going back to the original curved film problem, we see that

\[
e(h)(\Phi(h)) = \frac{1}{h} \int_{\Omega_h} W(\tilde{A}) d\tilde{x} = W(\tilde{A}) \frac{\text{vol}\,\tilde{\Omega}_h}{h} \rightarrow W(\tilde{A}) \text{area}\,\tilde{S}.
\]

Since \( e(h)(\Phi(h)) \leq e(h)(\Phi(h)) \), Lemma 4.2 applies, and it suffices to add 1 to the radius of the closed ball centered at 0 given by the Lemma to obtain the radius of the desired open ball. □

Corollary 4.1 Each subsequence of the family of minimizers \( \Phi(h) \) of \( e(h) \) over \( V_h \) such that

\[
\begin{align*}
\Phi(h) &\rightharpoonup \Phi \quad \text{weakly in } H^2(\Omega_1;\mathbb{R}^3), \\
\frac{1}{h}\Phi(h),3 &\rightharpoonup b \quad \text{weakly in } H^1(\Omega_1;\mathbb{R}^3), \\
\frac{1}{h^2}\Phi(h),33 &\rightharpoonup c \quad \text{weakly in } L^2(\Omega_1;\mathbb{R}^3),
\end{align*}
\]

is such that \( (\Phi, b, c) \in W_0 \) minimizes the limit energy

\[
e(0)(\Phi, b, c) = \int_{\Omega_1} \left\{ \kappa \left[ (\nabla^2_p \Phi + \nabla_p b \otimes e_3 + (\nabla_p b \otimes e_3)^T + c \otimes e_3 e_3) \otimes A_0 \right]^T \otimes A_0 \\
+ \left[ \nabla_p \Phi + b \otimes e_3 \right] \otimes B_0 \right\}^2 + W \left[ (\nabla_p \Phi + b \otimes e_3) A_0 \right] d_0 dx
\]

over \( W_0 \).

Proof This is the standard \( \Gamma \)-convergence argument to conclude that the weak limits are minimizers over \( W_0 \cap B \). To conclude, it suffices to note that minimizers of the limit energy over \( W_0 \) certainly belong to another ball and to choose the larger of the two. □
Remark 4.3 It can be shown that the convergence of minimizers in Corollary 4.1 is actually strong. This is due to the fact that the leading term in the energy is a positive definite quadratic form, hence strictly convex functional. The proof of this fact follows along the lines of [3] by comparing the energies of the minimizers with the energies of the recovery sequence. Since this proof is fairly computational and does not contribute crucial information, we do not include it here. It shows however that the $\Gamma$-limits for the weak and the strong topologies coincide in this case. □

So far, the zero thickness limit problem still appears to be three-dimensional, via the second Cosserat vector $c$. In fact, whereas the $\Gamma$-limit energy is indeed three-dimensional, its minimizers actually are two-dimensional.

Proposition 4.4 Let $(\varphi, b, c) \in W_0$ be a minimizer of $e(0)$. Then we have

$$c_{3,3} = 0$$

in the sense of $D'(\Omega_1; \mathbb{R}^3)$.

Proof Consider the reduced minimization problem: Find $c \in L^2(\Omega_1; \mathbb{R}^3)$ such that

$$e(0)(\varphi, b, c) = \min_{\xi \in L^2(\Omega_1; \mathbb{R}^3)} e(0)(\varphi, b, \xi).$$

Solutions of this problem verify the following Euler-Lagrange equation: For all $\xi \in L^2(\Omega_1; \mathbb{R}^3)$,

$$\int_{\Omega_1} \left( \left[ \left( \nabla^2_p \varphi + \nabla_p b \otimes e_3 + \left( \nabla_p b \otimes e_3 \right)^T \otimes e_3 \otimes e_3 \right) \otimes A_0 \right]^T \otimes A_0 + \left[ \nabla_p \varphi + b \otimes e_3 \right] \otimes B_0 \right) \cdot \left[ (\xi \otimes e_3 \otimes e_3) \otimes A_0 \right]^T \otimes A_0 \, dx = 0.$$

We deduce from this that at all Lebesgue points $x$ common to the various intertwining functions, which is a set of full measure in $\Omega_1$, we have

$$(Y(x) + \left[ (c(x) \otimes e_3 \otimes e_3) \otimes A_0(x) \right]^T \otimes A_0(x)) \cdot \left[ (e_n \otimes e_3 \otimes e_3) \otimes A_0(x) \right]^T \otimes A_0(x) = 0$$

for $n = 1, 2$ and 3, with

$$Y = \left( \left[ \left( \nabla^2_p \varphi + \nabla_p b \otimes e_3 + \left( \nabla_p b \otimes e_3 \right)^T \otimes e_3 \otimes e_3 \right) \otimes A_0 \right]^T \otimes A_0 + \left[ \nabla_p \varphi + b \otimes e_3 \right] \otimes B_0 \right).$$

Now, $Y \in L^2(\Omega_1; \mathcal{M}_{333})$ is such that $Y_{3,3} = 0$ in the sense of distributions. It can thus be identified with a function $\tilde{Y} \in L^2(\Omega; \mathbb{M}_{333})$. 16
If $\zeta \in \mathbb{R}^3$, we have
\[
\left( \left[ (\zeta \otimes e_3 \otimes e_3) \otimes A_0(x) \right]^T \otimes A_0(x) \right)_{ijk} = \zeta_i A_0(x)_{3j} A_0(x)_{3k}.
\]
Therefore,
\[
\left( \left[ (c(x) \otimes e_3 \otimes e_3) \otimes A_0(x) \right]^T \otimes A_0(x) \right) \cdot \left[ \left[ (e_n \otimes e_3 \otimes e_3) \otimes A_0(x) \right]^T \otimes A_0(x) \right) = c_n(x) A_0(x)_{3j} A_0(x)_{3k} A_0(x)_{3l} = c_n(x).
\]
Indeed, we recall that $A_0(x) = A(x)^{-1} = (a^1(x)|a^2(x)|a^3(x))^T$ and that $|a^3(x)| = 1$. Consequently, we can write
\[
c_n(x) = -\nabla_3(x) \cdot \left[ \left[ (e_n \otimes e_3 \otimes e_3) \otimes A_0(x) \right]^T \otimes A_0(x) \right),
\]
almost everywhere in $\Omega_1$. Since the right-hand side does not depend on $x_3$, we obtain the result. \hfill \Box

**Remark 4.4** We now see that the limit problem is two-dimensional. If we introduce
\[
\tilde{W}_0 = \left\{ (\tilde{\phi}, \tilde{b}, \tilde{c}) \in H^2(\omega; \mathbb{R}^3) \times H^1(\omega; \mathbb{R}^3) \times L^2(\omega; \mathbb{R}^3) \text{ such that} \right. \\
\tilde{\phi}(x) = \tilde{A}\psi(x_1, x_2), \tilde{b}(x) = \tilde{A}a_3(x_1, x_2) \text{ on } \partial \omega \left\},
\]
and the two-dimensional energy
\[
\bar{e}(0)(\tilde{\phi}, \tilde{b}, \tilde{c}) = \int_\omega \left\{ \kappa \left[ (\nabla^2 \tilde{\phi} + \nabla \tilde{b} \otimes e_3 + (\nabla \tilde{b} \otimes e_3)^T + \tilde{c} \otimes e_3 \otimes e_3) \otimes A_0 \right]^T \otimes A_0 \\
+ \left| \nabla \tilde{\phi} + \tilde{b} \otimes e_3 \right|^2 B_0 \right\} d_0 dx
\]
then the two-dimensional functions $(\tilde{\phi}, \tilde{b}, \tilde{c})$ associated with the minimizers $(\varphi, b, c)$ of $e(0)$ over $W_0$, minimize $\bar{e}(0)$ over $\tilde{W}_0$.

The function $\tilde{\phi}$ describes the deformation of the midsurface of the film, and $\tilde{b}$ is a Cosserat director field more or less describing the deformation of the normal fiber to the film. The contributions of its gradient are not bending terms however, but come from the interfacial energy and are thus related to phase changes. The significance of the second Cosserat vector field $\tilde{c}$ is of little importance, as will become clear shortly. \hfill \Box
Let us now see what becomes of our model in the case of a planar film, for which the simplest chart is \( \psi(x_1, x_2) = (x_1, x_2, 0)^T \). In this case, \( A_0 = \text{Id}, a_3 = e_3, B_0 = 0 \) and \( d_0 = 1 \). Noting that several tensors are clearly orthogonal to each other, we obtain

\[
\bar{e}(0)(\Phi, b, \bar{c}) = \int_\Omega \left\{ \kappa (|\nabla^2 \Phi|^2 + 2|\nabla \bar{b}|^2 + |\bar{c}|^2) + W(\nabla \Phi + \bar{b} \otimes e_3) \right\} \, dx.
\]

For this energy, minimizers must obviously be such that \( \bar{c} = 0 \), and we recover the model of Bhattacharya and James, see [3], with only one Cosserat vector field \( \bar{b} \). This is consistent with the fact that it is shown in [3] in the planar case that \( \frac{1}{\hbar} \varphi(h)_{33} \to 0 \) in \( L^2(\Omega_1; \mathbb{R}^3) \).

We will see in the next section that the second Cosserat vector field \( \bar{c} \) does not play any role in the minimization problem in the curved case either, and thus that the same convergence also holds true (we mentioned earlier that the convergence of minimizers is actually strong). However, extra terms related to the film curvature will remain in the model as opposed to the planar case.

## 5 The curved two-dimensional limit model

The \( \Gamma \)-convergence result we have obtained is through one chart \( \psi \). Obviously, it is the same through any other chart \( \psi' \) from an open subset \( \omega' \) of \( \mathbb{R}^2 \) into \( \widetilde{S} \). In this sense, the result is intrinsic to the surface \( S \) and it makes sense to try and write it directly on \( \widetilde{S} \). Note however, that the expression in a chart is more useful for numerical purposes than the intrinsic expression below.

We thus set for all \( \bar{x} \in \widetilde{S} \), \( x = \psi^{-1}(\bar{x}) \) and \( \Phi(\bar{x}) = \tilde{\Phi}(x), \bar{b}(\bar{x}) = \tilde{b}(x), \bar{c}(\bar{x}) = \tilde{c}(x) \), \( \bar{e}(0)(\tilde{\Phi}, \tilde{b}, \tilde{c}) = \bar{e}(0)(\Phi, b, c) \) and

\[
\tilde{W}_0 = \left\{ (\tilde{\Phi}, \tilde{b}, \tilde{c}) \in H^2(\widetilde{S}; \mathbb{R}^3) \times H^1(\widetilde{S}; \mathbb{R}^3) \times L^2(\widetilde{S}; \mathbb{R}^3) \text{ such that } \tilde{\Phi}(\bar{x}) = A \bar{x}, \tilde{b}(\bar{x}) = \tilde{A} \tilde{n}(\bar{x}) \text{ on } \bar{\partial} \widetilde{S} \right\},
\]

where \( \tilde{n} \) denotes the normal vector to \( \tilde{S} \) (for a given orientation) and the \( L^2 \) spaces are considered with respect to the two-dimensional Hausdorff measure \( d\tilde{\sigma} \) on \( \tilde{S} \). All functions defined on the surface are implicitly assumed to be extended to a tubular neighborhood of the surface by being constant on each normal fiber. Partial derivatives are computed on these extensions and then restricted to the surface.

**Proposition 5.1** We have

\[
\bar{e}(0)(\tilde{\Phi}, \tilde{b}, \tilde{c}) = \int_{\widetilde{S}} \left\{ \kappa \left( \nabla^2 \tilde{\Phi}(\bar{x}) + \nabla \tilde{b}(\bar{x}) \otimes \tilde{n}(\bar{x}) + (\nabla \tilde{b}(\bar{x}) \otimes \tilde{n}(\bar{x}))^T \right) + \tilde{c}(\bar{x}) \otimes \tilde{n}(\bar{x}) \otimes \tilde{n}(\bar{x}) + \tilde{b}(\bar{x}) \otimes \nabla \tilde{n}(\bar{x}) \right\}^2 + W \left[ \nabla \tilde{\Phi}(\bar{x}) + \tilde{b} \otimes \tilde{n}(\bar{x}) \right] \right\} d\tilde{\sigma}.
\]
Moreover, \((\bar{\phi}, \bar{b}, \bar{c})\) minimizes \(\bar{e}(0)\) over \(\bar{W}_0\).

**Proof** This is basically performing the flattening step backward. The aforementioned extension to the tubular neighborhood of \(\tilde{S}\) is simply effected by letting \(\bar{\phi}(\bar{x}) = \phi(\Psi^{-1}(\bar{x}))\) and similarly for \(\bar{b}\) and \(\bar{c}\). Let us take each term in turn. We have

\[
(\nabla^2 \bar{\phi}(\bar{x}))_{ijk} = \tilde{\phi}_{i,jk}(\bar{x})
\]

\[
= \tilde{\phi}_{l,lm}(\Psi^{-1}(x))\Psi_{m,k}^{-1}(x)\Psi_{l,j}^{-1}(x) + \tilde{\phi}_{l,l}(\Psi^{-1}(x))\Psi_{l,jk}^{-1}(x)
\]

\[
= ([\nabla^2 \bar{\phi}(\Psi^{-1}(\bar{x})) \otimes \nabla \Psi^{-1}(\bar{x})]^T \otimes \nabla \Psi^{-1}(\bar{x}) + \nabla \bar{\phi}(\Psi^{-1}(\bar{x})) \otimes \nabla^2 \Psi^{-1}(\bar{x}))_{ijk}.
\]

Now, if \(\bar{x} \in \tilde{S}\), then \(\nabla \Psi^{-1}(\bar{x}) = A_0(x)\) and \(\nabla^2 \Psi^{-1}(\bar{x}) = B_0(x)\). Moreover, as \(\bar{\phi}\) does not depend on \(x_3\), we have \(\nabla \bar{\phi} = \nabla_p \bar{\phi}\) and \(\nabla^2 \bar{\phi} = \nabla^2 \bar{\phi}\).

Next, we note that \(\bar{n}(\bar{x}) = A_0^T(x)e_3\). Hence,

\[
(\nabla \bar{b}(\bar{x}) \otimes \bar{n}(\bar{x}))_{ijk} = \bar{b}_{l,j}(\bar{x})\bar{n}_k(\bar{x})
\]

\[
= \bar{b}_{l,l}(\Psi^{-1}(\bar{x}))\Psi_{l,j}^{-1}(x)(A_0)_{mk}(x)(e_3)_m
\]

\[
= ([\nabla \bar{b}(\Psi^{-1}(\bar{x})) \otimes e_3]^T \otimes \nabla \Psi^{-1}(\bar{x})]^T \otimes A_0(x))_{ijk},
\]

and we conclude as before.

For the fourth term in the interfacial energy, the same remark shows that

\[
(\bar{c}(\bar{x}) \otimes \bar{n}(\bar{x}) \otimes \bar{n}(\bar{x}))_{ijk} = \bar{c}_{i}(\bar{x})\bar{n}_j(\bar{x})\bar{n}_k(\bar{x})
\]

\[
= \bar{c}_{i}(\Psi^{-1}(\bar{x}))(A_0)_{mj}(x)(e_3)_m(A_0)_{lk}(x)(e_3)_l
\]

\[
= ([\bar{c}(\bar{x}) \otimes e_3 \otimes e_3] \otimes A_0(x))^T \otimes A_0(x))_{ijk}.
\]

Finally, for the last term in the interfacial energy, we note that we can also write \(\bar{n}(\bar{x}) = \nabla \Psi(x)^{-T} e_3 = \nabla \Psi^{-1}(\bar{x})^T e_3\). Therefore

\[
(\bar{b}(\bar{x}) \otimes \nabla \bar{n}(\bar{x}))_{ijk} = \bar{b}_{l}(\bar{x})\bar{n}_{jk}(\bar{x}) = \bar{b}_{l}(\bar{x})\Psi_{m,jk}^{-1}(\bar{x}) = \bar{b}_{l}(\bar{x})\Psi_{m,jk}^{-1}(\bar{x})(e_3)_m
\]

\[
= (\bar{b}(\bar{x}) \otimes e_3)_{lm} \Psi_{m,jk}^{-1}(\bar{x})
\]

\[
= ([\bar{b}(\bar{x}) \otimes e_3] \otimes \nabla^2 \Psi^{-1}(\bar{x}))_{ijk},
\]

hence the result for \(\bar{x} \in \tilde{S}\).

The terms in the hyperelastic energy are similar and simpler. \(\square\)
Remark 5.1 In components, the limit energy on the surface reads
\[
\tilde{e}(0)(\tilde{\phi}, \tilde{b}, \tilde{c}) = \int_S \left\{ \kappa \sum_{i,j,k=1}^3 \left| \tilde{\phi}_{i,j,k}(\tilde{x}) + \tilde{b}_{i,j}(\tilde{x})\tilde{n}_k(\tilde{x}) + \tilde{b}_{i,k}(\tilde{x})\tilde{n}_j(\tilde{x}) \right|^2 + W\left[ (\tilde{\phi}_{i,j}(\tilde{x}) + \tilde{b}_i(\tilde{x})\tilde{n}_j(\tilde{x})) \right] \right\} d\tilde{\sigma}.
\]
Note that we have the expected symmetry \( \tilde{n}_{j,k} = \tilde{n}_{k,j} \).

Let us now proceed to show that the second Cosserat vector \( \tilde{c} \) vanishes in the minimization, and thus plays no actual role in the limit model, just as in the planar case.

**Proposition 5.2** We have
\[
\nabla \tilde{\phi}(\tilde{x})\tilde{n}(\tilde{x}) = \nabla \tilde{b}(\tilde{x})\tilde{n}(\tilde{x}) = 0, \quad (5.1)
\]
\[
\nabla \tilde{n}(\tilde{x})\tilde{n}(\tilde{x}) = 0, \quad (5.2)
\]
and
\[
\nabla^2 \tilde{\phi}(\tilde{x})\tilde{n}(\tilde{x})\tilde{n}(\tilde{x}) = 0. \quad (5.3)
\]

**Proof** We note that \( \Psi_{l,j}^{-1}\Psi_{3,j}^{-1} = \delta_{3l} \) (the lines of the matrix \( \nabla \Psi^{-1} \) constitute the contravariant basis of the three-dimensional curvilinear coordinate system associated with the tubular neighborhood mapping). Therefore
\[
(\nabla \tilde{\phi}(\tilde{x})\tilde{n}(\tilde{x}))_i = \tilde{\phi}_{i,l}(\tilde{x})\Psi_{l,j}^{-1}(\tilde{x})\tilde{n}_j(\tilde{x}) = \tilde{\phi}_{i,l}(\tilde{x})\Psi_{l,j}^{-1}(\tilde{x})\Psi_{3,j}^{-1}(\tilde{x}) = \tilde{\phi}_{i,3}(\tilde{x}) = 0,
\]
and the same for the second relation in (5.1).

We have \( \tilde{n}(\tilde{x}) = \Psi_{3,3}(\Psi^{-1}(\tilde{x})) \). Therefore
\[
(\nabla \tilde{n}(\tilde{x})\tilde{n}(\tilde{x}))_j = n_{j,k}(\tilde{x})\tilde{n}_k(\tilde{x}) = \Psi_{j,3}(\tilde{x})\Psi_{i,3}^{-1}(\tilde{x})\Psi_{3,k}(\tilde{x}) = \Psi_{j,3}(\tilde{x})\delta_{3l} = 0,
\]
since \( \Psi \) is affine in \( x_3 \), and (5.2) holds true.

Finally, since \( \tilde{\phi}_{i,j}(\tilde{x})\tilde{n}_j(\tilde{x}) = 0 \) by the first relation of (5.1), differentiating with respect to \( \tilde{x}_k \) and multiplying by \( \tilde{n}_k \), we obtain
\[
(\nabla^2 \tilde{\phi}(\tilde{x})\tilde{n}(\tilde{x})\tilde{n}(\tilde{x}))_i = \tilde{\phi}_{i,j,k}(\tilde{x})\tilde{n}_j(\tilde{x})\tilde{n}_k(\tilde{x}) = -\tilde{\phi}_{i,j}(\tilde{x})n_{j,k}(\tilde{x})\tilde{n}_k(\tilde{x}) = 0
\]
by (5.2), hence (5.3).

**Remark 5.2** Note that relation (5.2) does not follow from the fact that \( |\tilde{n}| = 1 \), which implies \( \nabla\tilde{n}^T\tilde{n} = 0 \), but from the special structure of the tubular neighborhood mapping.
Proof Let us show that the tensor \( \tilde{c}(\tilde{x}) \otimes \tilde{n}(\tilde{x}) \otimes \tilde{n}(\tilde{x}) \) is orthogonal to all other tensors in the interfacial energy term. We consider each of these tensors in a row.

First of all,
\[
\nabla^2 \tilde{\phi} \cdot (\tilde{c} \otimes \tilde{n} \otimes \tilde{n}) = \tilde{\phi}_{ij,k} \tilde{c}_i \tilde{n}_j \tilde{n}_k = (\nabla^2 \tilde{\phi} \tilde{n} \tilde{n}) \cdot \tilde{c} = 0,
\]
by (5.3). Next,
\[
(\nabla \tilde{b} \otimes \tilde{n}) \cdot (\tilde{c} \otimes \tilde{n} \otimes \tilde{n}) = \tilde{b}_{i,j,k} \tilde{c}_i \tilde{n}_j \tilde{n}_k = (\nabla \tilde{b} \tilde{n}) \cdot \tilde{c} = 0,
\]
by the second relation of (5.1). Thirdly,
\[
(\nabla \tilde{b} \otimes \tilde{n})^T \cdot (\tilde{c} \otimes \tilde{n} \otimes \tilde{n}) = \tilde{b}_{i,k,j} \tilde{c}_i \tilde{n}_j \tilde{n}_k = (\nabla \tilde{b} \tilde{n}) \cdot \tilde{c} = 0,
\]
for the same reason. Finally,
\[
(\tilde{b} \otimes \nabla \tilde{n}) \cdot (\tilde{c} \otimes \tilde{n} \otimes \tilde{n}) = \tilde{b}_{i,j,k} \tilde{c}_i \tilde{n}_j \tilde{n}_k = 0,
\]
either since \(|\tilde{n}| = 1\) or by (5.2), which gives the result.

The following result is then obvious.

**Corollary 5.2** The minimizers of the limit energy are such that \( \tilde{c} = 0 \).

There is thus no effect of the second Cosserat vector in the minimization and we may as well introduce a reduced energy in the deformation of the film and one Cosserat vector as follows
\[
\tilde{c}(0)(\tilde{\phi}, \tilde{b}) = \int_{\tilde{S}} \left\{ \upsilon \left| \nabla^2 \tilde{\phi}(\tilde{x}) + \nabla \tilde{b}(\tilde{x}) \otimes \tilde{n}(\tilde{x}) + (\nabla \tilde{b}(\tilde{x}) \otimes \nabla(\tilde{\phi}(\tilde{x}))^T + \tilde{b}(\tilde{x}) \otimes \nabla(\tilde{\phi})(\tilde{x}) \right|^2 
\right. 
+ \left. W \left[ \nabla \tilde{\phi}(\tilde{x}) + \tilde{b} \otimes \tilde{n}(\tilde{x}) \right] \right\}d\tilde{\sigma}.
\]
(5.4)
to be minimized over all couples \((\tilde{\phi}, \tilde{b})\) that satisfy the boundary conditions.

The reduced energy is quite similar to the energy derived in the planar case in [3], except for the presence of the variable vector \( \tilde{n} \) and the zero-order term \( \tilde{b}(\tilde{x}) \otimes \nabla(\tilde{\phi})(\tilde{x}) \) directly linked to the curvature of the film in the interfacial energy. This is again in sharp contrast with the purely elastic case, in which the results in the planar and curved cases are basically the same, apart from additional technicalities in the curved case, see [13, 14].
Remark 5.3 We could wonder whether the interfacial energy decouples further into a sum of squares, as in the planar case. This is not entirely the case in general since not all remaining tensors are orthogonal to each other. For instance
\[ \nabla^2 \tilde{\phi} \cdot (\nabla \tilde{b} \otimes \tilde{n}) = \tilde{\phi}_{i,j,k} \tilde{b}_i \tilde{n}_k = - \nabla \tilde{b} \cdot (\nabla \tilde{\phi} \nabla \tilde{n}) \]
has no reason to vanish. If we compute all the dot products and develop the square, we obtain the equivalent expression
\[ \tilde{e}(0)(\tilde{\phi}, \tilde{b}) = \int_{\mathcal{S}} \left\{ \kappa \left[ |\nabla^2 \tilde{\phi}|^2 + 2 |\nabla \tilde{b}|^2 + |\tilde{b} \otimes \nabla \tilde{n}|^2 - 4 \nabla \tilde{b} \cdot (\nabla \tilde{\phi} \nabla \tilde{n}) + 2 \tilde{b} \cdot (\nabla^2 \tilde{\phi} \nabla \tilde{n}) \right] + W \left[ \nabla \tilde{\phi} + \tilde{b} \otimes \tilde{n} \right] \right\} d\tilde{\sigma}, \]
which reduces to the Bhattacharya and James expression when \( \nabla \tilde{n} = 0 \), i.e., in the planar case.

As a non trivial example, let us consider the case of a spherical cap of radius \( r \) centered at the origin, so that \( \tilde{n}(\tilde{x}) = \tilde{x}/r \). Then we have \( \nabla \tilde{n} = (1 - \tilde{n} \otimes \tilde{n})/r \) and the expression
\[ \tilde{e}(0)(\tilde{\phi}, \tilde{b}) = \int_{\mathcal{S}} \left\{ \kappa \left[ |\nabla^2 \tilde{\phi}|^2 + 2 |\nabla \tilde{b}|^2 + 2 |\tilde{b}|^2/r^2 - 4 \nabla \tilde{b} \cdot \nabla \tilde{\phi}/r + 6 \tilde{b} \cdot \Delta \tilde{\phi}/r \right] + W \left[ \nabla \tilde{\phi} + \tilde{b} \otimes \tilde{n} \right] \right\} d\tilde{\sigma}, \]
(5.5)
Letting \( r \to +\infty \), we recover the planar case.

Remark 5.4 It is possible to directly obtain that \( c = 0 \) by pursuing the computations performed on the Euler-Lagrange equation in the proof of Proposition 4.4. The algebra involved seems however a little daunting, and it is easier to use the intrinsic expression on the surface and the minimization instead.

The limit energy retains the fundamental invariance of continuum mechanics.

Proposition 5.3 The limit energy is frame indifferent.

Proof We must rotate both deformation and Cosserat vector simultaneously. Let us thus be given a rotation matrix \( R \in SO(3) \). For all \( A \in \mathbb{M}_{33}, b \in \mathbb{R}^3 \) and \( n \in S^2 \), we have
\[ W(RA + (Rb) \otimes n) = W(RA + R(b \otimes n)) = W(R(A + b \otimes n)) = W(A + b \otimes n), \]
hence the material indifference of the elastic energy term. Concerning the interfacial energy, we note that
\[ \nabla^2(R\tilde{\phi}) = R \otimes \nabla^2 \tilde{\phi}, \quad \nabla(R\tilde{b}) \otimes \tilde{n} = (R \nabla \tilde{b}) \otimes \tilde{n} = R \otimes (\nabla \tilde{b} \otimes \tilde{n}), \]
and

\[(\nabla (R\tilde{b}) \otimes \tilde{n})^T = (R \otimes (\nabla \tilde{b} \otimes \tilde{n}))^T = R \otimes (\nabla \tilde{b} \otimes \tilde{n})^T, \quad (R\tilde{b}) \otimes N = R \otimes (\tilde{b} \otimes N)\]

for any \(N \in \mathbb{M}_{33}\) (standing for \(\nabla \tilde{n}\)). To conclude, it suffices to remark that for any tensor \(T \in \mathbb{M}_{333}\), we have \(|R \otimes T| = |T|\). Indeed

\[|R \otimes T|^2 = R_{il}T_{ljk}R_{im}T_{mjk} = (R^T)_{li}R_{im}T_{ljk}R_{im}T_{mjk} = \delta_{lm}T_{ljk}T_{mjk} = |T|^2,\]

since \(R\) is an orthogonal matrix.

Concerning material symmetry, we notice that the three-dimensional interfacial energy is isotropic. Indeed, if we let \(\tilde{\phi}_R(\tilde{x}) = \tilde{\phi}(R\tilde{x})\), then we have

\[\nabla^2 \tilde{\phi}_R(\tilde{x}) = (\nabla^2 \tilde{\phi}(R\tilde{x}) \otimes R)^T \otimes R,\]

and it is easily checked that \(|(T \otimes R)^T \otimes R| = |T|\) whenever \(R\) is an orthogonal matrix. Hence, any material symmetry possessed by the hyperelastic energy is retained by the whole martensitic energy. Interestingly enough, this feature is lost when passing to the thin film limit, as the curvature of the film interferes with the isotropy of the interfacial energy.

To see this, we must define what isotropy, and more generally material symmetry, means for a curved film. In three-dimensional continuum mechanics, a matrix \(A\) belongs to the material symmetry group at point \(x\) if the material response of the body at point \(x\) is undistinguishable from that of the same body homogeneously predeformed by \(A\) around \(x\). In terms of energy densities in hyperelasticity, this means invariance through right multiplication by \(A\).

We adopt the same point of view for thin films. However, due to curvature, films generally are not invariant under even homogeneous deformations that preserve the tangent plane at one point. Therefore, in order to preserve the thin film, we let \(\tilde{\phi}_R(\tilde{x}) = \tilde{\phi}(\tilde{\pi}(R(\tilde{x} - \tilde{x}_0) + \tilde{x}_0))\) where \(\tilde{x}_0\) is a given point in \(\tilde{S}\) and \(\tilde{\pi}\) denotes the orthogonal projection onto \(\tilde{S}\) (see Section 2) which may defined as

\[\tilde{\pi}(\tilde{x}) = \Psi(\Psi^{-1}(\tilde{x}) - (\Psi^{-1}(\tilde{x}) \cdot e_3)e_3).\]  

(5.6)

The deformation \(\tilde{\phi}_R\) is well defined in a neighborhood of \(\tilde{x}_0\). We also perform a similar rotation on \(\tilde{b}\) by letting \(\tilde{b}_R(\tilde{x}) = \tilde{b}(\tilde{\pi}(R(\tilde{x} - \tilde{x}_0) + \tilde{x}_0))\). We will see that the interfacial energy of the couple \((\tilde{\phi}_R, \tilde{b}_R)\) at \(\tilde{x}_0\) may actually depend on \(R\).

Lemma 5.1 We have

\[\nabla \tilde{\pi} = I - \tilde{n} \otimes \tilde{n}, \quad \nabla^2 \tilde{\pi} = -\nabla \tilde{n} \otimes \tilde{n} - (\nabla \tilde{n} \otimes \tilde{n})^T - \tilde{n} \otimes \nabla \tilde{n},\]

(5.7)

on \(\tilde{S}\).
Proof Let \( H(\tilde{x}) = \Psi^{-1}(\tilde{x}) - (\Psi^{-1}(\tilde{x}) \cdot e_3)e_3 \). In \( \tilde{\Omega}_h \), we have
\[
\tilde{\pi}_{i,j}(\tilde{x}) = \Psi_{i,\alpha}(H(\tilde{x}))\Psi_{\alpha,j}^{-1}(\tilde{x}),
\]
and
\[
\tilde{\pi}_{i,j,k}(\tilde{x}) = \Psi_{i,\alpha}\beta(H(\tilde{x}))\Psi_{\alpha,j,k}^{-1}(\tilde{x}) + \Psi_{i,\alpha}(H(\tilde{x}))\Psi_{\alpha,j,k}^{-1}(\tilde{x}).
\]
On \( \tilde{S} \), we have \( H(\tilde{x}) = \Psi^{-1}(\tilde{x}) \). Consequently,
\[
\tilde{\pi}_{i,j}(\tilde{x}) = \Psi_{i,\alpha}(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j}^{-1}(\tilde{x})
= \Psi_{i,\beta}(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j}^{-1}(\tilde{x}) - \Psi_{i,3}(\Psi^{-1}(\tilde{x}))\Psi_{3,j}^{-1}(\tilde{x})
= \delta_{i,j} - \tilde{n}_i(\tilde{x})\tilde{n}_j(\tilde{x}).
\]
Similarly,
\[
\tilde{\pi}_{i,j,k}(\tilde{x}) = \Psi_{i,\alpha}\beta(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j,k}^{-1}(\tilde{x}) + \Psi_{i,\alpha}(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j,k}^{-1}(\tilde{x})
- \Psi_{i,\alpha}(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j,k}^{-1}(\tilde{x}) - \Psi_{i,3}(\Psi^{-1}(\tilde{x}))\Psi_{3,j,k}^{-1}(\tilde{x})
- \Psi_{i,33}(\Psi^{-1}(\tilde{x}))\Psi_{3,j,k}^{-1}(\tilde{x}) - \Psi_{i,3}(\Psi^{-1}(\tilde{x}))\Psi_{3,j,k}^{-1}(\tilde{x}).
\]
Now, we note that
\[
\Psi_{i,\beta}(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j,k}^{-1}(\tilde{x}) + \Psi_{i,\beta}(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j,k}^{-1}(\tilde{x}) = \tilde{x}_{i,j,k} = 0,
\]
and that
\[
\Psi_{i,33}(\Psi^{-1}(\tilde{x})) = 0.
\]
Moreover, we have \( \tilde{n}(\tilde{x}) = \Psi_{3}(\Psi^{-1}(\tilde{x})) \), so that
\[
\Psi_{i,\alpha}(\Psi^{-1}(\tilde{x}))\Psi_{\alpha,j}^{-1}(\tilde{x}) = \tilde{n}_{i,j}(\tilde{x}),
\]
and we also already used the fact that
\[
\tilde{n}_{j,k}(\tilde{x}) = \Psi_{3,j,K}(\tilde{x}).
\]
Therefore, we obtain
\[
\tilde{\pi}_{i,j,k}(\tilde{x}) = -\tilde{n}_{i,j}(\tilde{x})\tilde{n}_k(\tilde{x}) - \tilde{n}_{i,k}(\tilde{x})\tilde{n}_j(\tilde{x}) - \tilde{n}_{i}(\tilde{x})\tilde{n}_{j,k}(\tilde{x}),
\]
hence the Lemma.
we obtain \( \tilde{\phi} \) since \( \tilde{\phi} \) suffices to let \( \tilde{\phi} \) be a tubular neighborhood of \( \Omega h \) in the sense of Lemma 4.2 is given by \( \tilde{\phi}(\tilde{x}) = \tilde{x}, \ b(\tilde{x}) = \tilde{n}(\tilde{x}) \). Since this deformation has zero interfacial energy for all \( h \), it is fairly clear that its limit interfacial energy is also zero, in view of the proof of Proposition 4.1. It then suffices to note that \( \nabla^2 \tilde{\phi} = \nabla^2 \pi \).

Remark 5.5 The second formula in Lemma 5.1 can be derived directly from the expression of the reduced energy (5.4). Indeed, the weak limit of the identity mapping on \( \Omega \) in the sense of Lemma 4.2 is given by \( \tilde{\phi}(\tilde{x}) = \tilde{x}, \ b(\tilde{x}) = \tilde{n}(\tilde{x}) \).

Without loss of generality, we may assume that \( \tilde{x}_0 = 0 \). Let us consider a rotation \( R \) such that \( R\tilde{n}(0) = \tilde{n}(0) \), so that the tangent plane at 0 is invariant under the action of \( R \) (these are the only matrices that may also belong to the symmetry group of the limit hyperelastic term). Let us now proceed to show that the limit interfacial energy is generally not isotropic, due to curvature. First of all, we compute the derivatives of rotated deformations.

Lemma 5.2 Let \( R \) be such that \( R\tilde{n}(0) = \tilde{n}(0) \) and define \( \tilde{\phi}_R(\tilde{x}) = \tilde{\phi}(R\tilde{x}) \). We have

\[
\nabla \tilde{\phi}_R(0) = \nabla \tilde{\phi}(0)R, \tag{5.8}
\]

and

\[
(\nabla^2 \tilde{\phi}_R)_{ij}(0) = \tilde{\phi}_{i,lm}(0) \left[ R_{lj}R_{mk} - \tilde{n}_m(0)(R_{lj}\tilde{n}_k(0) + R_{lk}\tilde{n}_j(0)) \right] - \tilde{\phi}_{i,l}(0)R_{lm} \left[ \tilde{n}_{m,j}(0)\tilde{n}_k(0) + \tilde{n}_{m,k}(0)\tilde{n}_j(0) \right]. \tag{5.9}
\]

Proof In order to compute the above derivatives, we first need to extend \( \tilde{\phi}_R \) to a tubular neighborhood of \( \tilde{S} \) by being constant on the normal fibers. For this, it suffices to let

\[
\tilde{\phi}_R(\tilde{x}) = \tilde{\phi}_R(\tilde{\pi}(\tilde{x})) = \tilde{\phi}(\tilde{\pi}(R\tilde{x})) = \tilde{\phi}(R\tilde{\pi}(\tilde{x})),
\]

since \( \tilde{\phi} \) is itself constant on the normal fibers. Differentiating this formula once, we obtain

\[
(\tilde{\phi}_R)_{i,j}(\tilde{x}) = \tilde{\phi}_{i,l}(R\tilde{\pi}(\tilde{x}))R_{lp}\tilde{\pi}_{p,j}(\tilde{x}),
\]

and twice

\[
(\tilde{\phi}_R)_{i,jk}(\tilde{x}) = \tilde{\phi}_{i,l}(R\tilde{\pi}(\tilde{x}))R_{lp}R_{mq}\tilde{\pi}_{p,j}(\tilde{x})(\tilde{\pi}_{q,k}(\tilde{x}) + \tilde{\phi}_{i,l}(R\tilde{\pi}(\tilde{x}))R_{lp}\tilde{\pi}_{p,jk}(\tilde{x}).
\]

Now, for \( \tilde{x} = 0 \), we have \( \tilde{\pi}_{p,j}(0) = \delta_{pj} - \tilde{n}_p(0)\tilde{n}_j(0) \). Therefore,

\[
(\tilde{\phi}_R)_{i,j}(0) = \tilde{\phi}_{i,l}(0)(R_{lj} - R_{lp}\tilde{n}_p(0)\tilde{n}_j(0)) = \tilde{\phi}_{i,l}(0)(R_{lj} - \tilde{n}_l(0)\tilde{n}_j(0)),
\]

since \( R\tilde{n}(0) = \tilde{n}(0) \). Recalling that \( \tilde{\phi}_{i,l}\tilde{n}_l = 0 \), we obtain formula (5.8). Similarly, we see that

\[
\tilde{\phi}_{i,lm}(0)R_{lp}R_{mq}\tilde{n}_{p,j}(0)\tilde{n}_{q,k}(0) = \tilde{\phi}_{i,lm}(0) \left[ R_{lj}R_{mk} - \tilde{n}_m(0)(R_{lj}\tilde{n}_k(0) + R_{lk}\tilde{n}_j(0)) \right],
\]

since \( \tilde{\phi}_{i,lm}\tilde{n}_l\tilde{n}_m = 0 \) and \( \tilde{\phi}_{i,lm} = \tilde{\phi}_{i,ml} \). For the second term, we use the second formula in (5.7), together with the fact that \( \tilde{\phi}_{i,l}R_{lp}\tilde{n}_p\tilde{n}_{j,k} = (\tilde{\phi}_{i,l}\tilde{n}_l)\tilde{n}_{j,k} = 0 \). \( \square \)
**Proposition 5.4** Assume that 0 is not an umbilical point of \( \tilde{S} \). Let \( \tilde{\varphi}(\tilde{x}) = \tilde{x} \) and \( \tilde{b}(\tilde{x}) = \tilde{n}(\tilde{x}) \). Then there exists a rotation \( R \) satisfying \( R\tilde{n}(0) = \tilde{n}(0) \) such that the interfacial energy density of \( (\tilde{\varphi}, \tilde{b}_R) \) at \( \tilde{x} = 0 \) is strictly positive.

**Proof** We have \( \tilde{\varphi}_R(\tilde{x}) = \tilde{\varphi}(R\tilde{\varphi}(\tilde{x})) \) and \( \tilde{b}_R(\tilde{x}) = \tilde{n}(R\tilde{\varphi}(\tilde{x})) \). All functions will be taken at \( \tilde{x} = 0 \), which we thus omit for brevity. We use the above formulas for each term in the energy.

\[
\begin{align*}
(\tilde{\varphi}_R)_{i,l} & = \tilde{\varphi}_{i,lm}[R_{lj}R_{mk} - \tilde{n}_m(R_{lj}\tilde{n}_k + R_{lk}\tilde{n}_j)] - \tilde{\varphi}_{i,l}R_{lm}[\tilde{n}_{mj}\tilde{n}_k + \tilde{n}_{mk}\tilde{n}_j] \\
& = - (\tilde{n}_{i,l}\tilde{n}_m + \tilde{n}_{i,ml}\tilde{n}_l + \tilde{n}_{i,lm}\tilde{n}_i)R_{lj}R_{mk} \\
& \quad + (\tilde{n}_{i,l}\tilde{n}_m + \tilde{n}_{i,ml}\tilde{n}_l + \tilde{n}_{i,lm}\tilde{n}_i)(R_{lj}\tilde{n}_k + R_{lk}\tilde{n}_j) \\
& \quad - (\delta_{ij} - \tilde{n}_{i,l}\tilde{n}_j)R_{lm}(\tilde{n}_{mj}\tilde{n}_k + \tilde{n}_{mk}\tilde{n}_j) \\
& = - \tilde{n}_{i,l}\tilde{n}_kR_{lj} - \tilde{n}_{i,lm}\tilde{n}_lR_{mk} - \tilde{n}_{i,lm}\tilde{n}_iR_{lj}R_{mk} \\
& \quad + \tilde{n}_{i,lm}\tilde{n}_iR_{lj}\tilde{n}_j + \tilde{n}_{i,lm}\tilde{n}_jR_{lk}\tilde{n}_j \\
& \quad - R_{lm}(\tilde{n}_{mj}\tilde{n}_k + \tilde{n}_{mk}\tilde{n}_j) \\
& = - \tilde{n}_{i,lm}\tilde{n}_jR_{lj}R_{mk} - R_{lm}(\tilde{n}_{mj}\tilde{n}_k + \tilde{n}_{mk}\tilde{n}_j),
\end{align*}
\]

since \( R^T\tilde{n} = \tilde{n} \; , \; |\tilde{n}|^2 = 1 \) and \( \tilde{n}_{i,l}\tilde{n}_l = 0 \), viz. Proposition 5.2. Similarly, by formula (5.8),

\[
(\tilde{b}_R)_{i,j} = \tilde{n}_{i,l}R_{lj}.
\]

Therefore

\[
(\nabla \tilde{b}_R \otimes \tilde{n})_{i,jk} = \tilde{n}_{i,lm}R_{lj}.
\]

By transposition, we obtain that \( (\nabla \tilde{b}_R \otimes \tilde{n})^T_{i,jk} = \tilde{n}_{i,jm}R_{lj} \). Finally, the last term in the interfacial energy is unchanged

\[
\tilde{b}_R \otimes \nabla \tilde{n} = \tilde{n} \otimes \nabla \tilde{n}.
\]

Putting all these terms together, we obtain for the interfacial energy density \( k(\tilde{\varphi}_R, \tilde{b}_R) \) at \( \tilde{x} = 0 \)

\[
k(\tilde{\varphi}_R, \tilde{b}_R) = k \sum_{i,j,k} - \tilde{n}_{i,lm}R_{lj}R_{mk} - R_{lm}(\tilde{n}_{mj}\tilde{n}_k + \tilde{n}_{mk}\tilde{n}_j) \\
+ \tilde{n}_{i,lm}\tilde{n}_iR_{lj}\tilde{n}_j + \tilde{n}_{i,lm}\tilde{n}_jR_{lk} + \tilde{n}_{i,lm}\tilde{n}_jR_{lk}.
\]

At this point, we are at liberty to choose a coordinate system such that \( \tilde{n} = e_3 \), so that \( R_{e_3} = R_{3e_3} = 0 \) and \( \tilde{n}_{3,j} = \tilde{n}_{j,3} = 0 \). Let \( k_{ijk} \) be the generic term in the sum above. For \( i = 3 \), we have

\[
k_{3,jk} = - \tilde{n}_{i,lm}R_{lj}R_{mk} + \tilde{n}_{j,k}.
\]
Therefore, if $j = 3$ or $k = 3$, it follows that $k_{3,jk} = 0$. If, on the other hand, $j = \alpha \leq 2$ and $k = \beta \leq 2$, we have

$$k_{3\alpha\beta} = -\vec{n}_{\lambda,\mu}R_{\lambda\alpha}R_{\mu\beta} + \vec{n}_{\alpha,\beta}. \quad (5.10)$$

Next, for $i = \alpha$,

$$k_{\alpha,jk} = -R_{\alpha\mu}(\vec{n}_{\mu,j}\vec{n}_k + \vec{n}_{\mu,k}\vec{n}_j) + \vec{n}_{\alpha,\lambda}\vec{n}_\lambda R_{\lambda,j} + \vec{n}_{\alpha,\lambda}\vec{n}_j R_{\lambda,k}. \quad (5.11)$$

Therefore, if $j = k = 3$ or $j = \beta$ and $k = \gamma$, $k_{\alpha,jk} = 0$. In the remaining cases $j = \beta$, $k = 3$ or $j = 3$, $k = \beta$, we obtain

$$k_{\alpha3\beta} = k_{\alpha\beta3} = -R_{\alpha\mu}\vec{n}_{\mu,\beta} + \vec{n}_{\alpha,\lambda}R_{\lambda,\beta}. \quad (5.11)$$

Going back to the problem at hand, we see that $k(\vec{\phi}_R, \vec{b}_R) = 0$ if and only if $k_{3\alpha\beta} = k_{\alpha3\beta} = 0$. In view of formulas (5.10) and (5.11), we may rewrite these equations as

$$\vec{n}_{\alpha,\beta} = \vec{n}_{\lambda,\mu}R_{\lambda\alpha}R_{\mu\beta},$$

$$\vec{n}_{\alpha,\beta} = \vec{n}_{\lambda,\mu}R_{\alpha\lambda}R_{\beta\mu}. \quad (5.11)$$

Let us set

$$R = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we see that $k(\vec{\phi}_R, \vec{b}_R) = 0$ is equivalent to

$$(\vec{n}_{2,2} - \vec{n}_{1,1}) \sin^2 \theta = \pm 2\vec{n}_{1,2} \cos \theta \sin \theta,$$

$$\vec{n}_{1,2} = \pm (\vec{n}_{2,2} - \vec{n}_{1,1}) \cos \theta \sin \theta.$$

Clearly, these two relations hold for all $\theta$ if and only if

$$\vec{n}_{1,1} = \vec{n}_{2,2},$$

$$\vec{n}_{1,2} = \vec{n}_{2,1} = 0.$$

It can be checked that the above equalities are characteristic of an umbilical point, hence the result. \Box

Proposition 5.4 shows that the interfacial energy of a curved film is not isotropic in general. The umbilical point condition is actually necessary and sufficient.

**Proposition 5.5** Assume that $0$ is an umbilical point of $\vec{S}$. Then the interfacial energy density is isotropic at $\vec{x} = 0$.  

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Proof We directly choose the same coordinate system as above and a couple of competing deformation and Cosserat vector \((\tilde{\varphi}, \tilde{b})\), which are thus such that \(\tilde{\varphi}_3 = \tilde{b}_3 = 0\). Moreover, the gradient of the normal vector has the form \(\nabla \tilde{n} = \Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) with \(\Lambda \in \mathbb{R}\). We use Lemma 5.2 and list all the terms appearing in the interfacial energy, which we denote by \(k_{ijk}(\tilde{\varphi}_R, \tilde{b}_R)\).

For \(j = 3, k = 3\), we have
\[
(\tilde{\varphi}_R)_{i,33} = \tilde{\varphi}_{i,lm}[R_{l3}R_{m3} - 2\tilde{n}_mR_{l3}] - 2\tilde{\varphi}_{i,l}R_{lm}\tilde{n}_{m,3} = 0,
\]
for the deformation and
\[
(\nabla \tilde{b}_R \otimes \tilde{n})_{i33} = \tilde{b}_{i,l}R_{l3} = \tilde{b}_{i,3} = 0,
\]
and
\[
(\tilde{b}_R \otimes \nabla \tilde{n})_{i33} = \tilde{b}_{i}\tilde{n}_{3,3} = 0,
\]
for the Cosserat vector. The corresponding term \(k_{i33}(\tilde{\varphi}_R, \tilde{b}_R) = 0\) is thus independent of \(R\).

For \(j = 3, k = \alpha\), we have
\[
(\tilde{\varphi}_R)_{i,3\alpha} = \tilde{\varphi}_{i,lm}R_{l3}R_{m\alpha} - \tilde{\varphi}_{i,\beta}R_{l3}R_{l\alpha} - \tilde{\varphi}_{i,3}R_{l\alpha}R_{lm}\tilde{n}_{m,\alpha}
\]
\[
= \tilde{\varphi}_{i,\beta}\tilde{R}_{\beta\alpha} - \tilde{\varphi}_{i,3\beta}R_{\beta\alpha} - \Lambda \tilde{\varphi}_{i,\beta}R_{\beta\alpha}
\]
\[
= -\Lambda \tilde{\varphi}_{i,\beta}R_{\beta\alpha}
\]
for the deformation and
\[
(\nabla \tilde{b}_R \otimes \tilde{n})_{i3\alpha} = \tilde{b}_{i,l}R_{l\alpha} = \tilde{b}_{i,\beta}R_{\beta\alpha}, \quad (\nabla \tilde{b}_R \otimes \tilde{n})_{i3\alpha}^T = \tilde{b}_{i,l}R_{l3}\tilde{n}_{\alpha} = 0,
\]
and
\[
(\tilde{b}_R \otimes \nabla \tilde{n})_{i3\alpha} = \tilde{b}_{i}\tilde{n}_{3,\alpha} = 0,
\]
for the Cosserat vector. Therefore
\[
\sum_{\alpha} k_{i3\alpha}(\tilde{\varphi}_R, \tilde{b}_R) = \sum_{\alpha} |\Lambda \tilde{\varphi}_{i,\beta}R_{\beta\alpha} + \tilde{b}_{i,\beta}R_{\beta\alpha}|^2
\]
\[
= \sum_{\alpha} |(-\Lambda \tilde{\varphi}_{i,\beta} + \tilde{b}_{i,\beta})R_{\beta\alpha}|^2
\]
\[
= \sum_{\alpha} |\Lambda \tilde{\varphi}_{i,\alpha} + \tilde{b}_{i,\alpha}|^2
\]
\[
= \sum_{\alpha} k_{i3\alpha}(\tilde{\varphi}, \tilde{b}).
\]
Finally, for \( j = \alpha \) and \( k = \beta \), we have
\[
(\tilde{\varphi}_R)_{i,\alpha\beta} = \tilde{\varphi}_{i,\lambda\rho} R_{\lambda\rho} R_{\alpha\beta}
\]
for the deformation and
\[
(\nabla \tilde{b}_R \otimes \tilde{n})_{i\alpha\beta} = \tilde{b}_{i,j} R_{\alpha\beta} = 0 = (\nabla \tilde{b}_R \otimes \tilde{n})^T_{i\alpha\beta},
\]
and
\[
(\tilde{b}_R \otimes \nabla \tilde{n})_{i\alpha\beta} = \Lambda \tilde{b}_{i,\delta_{\alpha\beta}},
\]
for the Cosserat vector. Therefore
\[
\sum_{\alpha,\beta} k_{i\alpha\beta}(\tilde{\varphi}_R, \tilde{b}_R) = \sum_{\alpha,\beta} |\tilde{\varphi}_{i,\lambda\rho} R_{\lambda\rho} R_{\alpha\beta} + \Lambda \tilde{b}_{i,\delta_{\alpha\beta}}|^2
\]
\[
= \sum_{\alpha,\beta} |(\tilde{\varphi}_{i,\lambda\rho} + \Lambda \tilde{b}_{i,\delta_{\alpha\beta}}) R_{\lambda\rho} R_{\alpha\beta}|^2
\]
\[
= \sum_{\alpha,\beta} |\tilde{\varphi}_{i,\alpha\beta} + \Lambda \tilde{b}_{i,\delta_{\alpha\beta}}|^2
\]
\[
= \sum_{\alpha,\beta} k_{i\alpha\beta}(\tilde{\varphi}, \tilde{b}),
\]

hence the isotropy in this case. \( \square \)

**Corollary 5.3** The only surfaces \( \tilde{S} \) such that the thin film interfacial energy is isotropic are either planar or spherical.

**Proof** It is a classical result in differential geometry that if all points of a surface are umbilical, then the surface is either part of a sphere or part of a plane. \( \square \)

**Remark 5.6** The isotropy in the planar case is obvious. It can also be easily checked directly on formula (5.5) in the spherical case. \( \square \)

**Corollary 5.4** Let \( R \in SO(3) \) be an element of the material symmetry group of the hyperelastic energy \( W \) and \( \tilde{x} \) and umbilical point of \( \tilde{S} \) such that \( R\tilde{n}(\tilde{x}) = \tilde{n}(\tilde{x}) \). Then the whole martensitic energy has symmetry \( R \) at \( \tilde{x} \).

**Proof** It suffices to consider the hyperelastic term. We have, for all \((\tilde{\varphi}, \tilde{b})\)
\[
W(\nabla \tilde{\varphi} R + \tilde{b} \otimes \tilde{n}) = W(\nabla \tilde{\varphi} R + \tilde{b} \otimes (R^T \tilde{n})) = W((\nabla \tilde{\varphi} + \tilde{b} \otimes \tilde{n}) R) = W(\nabla \tilde{\varphi} + \tilde{b} \otimes \tilde{n}),
\]
hence the result. \( \square \)
6 Vanishing interfacial energy

In this section, we follow the lead of [3] and wonder what happens if we decide to set \( \kappa = 0 \) in the limit thin film energy, since \( \kappa \) is typically very small compared with the elastic moduli, even though \( \kappa > 0 \) plays a crucial role in the \( \Gamma \)-limit analysis. The goal is to see whether it is possible to conceive nontrivial tunnel- and tent-like deformations made possible by phase changes and that could be used to design microactuators, see [3] for details. This kind of analysis relies crucially on the multiwell structure of the hyperelastic stored energy function, which has played no role up to now.

We thus recall that we are given a finite number of symmetric positive definite matrices \( U_0 = I, U_1, U_2, \ldots, U_n \) such that \( W(U_i) = 0 \) and \( W(F) > 0 \) for all \( F \notin \cup_{i=0}^{n} SO(3) U_i \). The deformation gradient \( U_0 = I \) corresponds to the austenite phase, which we thus assume to be the reference configuration, and \( U_i, i = 1, \ldots n \) to the different martensite variants. In this case, the integral energy is positive, and may be minimized by pointwise minimizing deformations whose gradients fall into the energy wells. This process is rather rigid in the three-dimensional case, but as is shown in [3], the thin planar film case offers much more flexibility, the one-dimensional case being even less constrained, see [11, 12].

Let us start with discussing single phase deformations.

**Proposition 6.1** Let \( U \) be a given distortion matrix. A deformation/Cosserat vector pair \( (\tilde{\varphi}, \tilde{b}) \) of the film corresponds to the phase \( U \) if and only if

\[
\tilde{b} = RU\tilde{n} \quad \text{and} \quad \nabla \tilde{\varphi} = RU(I - \tilde{n} \otimes \tilde{n}),
\]

for some rotation \( R \in SO(3) \).

**Proof** Assume that we are given a pair \( (\tilde{\varphi}, \tilde{b}) \) such that \( \nabla \tilde{\varphi} + \tilde{b} \otimes \tilde{n} = RU \) for some rotation \( R \). Multiplying on the right by \( \tilde{n} \), we first obtain \( \tilde{b} = RU\tilde{n} \) since \( \nabla \tilde{\varphi} \tilde{n} = 0 \). The result immediately follows as \( \nabla \tilde{\varphi} = RU - b \otimes \tilde{n} = RU - RU\tilde{n} \otimes \tilde{n} \).

**Remark 6.1** A necessary condition for (6.1) to hold is that

\[
\nabla \tilde{\varphi}^T \nabla \tilde{\varphi} = (I - \tilde{n} \otimes \tilde{n})C(I - \tilde{n} \otimes \tilde{n}),
\]

where \( C = U^2 \) is the corresponding strain tensor. Condition (6.2) is also sufficient if \( \tilde{n} \) is an eigenvector of \( C \). Otherwise, there is a polar factorization for \( \nabla \tilde{\varphi} \) as in the second relation in (6.1), but with \( U \) replaced by \( U' = [(I - \tilde{n} \otimes \tilde{n})C(I - \tilde{n} \otimes \tilde{n})]^{1/2} \).

We have \( U = U' \) if and only if \( \tilde{n} \) is an eigenvector of \( C \). Once this second relation is satisfied, \( \tilde{b} \) is reconstructed via the first relation in (6.1). \( \square \)

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The construction of single-phase deformations follows the same recipe as that of the planar case.

**Proposition 6.2** Let $U$ be a given distortion matrix and $\tilde{\theta}: \tilde{U} \tilde{S} \rightarrow \mathbb{R}^3$ an isometry. Then $\tilde{\varphi}(\tilde{x}) = \tilde{\theta}(U \tilde{\pi}(\tilde{x}))$ is a deformation of $\tilde{S}$ that satisfies condition (6.1). Conversely, if $\tilde{\varphi}$ satisfies condition (6.1), then $\tilde{\theta}(\tilde{y}) = \tilde{\varphi}(U^{-1} \tilde{\pi}_U(\tilde{y}))$, where $\tilde{\pi}_U$ denotes the orthogonal projection on $\tilde{U}$, is an isometry of $\tilde{US}$.

**Proof** Let $\tilde{n}_U$ be the normal vector to $\tilde{US}$, which is given by

$$\tilde{n}_U = \frac{\text{cof} U \tilde{n}}{\|\text{cof} U \tilde{n}\|}.$$ 

The fact that $\tilde{\theta}$ is an isometry amounts to

$$\nabla \tilde{\theta} = R(I - \tilde{n}_U \otimes \tilde{n}_U),$$

for some rotation matrix $R$. Let us thus compute the gradient of $\tilde{\varphi}$.

$$\nabla \tilde{\varphi}(\tilde{x}) = \nabla \tilde{\theta}(U \tilde{\pi}(\tilde{x})) U(I - \tilde{n} \otimes \tilde{n})$$

$$= R(I - \tilde{n}_U \otimes \tilde{n}_U) U(I - \tilde{n} \otimes \tilde{n})$$

$$= RU(I - \tilde{n} \otimes \tilde{n}) - K \tilde{n}_U \otimes \tilde{n}_U U(I - \tilde{n} \otimes \tilde{n}).$$

Now we note that

$$\tilde{n}_U \otimes \tilde{n}_U U = [\tilde{n}_U \otimes (U \tilde{n}_U)] = \frac{\det U}{\|\text{cof} U \tilde{n}\|} \tilde{n}_U \otimes \tilde{n}$$

and

$$\tilde{n}_U \otimes \tilde{n}_U U \tilde{n} \otimes \tilde{n} = [\tilde{n}_U \otimes (U \tilde{n}_U)] \tilde{n} \otimes \tilde{n} = \frac{\det U}{\|\text{cof} U \tilde{n}\|} \tilde{n}_U \otimes \tilde{n},$$

since $U$ is symmetric, hence the first part of the result.

Conversely, let us be given $\tilde{\varphi}$ that satisfies condition (6.1) and let us compute the gradient of $\tilde{\theta}$. We obtain

$$\nabla \tilde{\theta}(\tilde{y}) = \nabla \tilde{\varphi}(U^{-1} \tilde{\pi}_U(\tilde{y})) U^{-1}(I - \tilde{n}_U \otimes \tilde{n}_U)$$

$$= RU(I - \tilde{n} \otimes \tilde{n}) U^{-1}(I - \tilde{n}_U \otimes \tilde{n}_U)$$

$$= R(I - \tilde{n}_U \otimes \tilde{n}_U) - RU(\tilde{n} \otimes \tilde{n}) U^{-1}(I - \tilde{n}_U \otimes \tilde{n}_U).$$

Again we note that

$$U(\tilde{n} \otimes \tilde{n}) U^{-1} = \frac{\|\text{cof} U \tilde{n}\|}{\det U} (U \tilde{n}) \otimes \tilde{n}_U$$

and

$$U(\tilde{n} \otimes \tilde{n}) U^{-1}(\tilde{n}_U \otimes \tilde{n}_U) = \frac{\|\text{cof} U \tilde{n}\|}{\det U} ((U \tilde{n}) \otimes \tilde{n}_U) (\tilde{n}_U \otimes \tilde{n}_U) = \frac{\|\text{cof} U \tilde{n}\|}{\det U} (U \tilde{n}) \otimes \tilde{n}_U,$$

which establishes the proposition.  

\[\square\]
It follows that the construction of nontrivial single phase deformations hinges on the existence of nontrivial isometries for the surface $U\tilde{S}$, a question which entails considerations of geometric rigidity in differential geometry, see [19], chapter 12, for a review of results. Let us mention that convex surfaces minus a disc are known to be bendable, i.e., admit a continuous family of nontrivial isometries. For example, subsets of a sphere minus a disc can be isometrically deformed using rotation surfaces of constant curvature. Such isometries do not however take into account boundary conditions, or in our context, interface conditions between different phases, and thus seem hard to exploit.

Indeed, the tunnel and tents described in [3] in the planar case are made of a finite number of pieces of film, each of which is in a different phase. The question is how to patch such pieces together into a single film deformation.

In this respect, the analysis carried out in [3] remains valid here, namely an invariant line condition must be satisfied by the tangent to an interface curve, cf. Proposition 5.1 of the above mentioned article. On a curved surface, the invariant line condition will in general yield curves, and it is not clear whether a deformed isometry can be made to fit such a curve, given the paucity of information available regarding surface isometries.

A simple case however is that of cylinders, for which the construction of tunnels is quite similar to the planar case. Let us consider a simple example with $\omega = [-2,2]^2$ and $\tilde{S}$ the cylinder defined by $\psi(x) = (\psi_1(x_1), x_2, \psi_3(x_1))^T$. We may assume that $\psi_1'(x_1)^2 + \psi_3'(x_1)^2 = 1$. Let us be given a diagonal distortion matrix $U$ of the form

$$
U = \begin{pmatrix}
\mu & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \nu
\end{pmatrix}.
$$

We can construct a two-phase deformation using the above recipe by defining $\varphi = \psi$, which is to say $\varphi = id$ for $|x_1| > 1$, in which case the material is in the austenite phase. For $|x_1| < 1$, we first stretch the surface by $U$, which yields $U\psi(x) = (\mu\psi_1(x_1), x_2, \nu\psi_3(x_1))^T$. The length element along the base of the stretched cylinder is given by $dx^2 = (\mu^2\psi_1'(x_1)^2 + \nu^2\psi_3'(x_1)^2)\,dx_1$. Let

$$
L = \sqrt{(\psi_1(1) - \psi_1(-1))^2 + (\psi_3(1) - \psi_3(-1))^2}.
$$

Clearly, it is possible to join the two interface lines corresponding to $x_1 = \pm 1$ by an isometric deformation of the stretched cylinder if and only if

$$
L \leq \int_{-1}^{1} \sqrt{\mu^2\psi_1'(x_1)^2 + \nu^2\psi_3'(x_1)^2} \,dx_1.
$$

This construction leads to tunnels similar to those of [3] as shown below,
but also to possible “shortcuts” if \( \mu < 1 \) and/or \( \nu < 1 \), depending on the reference geometry of the film.

**Remark 6.2** It should be noted that all the above considerations concern a film for which the different variants are, in a sense, fixed in space, that is to say as if the film had been manufactured by carving it out of a solid block of martensitic material. In practice, such a film would presumably be deposited on a curved substrate, and it makes more sense to think that the variants would follow the geometry of the film. This can easily be modeled by allowing an energy density of the form \( W(\tilde{\pi}(\tilde{x}), F) \) with potential wells \( U_i(\tilde{\pi}(\tilde{x})) \). The \( \Gamma \)-convergence analysis is unchanged and the limit model is simply obtained by substituting this density in place of the homogeneous density we have considered up to now.

We could furthermore assume that \( \tilde{n}(\tilde{x}) \) is an eigenvector of \( U_i(\tilde{x}) \), for instance by letting \( U_i(\tilde{x}) = Q(\tilde{x})^T (\sum_{k=1}^{3} \mu_k e_k \otimes e_k) Q(\tilde{x}) \) and \( Q(\tilde{x})\tilde{n}(\tilde{x}) = e_3 \), in which case a necessary and sufficient condition for \( \tilde{\phi} \) to represent a deformation in phase \( i \) is that

\[
\nabla\tilde{\phi}^T(\tilde{x}) \nabla\tilde{\phi}(\tilde{x}) = (I - \tilde{n}(\tilde{x}) \otimes \tilde{n}(\tilde{x})) U_i(\tilde{x})^2 (I - \tilde{n}(\tilde{x}) \otimes \tilde{n}(\tilde{x})),
\]

as before. In this case, however, the generic construction of single-phase deformations using isometries no longer works. Let us show how another construction can be achieved in the same example as above. The computations are easier in the
chart domain. The covariant basis only depends on $x_1$ and is given by

$$a_1(x_1) = \begin{pmatrix} \psi'_1(x_1) \\ 0 \\ \psi'_3(x_1) \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } a_3(x_1) = \begin{pmatrix} -\psi'_3(x_1) \\ 0 \\ \psi'_1(x_1) \end{pmatrix}.$$ 

Note that this basis is orthonormal, hence equal to its contravariant basis. Let us assume that we have a deposited martensite variant of the form

$$U(x_1) = \mu a_1(x_1) \otimes a_1(x_1) + a_2 \otimes a_2 + \nu a_3(x_1) \otimes a_3(x_1)$$

that follows the surface. For $|x_1| < 1$, we define

$$\varphi(x) = \begin{pmatrix} \theta_1(\mu x_1) \\ x_2 \\ \theta_3(\mu x_1) \end{pmatrix}$$

where $\theta'_1(x_1)^2 + \theta'_3(x_1)^2 = 1$. Then it is easy to check that

$$\nabla \varphi A_0 = R(x_1) U(x_1) (I - a_3(x_1) \otimes a_3(x_1)),$$

where $R(x_1)$ is the rotation of axis $a_2$ that takes $a_1(x_1)$ to $(\theta'_1(x_1), 0, \theta'_3(x_1))^T$. Therefore, the construction is complete if we can join the three film parts into one continuous single film using a curve $\theta$ parametrized by arclength, which is clearly possible if and only if $L \leq 2\mu$. We then set $b(x_1) = a_3(x_1)$ in the austenite and $b(x_1) = \nu R(x_1) a_3(x_1)$ in the martensite.

It should be noted that the above construction is quite specific to the cylinder case, and that generalizing it to other surfaces may not be straightforward. □

**References**


